ANALYTIC GEOMETRY AND CALCULUS

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AND

CALCULUS

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PREFACE

From a subject once presented in more detail, analytic geometry has come to be regarded almost entirely as a preparation for the calculus. It is in recognition of this fact that the authors have fixed the aim of this book: to present, in a single volume, the essentials of both those subjects, in a manner adequate to the needs of the courses as offered in American colleges and technical schools.

The word "essentials" is used deliberately. The authors feel that some topics generally treated in current textbooks—topics such as poles and polars related to conics, diameters of a conic, oblique asymptotes, singular points of a curve—can be much more effectively treated in a second course on these subjects, and much more appropriately.

On the other hand, greater than usual care has been devoted to certain matters where ideas of first importance are involved—among them solid analytic geometry, the notion of a limit, continuity of functions, partial differentiation, and approximate integration. The treatment of the last named subject—to quote an instance—is strengthened by the presentation of the Gauss method and of the Euler-Maclaurin formula, an addition that imposes no severe strain on the student but puts within his hands a powerful aid in numerical approximation.

The chief problem that confronts an author of a first course such as this, however, is not the scope of its contents—that, after all, is, within certain limits, well-nigh prescribed—but rather the spirit in which he chooses to write.

The authors hesitate to use the word "rigor" in connection with an elementary course, but certainly the matter of care which is to be given to precision in one's definitions and proofs is something that cannot be lightly dismissed. A book may be made more pleasant to read by the device of skirting the edges of any proof that may become difficult, or by scant concern for exceptional cases that so often beset a statement. This is not the way, the authors believe, to introduce a student to the sub-

ject of analysis. A proof, here and there, is apt to involve difficulty, but no service is done the student by sidestepping such occasional difficulties. Habits of care and precision in one's mathematics must be instilled early if they are to take root. This belief has set the tone for the pages of this book, with the hope that it shall not be just a collection of topics but shall serve as a genuine introduction to analysis.

Every teacher has had the experience of confronting a tendency, on the part of the student, to glance at just enough of the text to pick out the formulas he needs in the solution of his problems and to slight the rest of the text. An effective way to check this tendency is to make the student responsible for the exposition in the text, by requiring him to contribute to that exposition, extending it here, drawing a corollary there, and generally taking his own part in supplying details and rounding out the argument of the text. To this end, the book is dotted with exercises, as distinct from problems, appearing wherever appropriate within the text. In general all these exercises should be worked. In so doing, the student becomes an active participant in the development of the text, a task that he will find interesting and profitable.

A book such as this must of necessity be also a drill book, and that function can be served only by including problems. The lists of problems throughout the book will be found to be numerous without stint, to be graded to a wide degree of difficulty, and to exploit in every possible way the ideas encountered in the text.

The authors wish to acknowledge the interest of their colleagues at Case School of Applied Science, especially that of Professor C. F. Thomas and Dr. C. C. Torrance, who have read with care and helpfully criticized several portions of this book.

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CASE SCHOOL OF APPLIED SCIENCE,
April, 1937.

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ANALYTIC GEOMETRY

CHAPTER I

INTRODUCTION

1. Coordinates on a Line. Let a line of indefinite extent be graduated with a scale showing the distance, in either direction, from a fixed point O of the line. We shall call O the *origin*, and the number representing the distance from O to any point P of the line, the *coordinate* of P. This coordinate will be reckoned as

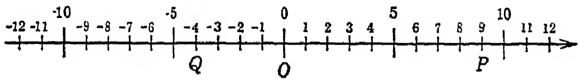


FIG. 1.

positive or negative according as the corresponding distance is measured to the right or to the left of the origin. Thus, the coordinates of the points P and Q are 9 and -4, respectively. The coordinate of the origin is obviously 0.

This arrangement establishes a correspondence between the real numbers and the points of our line, such that to each point of the line corresponds a unique real number (its coordinate) and to each real number x a point R on the line, *viz.*, the point of which x is the coordinate. Such a correspondence between real numbers and the points of a line is called a (one-dimensional) *system of coordinates.*

It will be convenient to designate a point of the line by stating its coordinate in parentheses. Thus, (u) shall represent the point having the real number u for its coordinate, and the notation S (u) will serve to indicate that the point is labeled S and has the coordinate u.

Let us note that in treating distances between points on a line, it is frequently desirable to consider *directed distances, i.e.*, distances considered as to numerical value and sign. We shall define the *directed* distance from P_1 (x_1) to P_2 (x_2) by the equality

$$P_1P_2 = x_2 - x_1.$$

This will, evidently, represent the directed distance P_1P_2 by a positive number if P_2 is to the right of P_1 (*i.e.*, if P_1P_2 is headed to

the right), and by a negative number if P_2 is to the left of P_1 (*i.e.*, if P_1P_2 is headed to the left). Likewise, the directed distance from P_2 to P_1 will then be given by

$$P_2P_1 = x_1 - x_2.$$

Reversing the direction of a directed distance, thus, results in reversing its sign. *I.e.*,

$$P_2P_1 = -P_1P_2.$$

Exercise 1. Prove that the point P (x) which divides the segment from P_1 (x_1) to P_2 (x_2) in the ratio $r_1:r_2$ (*i.e.*, so that $P_1P:PP_2 = r_1:r_2$, where the distances are directed), is located by the formula

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \qquad r_1 \neq -r_2. \tag{1}$$

HINT: $P_1P = x - x_1$, $PP_2 = x_2 - x$.

Exercise 2. As a corollary to Exercise 1 prove that the point P (x) midway between P_1 (x_1) and P_2 (x_2) is located by the formula

$$x = \frac{x_1 + x_2}{2}.$$

Problems

1. Find the directed distance from P_1 (2) to P_2 (7); also from Q_1 (-3) to Q_2 (2); also from R_1 (4) to R_2 (-2); also from S_1 (-2) to S_2 (-8).
Ans. $P_1P_2 = 5$; $R_1R_2 = -6$.

2. For the points P_1 (x_1), P_2 (x_2), P_3 (x_3) on a line, prove that if the distances in question are directed, then $P_1P_2 + P_2P_3 = P_1P_3$. Consider the case where P_2 is between P_1 and P_3; also where P_2 is on P_1P_3 produced.

3. Find the coordinates of the two points whose distances from P (5) are 6; also of the point whose directed distance from P (5) is 4; also of the point whose directed distance from P (5) is -8.

4. Find the point dividing the segment from P_1 (-2) to P_2 (7) in the ratio 1:4.
Ans. $P(-1/5)$.

5. Find the two points of trisection of the line segment joining P (-5) and Q (11).

6. What interpretation may be placed upon formula (1) of the text if $r_1:r_2$ is negative?

7. Find the point dividing the segment from P (0) to Q (5) in the ratio $-2:3$.
Ans. (-10).

8. Find the point dividing the segment from R (6) to S (-2) in the ratio -2.

9. Find x if the line segment from P_1 (x) to P_2 (6) is divided in the ratio 3:5 by the point P (-4).
Ans. $x = -10$.

10. Find the midpoint of the line segment joining P_1 (7) to P_2 (-5).

2. Cartesian Coordinates in a Plane. To establish a system of coordinates for points of a plane, we shall set up two lines in the plane, at any convenient angle to each other, each graduated as in Section 1 with their point of intersection used as a common origin. We shall call these lines the *coordinate axes* and designate one of them as the *x-axis*, and the other as the *y-axis*, with a definite direction on each line chosen as positive. The common origin for the two lines shall be known as the *origin* of the system. When the axes are at right angles (in which case the system is called *rectangular*), the positive direction on the x-axis is chosen to the

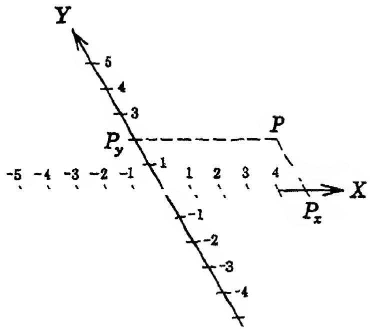


FIG. 2.

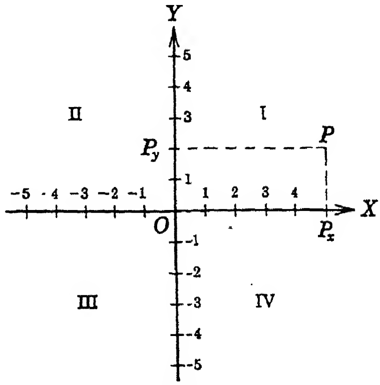


FIG. 3.

right of the origin, and on the y-axis, upward from the origin. Let, now, P be any point in the plane. Draw lines through P parallel to the axes, meeting the x-axis in the point P_x and the y-axis in P_y. If x is the coordinate of P_x (on OX) and y the coordinate of P_y (on OY), we shall say that the *coordinates* of the point P in the plane are x and y, and shall represent the point P by the notation (x,y). Thus, in Figs. 2 and 3 the point P is (5,2). An equivalent way of defining the coordinates of P is to say that they are the numbers representing the directed distances OP_x and OP_y. This system provides, then, a correspondence between the points of the plane and *pairs* of real numbers such that to each point of the plane corresponds a unique pair of real numbers, and to each such pair a unique point of the plane.

We have already seen how to obtain the two coordinates of a given point. To locate a point when its coordinates x and y are given, we locate P_x on OX by its coordinate x, and P_y on OY by its coordinate y, draw a line through P_x parallel to OY and a

line through P_y parallel to OX. Their intersection uniquely determines the point P (x,y). This two-dimensional system of coordinates is called the *Cartesian* system.*

The first coordinate, x, of a point is called its *abscissa,* and the second coordinate, y, its *ordinate.* The four parts into which the axes divide the plane are called quadrants and are numbered as shown in Fig. 3.

Unless otherwise stated, the use of Cartesian coordinates throughout this book will imply a rectangular system.

Problems

1. Plot the points (3,7), (−7,3), (0,5), (6,−2), (−1,−8), (0,0) (7/2,−9), (−2,0).

2. Draw the triangles having the following sets of points as vertices: (*a*) (1,2), (6,−5), (−3,−4). (*b*) (2,0), (5,8), (−6,3). (*c*) (0,−5), (−1,−8), (−3,6).

3. (*a*) If the abscissa of a point is 3, what can be said about its position?
(*b*) If the ordinate of a point is −5, what can be said about its position?

4. What is true of the coordinates x and y for every point, (*a*) on the y-axis? (*b*) on the x-axis? (*c*) on the line bisecting the angle between the axes and lying in quadrants I and III? (*d*) on the line bisecting the angle between the axes and lying in quadrants II and IV?

5. Two distinct points on a line each have the abscissa −6. What is known concerning the coordinates of all other points on the line? about the position of the line?

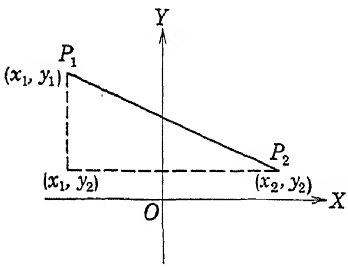


FIG. 4.

6. (*a*) The end points of the base of an isosceles triangle are (1,2) and (5,2). The length of its altitude is 3. Find the coordinates of its vertex.

(*b*) Two opposite vertices of a square are (−4,3) and (−4,9). Find the other two vertices.

(*c*) One vertex of a square is the origin, and its center is (−3,3). Find all the other vertices.

3. Length of a Line Segment. Since the locations of two points P_1 and P_2 are determined by their coordinates (x_1,y_1) and (x_2,y_2), these coordinates must also determine the distance between the points. To find this distance, draw a line through P_1 parallel to OY and a line through P_2 parallel to OX as in Fig. 4.

* After René Descartes (1596–1650), a distinguished French mathematician and philosopher, who is credited with the invention of analytic geometry.

(A line through P_1 parallel to OX and a line through P_2 parallel to OY would do as well.) The distance sought is seen to be the hypotenuse of a right triangle in which the perpendicular sides are of lengths $|x_2 - x_1|$ and $|y_2 - y_1|$.* Hence, by the Pythagorean theorem,

$$P_1P_2 = \sqrt{[|x_2 - x_1|]^2 + [|y_2 - y_1|]^2},$$

or

$$P_1P_2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

Problems

1. Find the lengths of the lines joining the pairs of points:
(*a*) (2,1), (−3,5).
(*b*) (0,4), (−6,2).
(*c*) (5,11), (−1,0).
(*d*) (3, −6), (−17,15).
Ans. (*a*) $\sqrt{41}$.

2. Show that the triangles with the following vertices are isosceles:
(*a*) (−6,5), (−2,4), (−5,1).
(*b*) (2,3), (−3,0), (5,8).
(*c*) (3, −2), (4,6), (−4,2).

3. Show that the triangles with the following vertices are right triangles and find their areas: (*a*) (3, −6), (8, −2), (−1, −1). (*b*) (3,1), (4,3), (6,2).

4. A circle with center at (−4,2) passes through (4,3). Find which of the following points lie on the circle: (−11,6), (−4,10), (0,9).

5. Show that the points (13,10), (21, −5), (6, −13) and (−2,2) are the vertices of a square. Find the length of a diagonal.

6. Show that the points (−3,2), (1, −2) and (9, −10) lie on a line.

7. Find whether or not the points (14,7), (2,2), and (−4, −1) lie on a line.

8. Find the value of z if the points $(z,4)$ and $(-4,z)$ are $2\sqrt{58}$ units apart.
Ans. $z = \pm 10$.

9. The abscissa of a point is (−6) and its distance from (1,3) is $\sqrt{74}$. Find its ordinate. *Ans.* 8 or −2.

10. Find the point on the x-axis which is equidistant from (0, −2) and (6,4).

11. Find the center of the circle passing through the points (2,6), (5,2), and (0,3).

12. If two vertices of an equilateral triangle are (−4,3) and (0,0), find the third vertex.

13. Show that if θ is the angle from OX to OY, the distance between (x_1,y_1) and (x_2,y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\theta}.$$

---

* The symbol $|m|$ designates the *absolute value* of m, which is defined by

$$|m| = m \text{ if } m \text{ is positive, or zero,}$$
$$|m| = -m \text{ if } m \text{ is negative.}$$

Thus $|m|$ is always positive or zero, and $[|m|]^2 = m^2$. For example $|7| = 7$; $|-2/3| = 2/3$, and $[|-3|]^2 = 9$.

14. If the angle between the axes is 120°, find the distance between the points (8,1) and (3,6). *Ans.* $5\sqrt{3}$.

15. If the angle between the axes is 60°, find the distance between $(-2,6)$ and $(4,-5)$.

16. If the distance from (2,3) to the origin is $\sqrt{19}$ and the axes are oblique, find the angle between the axes. *Ans.* 60°.

17. Prove, by analytic geometry, that the diagonals of a square are equal. HINT: Assume two adjacent sides of the square for coordinate axes, and an arbitrary value, say a, for the length of the side.

18. Prove analytically that the diagonals of a rectangle are equal.

4. Point of Division of a Segment. Given two points, P_1 (x_1, y_1) and P_2 (x_2, y_2), suppose we wish to find the coordinates (x,y) of the point P which divides the segment P_1P_2 in the ratio $r_1:r_2$ (*i.e.*, so that $P_1P:PP_2 = r_1:r_2$). Let us project P_1, P, and P_2 upon the x-axis by lines parallel to the y-axis meeting the x-axis in the points Q_1, Q, and Q_2, respectively, as shown in Figs. 5*A* and 5*B*. It is

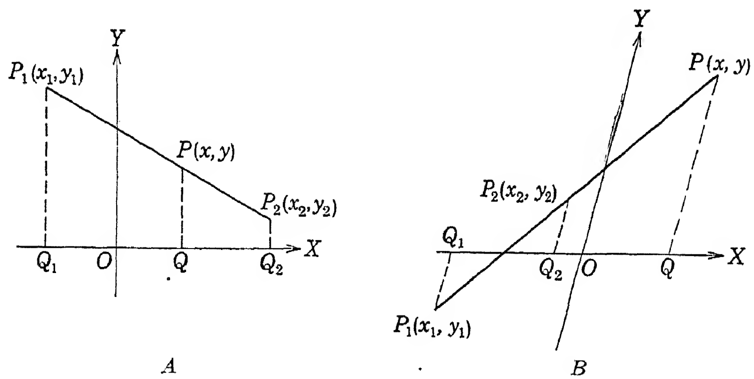


FIG. 5.

clear, by geometry, that the ratios $P_1P:PP_2$ and $Q_1Q:QQ_2$ are equal numerically. If the segments are considered as directed, these ratios also agree in sign. It follows that the point Q divides the segment Q_1Q_2 in the ratio $r_1:r_2$ and that, using the result of formula (1), page 4,

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}. \tag{2$_1$}$$

Similarly, by projecting the points P_1, P, and P_2 upon the y-axis, the student may obtain the formula

$$y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2} \tag{2$_2$}$$

Problems

1. Find the coordinates of the point P which divides the segment P_1P_2 in the ratio $r_1:r_2$ for each of the cases below:

(*a*) P_1 (0,0), P_2 (4, -12), $r_1:r_2 = 1:3$. *Ans.* $(1,-3)$.

(*b*) P_1 $(2,-1)$, P_2 (8, 14), $r_1:r_2 = 3:2$.

(*c*) P_1 $(-6,3)$, P_2 $(1,-11)$, $r_1:r_2 = 4:3$.

(*d*) P_1 (2,6), P_2 (3,10), $r_1:r_2 = -5:4$. *Ans.* (7,26).

(*e*) P_1 $(-2,-6)$, P_2 $(-5,-2)$, $r_1:r_2 = -2:3$.

2. Obtain, as a corollary to formulas (2_1) and (2_2), the coordinates of the midpoint of the segment joining P_1 (x_1,y_1) and P_2 (x_2,y_2) as

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}.$$

3. Find the coordinates of the midpoints of the sides of the triangle that has its vertices at (2,5), $(8,-7)$, and $(-2,1)$.

4. In the triangle of Exercise 3, find the coordinates of the point on each median which is $\frac{2}{3}$ of the way from the vertex to the midpoint of the side opposite. What theorem concerning the medians of a triangle is verified?

5. Find the coordinates of the four midpoints of the sides of the quadrilateral that has its vertices at (0,10), $(-6,2)$, $(-2,-8)$, and $(12,-2)$.

6. Join the consecutive midpoints found in Exercise 5 and determine the lengths of the four sides of the quadrilateral formed. What special form of quadrilateral is it?

7. Prove analytically that the three medians of a triangle have a point in common, which is $\frac{2}{3}$ of the distance, on each, from the vertex to the midpoint of the side opposite.

HINT: Assume the origin at one of the vertices, the x-axis as coinciding with one of the sides through that vertex [hence a second vertex, say, as $(a,0)$] and the third vertex, say, as (b,c), where a, b, and c are arbitrary numbers.

8. Prove the proposition of Exercise 7 by using oblique axes and taking the points (0,0), $(a,0)$, and $(0,b)$ as the vertices.

9. Prove analytically that the four midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

10. Prove that the points $(-7,11)$, $(-10,5)$, $(-2,1)$ and (1,7) are the vertices of a rectangle. By finding the midpoints of the diagonals, verify that they bisect each other.

11. Prove analytically that the diagonals of any rectangle bisect each other.

12. Given a parallelogram with an acute angle of 60° and its sides 10 and 16 units long, verify analytically that its diagonals bisect each other.

HINT: Assume one vertex at (0,0), one at (16,0), and one at (0,10) on a system of oblique axes.

13. Prove, using oblique axes, that the diagonals of any parallelogram bisect each other.

14. In what ratio does the x-axis divide the line segment from $(-5,8)$ to $(1,-4)$? Ans. 2:1.

15. In what ratio does the point $(-1,1)$ divide the segment drawn from $(-7,-2)$ to $(9,6)$? Find by two distinct methods.

16. The point $(3,-5)$ bisects a line segment. If one end is at $(4,3)$, find the other end.

17. Find the area of the isosceles triangle with vertices at $(4,2)$, $(-6,-1)$, and $(1,-8)$. Ans. $^{91}\!/_2$.

18. The segment P_1P_2 is divided by the point P in the ratio $2:3$. If P_1 is $(-8,9)$ and P is $(0,6)$, find the coordinates of P_2. Do it as a problem in internal division and also as a problem in external division.

19. The line segment joining the points $(-6,0)$ and $(0,-5)$ is produced past the latter point a distance twice its own length. Find the coordinates of the new end point. Do it by two methods as in Exercise 18.

Ans. $(12,-15)$.

20. A parallelogram has vertices at $(3,2)$ and $(4,-3)$ and its center at $(-1,0)$. Find the other two vertices.

21. Show that if $P_1P/P_1P_2 = r$, the formulas for the coordinates of the point of division, P, become

$$x = x_1 + r(x_2 - x_1), \qquad y = y_1 + r(y_2 - y_1).$$

22. In the formulas (2_1) and (2_2), x and y become undefined for the case $r_1:r_2 = -1$. What happens to the point of division, P, when P_1 and P_2 are fixed and the ratio r_1/r_2 is given a series of values approaching -1; say, a series like $-\frac{9}{10}, -\frac{99}{100}, \ldots$, and also the series $-\frac{11}{10}, -\frac{101}{100}, \ldots$ (*i.e.*, in formal notation, as $r_1/r_2 \to -1$)?

23. What happens to the point P of division as r_1/r_2 is given a series of positive values that increase beyond bound (*i.e.* as $r_1/r_2 \to \infty$)?

24. What happens to the point of division, P, as $r_1/r_2 \to -\infty$ (*i.e.*, r_1/r_2 is given a series of negative values that increase numerically beyond bound)?

25. Study the behavior of the point P, as P_1 and P_2 remain fixed, while the value of r in the formulas of Exercise 21 varies from 0 to 1 and on to $+\infty$; also from 0 to -1 and on to $-\infty$.

5. Inclination and Slope. By the *inclination* of a line, in the plane of a rectangular coordinate system, is meant the least angle, positive or zero, measured from the positive x-axis to the line. The tangent of that angle is called the *slope* of the line. It will be noted that the inclination of a line is uniquely determined by its slope, in view of the fact that the angle is always in the first or second quadrant. Note also that when a line is parallel to OY, its slope is undefined ($\tan 90° = \infty$).

Inasmuch as the position of a line in the plane is determined by any two of its points, our first inquiry is about the value of the slope in terms of the coordinates (x_1,y_1) and (x_2,y_2) of two points known to be on the line. A formula is readily obtained if we draw lines through P_1 and P_2 parallel to the axes, as in Figs. 6*A*

and 6*B*. For each inclination angle A in the figure we have

$$\tan A = \frac{QP_2}{P_1Q} \qquad (P_1Q \text{ and } QP_2 \text{ directed lines})$$

and since $P_1Q = x_2 - x_1$ and $QP_2 = y_2 - y_1$, we obtain, designating the slope of the line P_1P_2 by m ($\therefore m = \tan A$)

$$m = \frac{y_2 - y_1}{x_2 - x_1}. \tag{3}$$

The student will easily see that if the labeling of the two points on

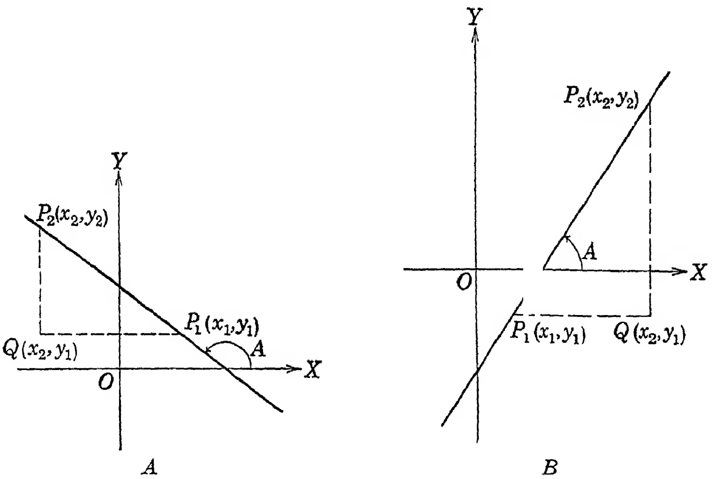


FIG.

the line be interchanged, the above argument leads to

$$m = \frac{y_1 - y_2}{x_1 - x_2},$$

a value equal to that in formula (3), as it should be. The fraction in (3) ceases, of course, to define a number when $x_1 = x_2$, which occurs in the exceptional case, noted above, of a line parallel to OY.

Exercise 1. Prove: if l_1 and l_2 are two lines whose slopes are m_1 and m_2, then (*a*) the lines l_1 and l_2 are parallel if, and only if, $m_1 = m_2$; (*b*) the lines l_1 and l_2 are perpendicular if, and only if, $m_1 = -1/m_2$.

Problems

1. Find the slopes of the lines through the following pairs of points:

(*a*) (2,0), (6,8). *Ans.* 2.
(*b*) (−1,5), (4,5),
(*c*) (−3,−4), (−8,2),
(*d*) (0,0), (h,k),
(*e*) (3,−6), (−6,3),
(*f*) (5,2), (5,7).

2. For each of the lines of Exercise 1, find the slope of a line perpendicular to it.

3. The inclination of a line in 135°. State the slope of any line parallel to it; also of any line perpendicular to it.

4. Prove, using slopes, that the points $(5,-2)$, $(8,8)$, and $(-5,1)$ are the vertices of a right triangle.

5. Prove, by means of slopes, that the three points $(8,-1)$, $(12,6)$, and $(4,-8)$ are on the same straight line.

6. Prove, by means of slopes, that the four points $(4,-5)$, $(10,-3)$, $(9,0)$, and $(3,-2)$ are the vertices of a rectangle.

7. Prove, by means of slopes alone, that the four points $(-3,1)$, $(0,3)$, $(-2,6)$, and $(-5,4)$ are the vertices of a square.

8. Given that the point $(1,3)$ lies on a line whose slope is $2/3$, write down the coordinates of three other points on the line.

9. A line of inclination 60° passes through the point $(5,3)$ and $(x,-2)$. Find x. *Ans.* $5\left(1 - \frac{\sqrt{3}}{3}\right)$.

10. Given that the point, D, divides AB and E divides CB, both in the ratio $r_1:r_2$, prove that DE is parallel to AC, and that $DE:AC = r_2:(r_1 + r_2)$.

11. Prove analytically that the line joining the midpoints of the nonparallel sides of a trapezoid is parallel to the bases and equal, in length, to half of their sum.

12. Prove, by means of slopes, that the midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

13. Three consecutive vertices of a parallelogram are $(2,-1)$, $(5,-7)$, and $(3,4)$. Find the fourth vertex, in two different ways. *Ans.* $(0,10)$.

14. Verify, by means of slopes, that the points (x_1,y_1), (x_2,y_2), and $\left(\frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \frac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$ lie on a line.

15. Prove analytically that the diagonals of a rhombus are perpendicular.

6. Angles between Lines. Let the inclinations of two lines be given as α_1 and α_2, and let the angle, measured in the counterclockwise direction *from* the first line *to* the second, be the positive angle θ. Then, $\alpha_2 = \alpha_1 + \theta$ as in Fig. 7*A* or $\alpha_2 = \alpha_1 + \theta - 180°$ as in Fig. 7*B*. From these, either

$$\theta = \alpha_2 - \alpha_1$$

or

$$\theta = \alpha_2 - \alpha_1 + 180°.$$

Hence, in either case, since $\tan(A + 180°) = \tan A$ for any angle A, we have

$$\tan\theta = \tan(\alpha_2 - \alpha_1) = \frac{\tan\alpha_2 - \tan\alpha_1}{1 + \tan\alpha_2\tan\alpha_1}.$$

If we call the slopes of the two lines m_1 and m_2, respectively, we now have

$$\tan \theta = \frac{m_2 - m_1}{1 + m_2 m_1}. \tag{4}$$

Exercise 1 of the preceding section can now be argued as a corollary to (4). Let the student do it as an exercise.

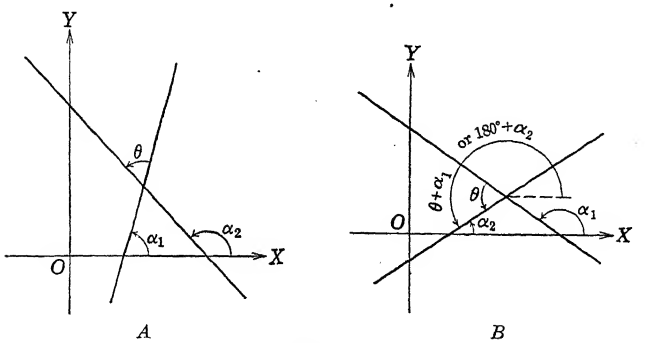


FIG. 7.

Problems

1. Find the angle from the first line designated, to the second, in each of the following cases:

(*a*) The line through $(-3,2)$ and $(1,6)$; a line of inclination 45°.

(*b*) A line of slope 3; a line of slope -2. *Ans.* $\theta = 45°$.

(*c*) The line through $(0,0)$ and $(3,5)$; a line of slope $\frac{2}{3}$.

(*d*) The line through $(1,1)$ and $(4,-2)$; a line of inclination 90°.

(*e*) A line of slope 4; a line parallel to OX.

2. Show that the triangle with vertices at $(4,2)$, $(-6,-1)$ and $(1,-8)$ is isosceles, by proving two of its angles equal.

3. Find the area of the triangle with vertices at $(3,1)$, $(4,9)$, and $(7,-2)$, by using the formula, *area* $= \frac{1}{2}ab \sin C$, where a and b are two sides and C is their included angle. *Ans.* $\frac{35}{2}$.

4. Show that the quadrilateral with vertices at $(4,5)$, $(-1,2)$, $(2,-2)$, and $(7,1)$ is a parallelogram and find its area.

5. The angle from a line of slope 2 to a line joining the point (x,y) to the point $(1,3)$ is 45°. Show that x and y are related by the equality $3x + y = 6$.

6. Show that if lines are drawn from the point $(1,3)$ to each of the points $(0,\frac{118}{47})$, $(\frac{1}{2},0)$, and $(4,0)$, then the second line bisects one of the angles between the first and third.

7. Show that the angle φ measured in the positive direction from the line with slope m_2 to the line with slope m_1 is given by $\tan \varphi = \dfrac{m_1 - m_2}{1 + m_1 m_2}$.

7. Polar Coordinates. In addition to Cartesian coordinates dealt with up to this point, we now introduce another system—that of *polar coordinates.* From a fixed point O, called the *pole,* a line is drawn in a definite direction OX, this line to be called the *polar axis.* If, with OX as the initial side, we construct an angle θ and then lay off on the terminal side OR a distance $OP = r$, we define r and θ as the *polar coordinates* of P and designate P by (r,θ). The coordinate r is called the *radius vector* of P, and θ its *polar angle.* The polar angle (and the number, θ, representing it) is considered positive if measured in the counterclockwise direction from OX, otherwise negative. The radius vector is considered as a directed line segment from the pole to the point P. When that direction is the same as that of the terminal side of θ, r is positive. If the direction of the radius vector is that of the terminal side of θ produced through the pole, r is negative. Thus, in Fig. 8, Q has coordinates $(7,\theta)$ and S is $(-3,\theta)$.

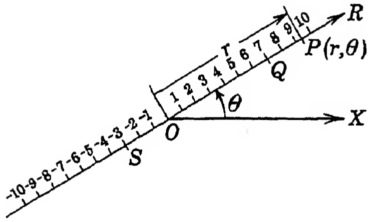


FIG. 8.

Note that in this system any real number r and any real angle θ still determine a unique point in the plane; but to a given point in the plane correspond more than one set of coordinates. Thus, if the angle θ in Fig. 8 is, say, 30°, then point S can be designated in any one of the ways: $(-3,30°)$, $(3,210°)$, $(3,-150°)$, $(-3,-330°)$, $(-3,390°)$, $(3,570°)$, etc.

Let, now, a system of both rectangular and polar coordinates be set up, where the polar axis is identical with the x-axis, and the pole with the origin. Then if a point P, in the plane, is (x,y) in the rectangular system and (r,θ) in the other, the relations,

$$\begin{aligned} x &= r \cos \theta \\ y &= r \sin \theta \end{aligned} \tag{5}$$

and

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \tan \theta &= \frac{y}{x}, \end{aligned} \tag{6}$$

are immediately seen to hold (see Fig. 9). Since the polar coor-

dinates of a point can always be written with r positive or zero, we may safely replace (6_1) by the equation

$$r = \sqrt{x^2 + y^2}.$$

We may use as an expression for θ either one of

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \cos^{-1}\left(\frac{x}{\sqrt{x^2 + y^2}}\right)$$

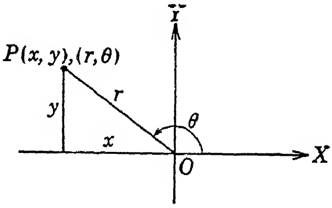


FIG. 9.

or any equivalent inverse trigonometric function, but two are needed in order to determine the quadrant of θ uniquely.

Problems

1. Plot the points whose polar coordinates are:

(a) $(5,60°)$, (d) $(-2,2\pi/3)$,* (g) $(6,3\pi/2)$,*
(b) $(3,\pi)$,* (e) $(-1,-150°)$, (h) $(-1,2)$,*
(c) $(4,240°)$, (f) $(0,40°)$, (i) $(\pi,\pi/2)$.*

2. Give three more sets of coordinates for each of the points in 1(a), (c), (d), and (e).

3. Give several sets of polar coordinates of the pole.

4. What is the locus of all points for which (a) $r = 5$? (b) $r = -5$?

5. What is the locus of all points for which (a) $\theta = 60°$? (b) $\theta = 240°$? (c) $\theta = -60°$?

6. What can you state about the polar coordinates of every point (a) on the polar axis? (b) on the 90° line (*i.e.*, the line through the pole making 90° with the polar axis)? (c) on the circle with center at the pole and radius 2? (d) on the line parallel to the polar axis and 4 units above it?

7. Find the rectangular coordinates of the points whose polar coordinates are given below (the arrangement between the two systems of coordinates is understood to be as in Fig. 9):

(a) $(3,300°)$, (c) $(-4,\pi)$, (e) $(2,-3\pi/2)$,
(b) $(2,3\pi/4)$, (d) $(-5,7\pi/4)$, (f) $(-1,-120°)$.

8. Find the polar coordinates of the points which are given below by their rectangular coordinates:

(a) $(3,3\sqrt{2})$, (c) $(-1,\sqrt{3})$, (e) $(2,-3)$,
(b) $(-4,0)$, (d) $(0,2)$, (f) $(3,\pi)$.

9. Find the distance between the pair of points whose polar coordinates are

(a) $(5,0)$, $(3,-\pi/2)$. (b) $(4,\pi/3)$, $(3,2\pi/3)$. (c) $(3,135°)$, $(-2,\pi)$.

Ans. (b) $\sqrt{13}$.

10. Prove that the distance between the two points whose polar coordinates are (r_1,θ_1), and (r_2,θ_2) is given by the formula

* If no units are designated for the polar angle of a point, the radian is understood.

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_2 - \theta_1)}.$$

11. Find the area of the triangle whose vertices are the pole and the points $(6,\pi/4)$ and $(5,0)$. *Ans.* $15\sqrt{2}/2$.

12. Show that the area of the triangle whose vertices are the pole and the points (r_1,θ_1) and (r_2,θ_2) is one-half of the expression $|r_1r_2 \sin(\theta_2 - \theta_1)|$.

13. Find the length of the perpendicular drawn from the pole to the line joining $(5,45°)$ and $(5,75°)$.

8. Translation and Rotation of Axes. Let P have the coordinates (x,y) with respect to a given pair of axes, OX and OY. If we now replace the given axes by another pair, $O'X'$ and $O'Y'$ parallel to OX and OY, respectively, the coordinates of P with respect to the new axes will evidently be a different pair of numbers than x and y. This method of replacing one set of axes by another is called *translation* of axes. Let us call the new coordinates of P x' and y' and let the new origin, O', be (h,k) with respect to OX and OY. [It is obviously $(0,0)$ with respect to $O'X'$ and $O'Y'$.] To obtain the relation between the two pairs of coordinates, project P upon both sets of axes as in Fig. 10. If these projections are S, Q, T, and R, then

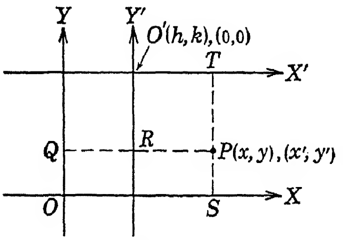


FIG. 10.

$$x = QP = QR + RP = h + x',$$
$$y = SP = ST - PT = ST + TP = k + y'.$$

The relation sought is embodied, then, in formulas

$$x = x' + h,\qquad y = y' + k. \tag{7}$$

Another change of axes frequently employed in analytic geometry is the *rotation* of OX and OY through the same angle about O. If we call, again, the coordinates of P with respect to OX and OY, x and y, and its coordinates with respect to OX' and OY', x' and y', we obtain the relation between the two pairs of coordinates of P as follows. Let the polar coordinates of P with respect to OX' as the polar axis and O as the pole be r and α. Then, if θ is the angle through which the axes have been rotated, the polar coordinates of the same point, with respect to OX as the polar axis and

O as the pole are, evidently, r and $\alpha + \theta$. Hence

$$
\begin{aligned}
x &= r \cos (\alpha + \theta), \\
y &= r \sin (\alpha + \theta), \\
x' &= r \cos \alpha, \\
y' &= r \sin \alpha.
\end{aligned}
$$

Using a well-known trigonometric relation, we write the first two equations as

$$
\begin{aligned}
x = r(\cos \alpha \cos \theta - \sin \alpha \sin \theta) &= \\
(r \cos \alpha) \cos \theta &- (r \sin \alpha) \sin \theta, \\
y = r(\sin \alpha \cos \theta + \cos \alpha \sin \theta) &= \\
(r \sin \alpha) \cos \theta &+ (r \cos \alpha) \sin \theta.
\end{aligned}
$$

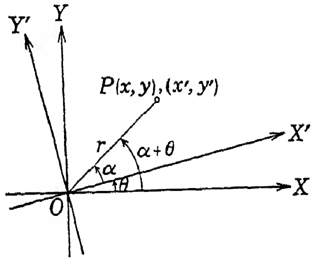


FIG. 11.

Using the last two of the above set of four equations, we obtain finally

$$
\begin{aligned}
x &= x' \cos \theta - y' \sin \theta, \\
y &= x' \sin \theta + y' \cos \theta,
\end{aligned}
\tag{8}
$$

which is the relation sought.

Problems

1. The axes are translated, with the new origin at (3,2). Find, in each case below, the old coordinates of the point whose new coordinates are given: (*a*) (5,2); (*b*) $(-1,4)$; (*c*) $(3,-1)$; (*d*) $(-6,-7)$; (*e*) $(0,-4)$; (*f*) (0,0).

Ans. (*a*) (8,4).

2. The axes are translated, with the new origin at $(-1,3)$. Find, in each case below, the new coordinates of the point whose old coordinates are given: (*a*) $(-1,6)$; (*b*) $(-3,9)$; (*c*) $(4,-5)$; (*d*) $(7,-1)$; (*e*) (5,3); (*f*) (0,0).

Ans. (*a*) (0,3).

3. By translation of axes, the point (4,1) becomes $(6,-2)$. What are the new coordinates of the point $(-3,7)$? What point becomes $(5,-3)$?

4. Verify, by Eqs. (7) of the text that the distance between two points and the slope of the line joining them is unchanged by a translation of axes.

5. The axes are rotated counterclockwise through an angle of 60°. Find, in each case below, the coordinates of the point whose new coordinates are given: (*a*) (5,1); (*b*) (3,0); (*c*) $(0,-2)$; (*d*) $(-4\sqrt{3},4)$; (*e*) (0,0).

Ans. (*b*) $(3/2, 3\sqrt{3}/2)$.

6. Solve Eqs. (8) of the text explicitly for x' and y', thus expressing the new coordinates in terms of the old coordinates and the angle of rotation.

Ans. $x' = x \cos \theta + y \sin \theta$, $y' = -x \sin \theta + y \cos \theta$.

7. The axes are rotated through an angle of 45°. Find, in each case below, the new coordinates of the point whose old coordinates are given:

(*a*) $(-3,0)$; (*b*) $(2,-2\sqrt{3})$; (*c*) $(0,1)$; (*d*) $(-2,-2)$.

Ans. (*a*) $\left(\frac{-3\sqrt{2}}{2},\frac{3\sqrt{2}}{2}\right)$.

8. Through what angle have the axes been rotated if the point: (*a*) $(2,-1)$ becomes $(-1,-2)$? (*b*) $(8,-6)$ becomes $(10,0)$? (*c*) $(-2,4)$ becomes $(3\sqrt{2},-\sqrt{2})$? *Ans.* (*c*) 135°.

9. Show that the origin is the only point whose coordinates remain unchanged when the axes are rotated through a positive angle less than 2π.

10. A certain line has a slope equal to 1. What is its slope after the axes have been rotated through a 15° angle?

11. The axes are rotated through an angle $\theta = \tan^{-1} S$. If the slope of a line is m_1 in the old system of axes, express m_2, its slope in the new system, in terms of m_1 and S.

12. Find the new coordinates of the point $(6,-2)$

(*a*) after the axes have been translated, with new origin at $(3,1)$, and then rotated, about the new origin, through an angle 60°.

Ans. $\left(\frac{3(1-\sqrt{3})}{2},\frac{-3(1+\sqrt{3})}{2}\right)$.

(*b*) after the axes have been rotated through an angle of 60° and then translated, with new origin at the point which is $(3,1)$ in the rotated system of axes.

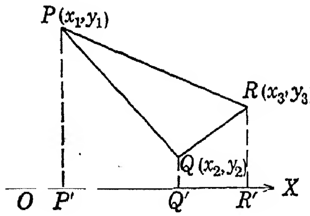


FIG. 12.

9. Area of a Triangle. Let it be our problem to find the area of a triangle in terms of the coordinates of its vertices. To solve the problem, project the vertices upon OX, and the area sought is seen, in Fig. 12, to equal the area of the trapezoid $P'PRR'$ less the areas of the trapezoids $P'PQQ'$ and $Q'QRR'$. In other words,

$$\Delta PQR = \frac{(P'P + R'R)P'R'}{2} - \frac{(P'P + Q'Q)P'Q'}{2} - \frac{(Q'Q + R'R)Q'R'}{2}$$

$$= \frac{(y_1 + y_3)(x_3 - x_1)}{2} - \frac{(y_1 + y_2)(x_2 - x_1)}{2} - \frac{(y_2 + y_3)(x_3 - x_2)}{2}$$

$$= \tfrac{1}{2}(x_1y_2 - x_1y_3 + x_2y_3 - x_2y_1 + x_3y_1 - x_3y_2).$$

This expression for the area may be thrown into a form easier to memorize, *viz.*,

$$\text{Area } \Delta PQR = \tfrac{1}{2}[x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)]. \tag{9}$$

Note that the order of the subscripts in the first term, in the brackets above, is the natural order, 1, 2, 3, and that, to pass to the next term, the subscripts are advanced cyclically each time.*

The student who is familiar with third-order determinants will best remember the expression for the area in the form

$$\text{Area } \Delta PQR = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \tag{10}$$

Exercise 1. Prove that if the vertices of the triangle are relabeled in such a way that the order (x_1,y_1), (x_2,y_2), (x_3,y_3) is still counterclockwise, the expression (9) still gives the area.

Exercise 2. Prove that if the coordinates of the vertices of the triangle are so designated that the order (x_1,y_1), (x_2,y_2), (x_3,y_3) is clockwise the expression in (9) is the negative of the area.

Exercise 3. Derive formula (9) from a figure in which the area of the triangle is bounded above by two sides and below by one, again keeping all points in the first quadrant and the order of the vertices counterclockwise.

Exercise 4. Discard the assumption that the vertices of the triangle are all in the first quadrant and derive formula (9). Do this in three steps as follows:

(*a*) Show that the right-hand member of (9) is unchanged if a constant n is added to each of x_1, x_2, and x_3.

(*b*) Show that the right-hand member of (9) is unchanged if a constant k is added to each of y_1, y_2, and y_3.

(*c*) By translating the axes so that the new coordinates of the vertices are all positive and using the results of (*a*) and (*b*), show that expression (9) holds regardless of the location of the vertices, provided P, Q, R form a counterclockwise order.

Problems

1. Find the areas of the triangles whose vertices have the following coordinates:

(*a*) $(5,1)$, $(1,3)$, $(-1,-2)$. *Ans.* 12.
(*b*) $(3,4)$, $(2,0)$, $(0,1)$. *Ans.* $\frac{9}{2}$.
(*c*) $(2,3)$, $(-5,-2)$, $(8,-2)$.
(*d*) $(7,1)$, $(5,5)$, $(-6,7)$.
(*e*) $(3,3)$, $(3,-2)$, $(12,4)$.
(*f*) $(1,1)$, $(10,8)$, $(4,6)$.
(*g*) $(0,0)$, $(-2,4)$, $(-5,-1)$.
(*h*) $(6,8)$, $(9,-1)$, $(1,3)$.

2. Find the distance:

(*a*) Between the point $(5,4)$ and the line joining $(0,2)$ and $(3,-1)$. *Ans.* $7\sqrt{2}/2$.

* Three or more things are said to advance (or permute) cyclically when the first is replaced by the second, the second by the third, etc., while the last is replaced by the first.

(*b*) Between the point $(-3,2)$ and the line of slope 2 passing through $(1,1)$.

3. By dividing them into triangles, find the areas of the quadrilaterals whose vertices are

(*a*) $(2,6)$, $(0,3)$, $(-2,4)$, $(0,8)$.
(*b*) $(2,-3)$, $(5,4)$, $(3,6)$, $(-4,2)$.
(*c*) $(-7,3)$, $(-10,11)$, $(0,8)$, $(0,3)$.
(*d*) $(-4,-2)$, $(3,-3)$, $(8,0)$, $(2,2)$.

Ans. (*b*) $39\frac{1}{2}$

4. Find whether or not the points lie on a straight line:

(*a*) $(11,0)$, $(8,-1)$, $(-1,-4)$.
(*b*) $(0,0)$, $(12,9)$, $(4,4)$.
(*c*) $(-3,10)$, $(7,-10)$, $(5,-6)$.

5. Prove analytically that the lines joining the midpoints of the sides of a triangle divide the triangle into four equal parts.

6. Prove analytically that the parallelogram formed by joining the consecutive midpoints of the sides of a quadrilateral has an area equal to half the area of the quadrilateral.

7. Prove that the area of the parallelogram three of whose vertices are P_1 (x_1,y_1), P_2 (x_2,y_2), and P_3 (x_3,y_3) is given by

$$x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2),$$

or the negative of it.

8. Derive formula (9) of the text in the following manner: Translate the axes, with new origin at P_1. The new coordinates of the vertices are P_1 $(0,0)$, P_2 $(x_2 - x_1,\ y_2 - y_1)$, P_3 $(x_3 - x_1,\ y_3 - y_1)$. Rotate the new set of axes so that P_1P_2 becomes the x-axis. Show that the angle θ of rotation is given by $\cos\theta = (x_2 - x_1)/P_1P_2$, $\sin\theta = (y_2 - y_1)/P_1P_2$. Show also that the altitude of the triangle (the new ordinate of P_3) is given by the expression $-(x_3 - x_1)\sin\theta + (y_3 - y_1)\cos\theta$. (See answer to Prob. 6 of the preceding section.) Finally, substitute the values of $\sin\theta$ and $\cos\theta$, and take half the product of the base and altitude.

9. Show that if each vertex of a polygon has rational coordinates, its area is given by a rational number.

10. Derive formula (9) in the following manner: Translate the axes as in Prob. 8. Let the polar coordinates of P_2 be now (r_1,θ_1) and of P_3, (r_2,θ_2). Employ the expression $\frac{1}{2}r_1r_2\sin(\theta_2 - \theta_1)$ for the area of the triangle (see Prob. 12 of Sec. 7). Use the formula $\sin(\theta_2 - \theta_1) = \sin\theta_2\cos\theta_1 - \cos\theta_2 \sin\theta_1$, and the relations between the rectangular and polar coordinates of a point.

11. Derive formulas (2_1) and (2_2) of Sec. 4, as follows: If

$$P_1P/PP_2 = r_1/r_2,$$

then area P_1PS/area $PP_2S = r_1/r_2$, where S (x_3,y_3) is any point whatever, not on line P_1P_2. Show that this condition is expressed by

$$\frac{x_1(y - y_3) + x(y_3 - y_1) + x_3(y_1 - y)}{x(y_2 - y_3) + x_2(y_3 - y) + x_3(y - y_2)} = \frac{r_1}{r_2},$$

and, finally, show that this equation is satisfied, regardless of the values of x_3 and y_3, if $x = \dfrac{r_1x_2 + r_2x_1}{r_1 + r_2}$, $y = \dfrac{r_1}{r_1 + r_2}$

12. Obtain the result of Exercise 2 of the text, using the determinant notation for the area of a triangle.

13. Using the determinant notation for the area of a triangle, prove that three points lie on a straight line if the difference of the abscissa and ordinate is the same for all three points.

14. Find the area of the triangle with vertices at $(-3,1)$, $(0,3)$, and $(-1,6)$ by employing the formula: Area $= \frac{1}{2}ab \sin \theta$, where a and b are two sides of the triangle and θ is the angle between them.

15. Find the area of the triangle with vertices at $(0,-2)$, $(3,1)$, and $(5,-2)$ by the method of Prob. 14.

CHAPTER II

GRAPHS AND LOCI

10. Correspondence between Loci and Equations. The notion of a correspondence between a point in the plane and a pair of real numbers can be extended to a more general kind of correspondence, *viz.*, between a geometric locus and an equation. We begin with an illustration.

Let a circle be given, with its center at $(1, -2)$ and radius equal to 4. This circle is the locus of points endowed with a certain property, *viz.*, that their distances from the point $(1, -2)$ are all equal to the same number, 4. If we designate any one of the points on the circle as (x,y) and recall that the distance from (x,y) to $(1, -2)$ is given by

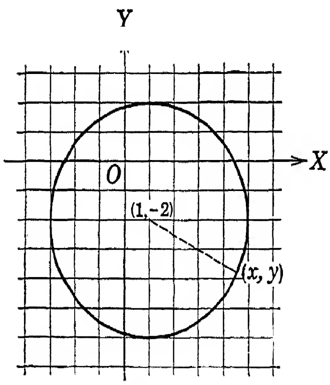


FIG. 12.

$$\sqrt{(x - 1)^2 + (y + 2)^2},$$

we see that the coordinates (x,y) of any point on the circle satisfy the equation

$$\sqrt{(x - 1)^2 + (y + 2)^2} = 4,$$

or, what amounts to the same,

$$(x - 1)^2 + (y + 2)^2 = 16. \qquad (11)$$

Conversely, let a point (x,y) be such that its coordinates, x and y, satisfy Eq. (11). Its distance from the point $(1, -2)$ is then equal to 4, and, hence, it lies on the circle.

There is, thus, a correspondence between the circle in question and Eq. (11), the correspondence having the following two properties: first, the coordinates of every point on the circle satisfy the equation; second, every point having coordinates that satisfy the equation lies on the circle. We say that the circle, with center

at $(1,-2)$ and radius equal to 4, is the *graph* of Eq. (11), in view of the following

Definition. The graph of an equation is the locus of the points whose coordinates satisfy the equation. Conversely, the equation of a given curve is an equation satisfied by the coordinates of every point on the curve and by the coordinates of no other points.

This correspondence between equations in two variables* and geometric loci will, indeed, form the central subject of analytic geometry. That is to say, the main investigations, in our study of the subject, will take the form of one or the other of the problems:

I. Given an equation in two variables, to obtain the corresponding geometric locus (the graph of the equation), along with its properties.

II. Given a geometric locus whose points possess some common property (shared by no other points), to find the corresponding equation.

In the latter case the equation, in turn, may help us in discovering or studying other properties of the locus.

Problems

1. Determine which of the following points lie on the locus represented by the equation $y - 2x + x^2 = 1$. (*a*) $(0,1)$; (*b*) $(2,1)$; (*c*) $(-1,-3)$; (*d*) $(4,-7)$; (*e*) $(-3,-12)$.

2. Determine which of the following loci pass through the origin. The locus, in each case, is represented by its equation. (*a*) $x^2 + 2y - 1 = 0$, (*b*) $3x - 4y = 0$, (*c*) $y^2 - 3x + xy = 0$, (*d*) $2x + 3y - 1 = 0$.

3. Determine which of the following points lie on the locus represented by the equation $r = 2(1 - 3 \cos \theta)$. (*a*) $(-4,0)$; (*b*) $(2,\pi/2)$; (*c*) $(10, 180°)$; (*d*) $(2 + 3\sqrt{3}, 225°)$; (*e*) $(2,-90°)$.

4. Determine which of the following loci pass through the pole. The locus, in each case, is represented by its equation. (*a*) $r \sin \theta = 2$; (*b*) $r = 1 + \sin 2\theta$; (*c*) $r = 2$; (*d*) $r = 3(1 - \cos \theta)$; (*e*) $r = 2 \sin \theta$.

5. (*a*) What must be the value of A in order that the locus represented by the equation $Ax + 3y - 5 = 0$ pass through $(2,-1)$?

(*b*) What must be the values of A and B in order that the locus represented by the equation $Ax^2 + Bxy - 3y^2 = 4$ pass through $(-1,0)$ and $(1,1)$? In order that it pass through $(-1,0)$ and $(0,0)$?

* By a *variable* is meant a quantity which, in a given discussion, can take on more than one value. Thus, in the case of the circle just discussed, the coordinates of a point on it clearly take on more than one pair of values as we pass from point to point, and hence these coordinates are variables. The coordinates of its center, on the other hand, have but one pair of values throughout any discussion involving that circle, and hence are said to be *constants*.

(c) What must be the value of A in order that the locus represented by the equation $r = A(1 - 2 \sin \theta)$ pass through $(2,30°)$?

(d) What must be the values of A and B in order that the locus represented by the equation $r^2 = A \sin \theta + B \cos \theta$ pass through $(-2,90°)$ and $(1,180°)$?

6. Show that the locus represented by the equation $x^2 + y^2 - 6x = 2$ is also represented by the equation $r^2 - 6r \cos \theta = 2$.

NOTE. We may speak of the first equation, *i.e.*, the equation satisfied by the rectangular coordinates of every point on the locus, as its rectangular equation. We call the second equation, *i.e.*, the one satisfied by the polar coordinates of every point on the locus, its polar equation.

7. Find the polar equation of the locus whose rectangular equation is

(a) $y = x$. (c) $x^2 + y^2 - 2y = 0$.

(b) $\dfrac{x}{y} + \dfrac{y}{x} = 3$. (d) $\dfrac{y}{x} = \tan (\log_{10} \sqrt{x^2 + y^2})$.

8. Find the rectangular equation of the locus whose polar equation is

(a) $r = -3 \csc \theta$. (d) $r^2 = \sin 2\theta$.

(b) $r = 4 \cos \theta$. (e) $r = \dfrac{2}{1 + 4 \cos \theta}$.

(c) $r(\sin \theta \cos \theta + 2 \cos^2 \theta) = \sin \theta$.

9. (a) If $(4,y)$ is a point on the locus represented by $x^2 + 3y - x = 1$, find y.

(b) If $(x,-2)$ is a point on the locus represented by $x^2 + 4y^2 = 25$, find x.

(c) A point is on the locus represented by $r = 2 \sin 3\theta$. If its polar angle is 60°, find its radius vector.

(d) A point is on the locus represented by $r = 2(3 - 4 \sin \theta)$. Its radius vector is 2. Find its polar angle.

10. (a) Show that $x^2 + (y - 3)^2 = -4$ represents no real locus.

(b) Show that the locus represented by $x^2 + (y - 3)^2 = 0$ is a single point.

11. (a) If $x = a$, $y = b$ is a common solution of two equations in x and y, what can be said about the point (a,b)?

(b) What point is common to the two loci whose equations are $x - 2y = 5$ and $2x + 3y = 1$?

(c) What point is common to the two loci whose equations are $6y = x^2 - x$ and $3y = x$?

(d) What point is common to the two loci whose equations are $y = x + 1$ and $2y = x^2 + x + 6$?

(e) What point is common to the two loci whose equations are $x^2 + 4y^2 = 16$ and $x + y = 6$?

(f) What point is common to the two loci whose equations are $r = 4 \cos \theta$ and $r = 2$?

(g) What point is common to the two loci whose equations are $r = 2 + \sin \theta$ and $r = -\sin \theta$?

11. Plotting the Graph of an Equation. Given, now, an equation in two variables x and y, let us turn to the problem of obtain-

ing its graph. It may well happen that the nature of the graph can be told directly from the equation, *e.g.*, in the case of the equation $y = x$. The graph of this is the locus of all points whose two coordinates are equal (in value and sign). By elementary geometry, it is at once recognized as the straight line bisecting the angle between the axes, in the first and third quadrants. Again, in the case of $x^2 + (y + 1)^2 = 9$, it is seen that the left-hand member is the square of the distance from (x,y) to $(0,-1)$. Hence, the graph is the locus of points whose distance from $(0,-1)$ is 3; in other words, the circle of radius 3, with center at $(0,-1)$.

For an equation given at random it is not, however, to be expected that the corresponding graph be identified at sight. To illustrate the method of plotting the locus in such a case, consider the equation

$$= x^2 - x + 1. \tag{12}$$

Evidently, it is possible to pick out as many solutions of this equation as one pleases. To order the work, we may assign values to x arbitrarily and compute the corresponding values of y by this equation. If we let x take on, *e.g.*, the values 0, 1, 2, 3, 4, -1, -2, -3 and compute the corresponding values of y, we obtain the pairs of values displayed in the table

x	0	1	2	3	4	-1	-2	-3
y	1	1	3	7	13	3	7	13

If we recall that by the graph of Eq. (12) we understand the locus of points whose coordinates are solutions of the equation, we interpret our table to mean that the points (0,1), (1,1), (2,3), (3,7), (4,13), $(-1,3)$, $(-2,7)$, and $(-3,13)$ are points on the proposed graph. These points are plotted in Fig. 14, and the locus to which they belong appears, with tolerable clearness, to be the curve drawn in that figure.

It is evident that the method of this illustration—obtaining a number of solutions of the given equation, plotting the corresponding points, and drawing a curve through the points plotted—is essentially a method of approximation. The quality of the approximation will, of course, depend on the number of points plotted, the care with which they are plotted, and the care with

which the locus that appears to go through them is drawn.

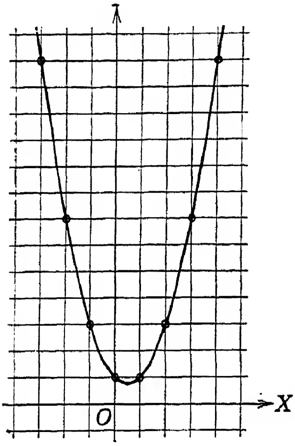


FIG. 14.

As another illustration, consider the equation

$$x^2 - 2y + y^2 - 3 = 0. \qquad (13)$$

For the purpose of obtaining solutions of this equation it is, evidently, advisable to solve it for x and to compute values of x corresponding to values assigned arbitrarily to y. Thus

$$x = \pm\sqrt{3 + 2y - y^2},$$

and if we set y equal, in turn, to 0, 1, 2, 3, 4, -1, -2 the corresponding solutions of Eq. (13) are shown in the following table:

x	$\pm\sqrt{3}$	± 2	$\pm\sqrt{3}$	0	Complex	0	Complex
y	0	1	2	3	4	-1	-2

Note that a pair of values of x and y which are not both real fails to determine a point in the plane. Hence the table displayed exhibits the points $(\sqrt{3},0)$, $(-\sqrt{3},0)$, $(2,1)$, $(-2,1)$, $(\sqrt{3},2)$, $(-\sqrt{3},2)$, $(0,3)$, and $(0,-1)$ as points on the proposed locus. These points plotted, the locus in question is seen to have the form shown in Fig. 15.

If the equation of the proposed locus is stated in polar coordinates, the above procedure applies, of course, equally well. Thus, for the equation

$$r = 2 \cos \theta$$

we obtain a number of solutions by assigning to θ values, say, 0°, 30°, 60°, 90°, 120°, 150°, 180°, 210°, 240°, 270°, 300°, 330°, 360° and exhibiting the corresponding values of r in the table

r	2	$\sqrt{3}$	1	0	-1	$-\sqrt{3}$	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°

Note that the thirteen pairs of values of r and θ displayed in the table serve to plot only six distinct points. These points outline the curve shown in Fig. 16.

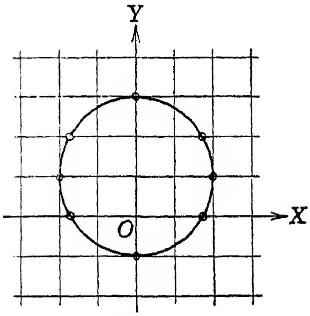


FIG. 15.

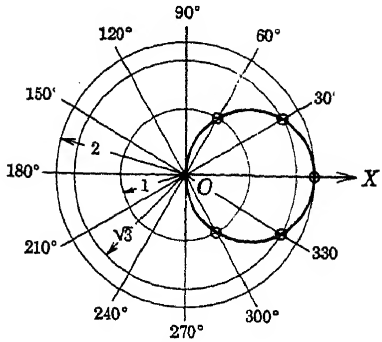


FIG. 16.

Problems

1. Plot the curve represented by each of the following equations:
(*a*) $x = 2$. (*f*) $x - 3y = 0$. (*k*) $y = 2x^2 + x - 1$.
(*b*) $y = -3$. (*g*) $3x + 2y = 6$. (*l*) $x = y^2 + 3y$.
(*c*) $y = 0$. (*h*) $4x - 3y = 8$. (*m*) $(x - 3)^2 + (y - 1)^2 = 9$.
(*d*) $x = 0$. (*i*) $y = x^2$. (*n*) $y^2 + y - x = 1$.
(*e*) $2x + y = 8$. (*j*) $\sqrt{y} = x$. (*o*) $2x^2 + 3x - 2y = 2$.

2. Plot the curve represented by each of the following equations: (*a*) $r = 2$. (*b*) $r = -3$. (*c*) $\theta = 60°$. (*d*) $\theta = \pi/2$. (*e*) $\theta = -120°$. (*f*) $r = 3 \sin \theta$. (*g*) $r = 1 + \cos \theta$. (*h*) $r = 4 - 2 \sin \theta$. (*i*) $r = -4 \cos \theta$. (*j*) $r = \sin 2\theta$. (*k*) $r = 3 \cos 2\theta$. (*l*) $r = 1 + 2 \cos \theta$. (*m*) $r = 2 - 4 \sin \theta$.

3. Show that the curve whose polar equation is $r = 4 \sin \theta$ has the rectangular equation $x^2 + (y - 2)^2 = 4$. State the nature of the curve from the latter equation, and plot it.

4. Proceed as in Prob. 3 for the equation $r = 2 \cos \theta$ treated in the text above.

12. Symmetry. An effective aid, in plotting a curve from its equation, is to examine it for an important characteristic called *symmetry*. We define this property as follows:

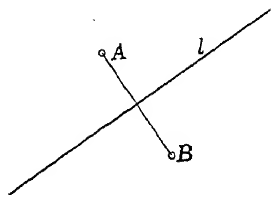


FIG. 17.

Two points, A and B, are said to be symmetrical with respect to a line l if the segment AB is perpendicularly bisected by l.

A curve is said to be symmetrical with respect to a line l if every perpendicular to l which meets the curve in a point A meets it also in the point B which is symmetrical to A with respect to l.

In case a curve C is symmetrical with respect to a line l we also say that the line l is an *axis of symmetry* for the curve C.

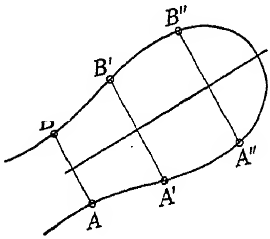


FIG. 18.

Evidently, the point symmetrical to (x,y) with respect to the x-axis has the coordinates $(x,-y)$, and the point symmetrical to (x,y) with respect to the y-axis is the point $(-x,y)$. Let, now, the equation of a curve be such that whenever it is satisfied by two numbers x and y, it is also satisfied by the numbers x and $-y$. Geometrically, this means that if the point (x,y) is on the curve the point $(x,-y)$ is also on the curve, and, in view of the above definition, the curve is symmetrical to the x-axis. In like manner, if whenever the numbers x and y satisfy the equation, the numbers $-x$ and y also satisfy it, the curve is symmetrical to the y-axis. Thus, the curve represented by the equation $y^2 = 2x - 3$ has the x-axis as an axis of symmetry, while the equation $x^2 + 5y^4 = 8$ represents a curve for which both coordinate axes are axes of symmetry.

Exercise 1. Prove that, if whenever an equation is satisfied by the numbers x and y, it is also satisfied by the numbers $2h - x$ and y, then the curve is symmetrical with respect to the line $x = h$.

Exercise 2. Prove that if an equation is such that whenever the coordinates (x,y) satisfy it the coordinates $(x, 2k - y)$ also satisfy it, the corresponding curve is symmetrical with respect to the line $y = k$.

Symmetry also obtains with respect to a point P in view of the following definitions:

Two points, A and B, are said to be symmetrical with respect to the point P if P is the midpoint of the segment AB.

A curve is said to be symmetrical with respect to the point P if every line through P which meets the curve in a point A also meets it in a point B which is symmetrical to A with respect to P.

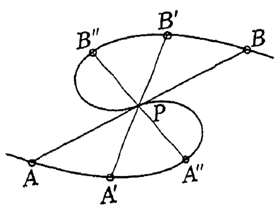


FIG. 19.

If a curve C is symmetrical with respect to a point P we shall also speak of P as a *center of symmetry* for the curve C.

Evidently, the point symmetrical to (x,y) with respect to the origin is the point $(-x,-y)$. If, now, an equation of a curve be such that whenever two numbers x and y satisfy it, the

two numbers $-x$ and $-y$ also satisfy it, the origin is a center of symmetry for the curve.

Exercise 3. Prove that if an equation of a curve is such that whenever the coordinates (x,y) satisfy it, the coordinates $(2h - x, 2k - y)$ also satisfy it, then the curve is symmetrical with respect to the point (h,k).

Exercise 4. Prove that an equation containing only even powers of x represents a curve which is symmetrical with respect to the y-axis.

Exercise 5. Prove that an equation containing only even powers of y represents a curve which is symmetrical with respect to the x-axis.

Problems

1. Find the points symmetrical to each of the following with respect to the coordinate axes and the origin:

(*a*) $(4, -2)$. (*c*) $(4,0)$. (*e*) $(0,0)$.
(*b*) $(3,1)$. (*d*) $(0, -3)$. (*f*) $(-2, -5)$.

2. Find the points symmetrical to each of the following with respect to the line $x = 3$, with respect to the line $y = -4$, and with respect to the point $(3, -4)$.

(*a*) $(2,1)$. (*c*) $(0, -2)$. (*e*) (u,v).
(*b*) $(3,4)$. (*d*) $(3, -2)$. (*f*) $(-5,4)$.

3. Test the curve represented by each of the following equations for symmetry with respect to each coordinate axis and the origin:

(*a*) $x^2 + 2y = 1$.
(*b*) $x^3 = 2y$.
(*c*) $x^2 - x^4 = 3y$.
(*d*) $x + y^4 = 1$.
(*e*) $x + y^2 - y = 0$.
(*f*) $x^4 + y^4 = 2$.
(*g*) $y = 2 \cos x$.
(*h*) $y = 3 \sin x$.
(*i*) $x^2 + x = 2y$

4. Verify in each case below that the curve has the line indicated as an axis of symmetry.

(*a*) $(x - 2)^2 = 3y$; line $x = 2$.
(*b*) $(y - 3)^2 = 3x + 1$; line $y = 3$.
(*c*) $(x + 1)^2 - y = 0$; line $x = -1$.
(*d*) $(y + 4)^2 - x + 2 = 0$; line $y = -4$.
(*e*) $(x + 4)^2 - 2(y + 2)^2 = 1$; lines $x = -4$ and $y = -2$.

5. Show that a curve symmetrical to both coordinate axes is also symmetrical to the origin; also that a curve symmetrical to any two perpendicular lines is symmetrical to their point of intersection.

6. Show that the points (r,θ) and $(r, -\theta)$ are symmetrical with respect to the polar axis, and that the points (r,θ) and $(r, \pi - \theta)$ are symmetrical to the line $\theta = 90°$; also that the points (r,θ) and $(-r,\theta)$ are symmetrical with respect to the pole.

7. State a set of polar coordinates for the points symmetrical to each of the following with respect to the polar axis, the 90° line, and the pole.

(*a*) $(3,60°)$. (*c*) $(4,0°)$.
(*b*) $(-5,120°)$. (*d*) $(-2,90°)$.

8. Test for symmetry with respect to the polar axis, the line $\theta = 90°$, and the pole, the curves represented by the following equations:

(a) $r = 2 \cos \theta$.
(b) $r = 3 \sin \theta$.
(c) $r = \tan \theta$.
(d) $r^2 = 2 \sin \theta$.
(e) $r = \sin \theta + 2 \cos$

9. A circle is symmetrical with respect to how many points? With respect to how many lines?

13. Aids in Plotting. (*A*) *Extent.* Inasmuch as points on a curve are determined only by real values of x and y satisfying its equation, it is well to test just what real values of x determine real values of y by a given equation, and what real values of y determine real values of x. Thus, if the equation is

$$4x^2 + y^2 = 25,$$

we may throw it into either of the forms

$$y = \pm\sqrt{25 - 4x^2},$$
$$x = \pm\sqrt{25 - y^2}$$

From these, it is evident that y assumes real values when, and only when, $25 - 4x^2$ is positive or zero, *i.e.*, when $x^2 \leqq 25/4$, or $-5/2 \leqq x \leqq 5/2$; likewise that x is real when, and only when, $-5 \leqq y \leqq 5$. Graphically, this means that the curve *extends* between the two parallels to the y-axis whose equations are $x = 5/2$ and $x = -5/2$, and between the two parallels to the x-axis whose equations are $y = -5$ and $y = 5$. If we notice that when $x = 0$, $y = \pm 5$ and when $y = 0$, $x = \pm 5/2$, these four points, along with the fact that the curve is symmetrical with respect to each of the coordinate axes and has the extent indicated, determine the graph of the equation quite readily as shown in Fig. 20.

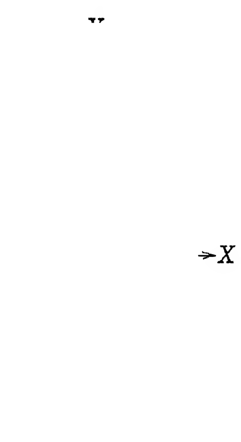


FIG. 20.

For another illustration, consider the equation

$$x^2 + y - y^3 = 0.$$

The student will find it prohibitive to solve it for y, but the equation is easily solved for x, giving

$$x = \pm\sqrt{y^3 - y} = \pm\sqrt{y(y + 1)(y - 1)}.$$

In this equation, x is evidently real if, and only if, the product under the radical sign is positive or zero. To determine what values of y make the product positive, notice that the first factor has opposite signs in the two cases, $y < 0$ and $y > 0$; that the second factor has opposite signs in the two cases $y < -1$ and $y > -1$; that the third factor has opposite signs in the two cases $y < 1$ and $y > 1$. We may display this in the table below.

	y	$y + 1$	$y - 1$	Product
$y < -1$	$-$	$-$	$-$	$-$
$-1 < y < 0$	$-$	$+$	$-$	$+$
$0 < y < 1$	$+$	$+$	$-$	$-$
$y > 1$	$+$	$+$	$+$	$+$

We see, then, that x is real when $-1 < y < 0$ and when $y > 1$. It is also real, and zero, when $y = -1$, or 0 or 1. The curve, then, extends from the line $y = -1$ to the line $y = 0$, and also above the line $y = 1$. The three points $(0,-1)$, $(0,0)$, and $(0,1)$, just noticed, and a few other points conveniently determined, along with the fact of symmetry to the y-axis and the extent just indicated, determine the graph of the equation as shown in Fig. 21.

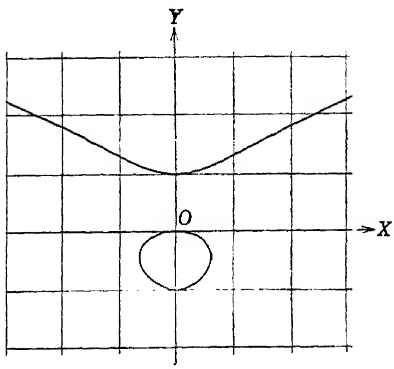


FIG. 21.

(*B*) *Intercepts.* We define an *x-intercept* of a curve as *the abscissa of a point which the curve has in common with the x-axis;* a *y-intercept* as *the ordinate of a point which the curve has in common with the y-axis.* In other words, if $(a,0)$ is a point on a curve, it is said to have an x-intercept equal to a; if $(0,b)$ is a point on the curve, it has a y-intercept equal to b. A curve may evidently have more than one intercept on either axis.

The method of finding the intercepts of a curve, quite obviously, is to set $y = 0$ (and the corresponding values of x, if real, are

the values of the x-intercept) and then set $x = 0$ (the corresponding values of y, if real, representing the y-intercepts).

In the first illustration in (A) we thus found, in effect, the x-intercepts to be $5/2$ and $-5/2$, the y-intercepts 5 and -5. In the second illustration we found the x-intercept to be zero and the y-intercepts to have the values -1, 0, 1.

(C) *Asymptotes.* The question we raise under this heading is whether there exist finite values of x which make y infinite, and similarly, whether there exist finite values of y which make x infinite.

To illustrate, consider the equation

$$xy - y = 6.$$

Solving it for y and for x, we obtain

$$y = \quad x - 1 \tag{14}$$

and

$$x = \frac{6 + y}{y} \tag{15}$$

Form (14) indicates that there is no value of y corresponding to $x = 1$, inasmuch as the computation of y in that case involves division by zero, a process excluded in mathematics. But y can be computed for values of x different from 1, however close to it. Furthermore, the values of y grow beyond bound, numerically, as x is assigned values that differ less and less from 1. That is what we mean, and that is all that we mean, when we say that y becomes infinite for $x = 1$, or when we use the symbol ∞ for infinity, which the student has probably met before, and write: $y = \infty$ for $x = 1$. Conversely, as y takes on values greater and greater numerically, x takes on values ever closer to 1. We express this briefly as: $x = 1$ for $y = \infty$.

Graphically, this means that the curve has no point in common with the line $x = 1$, but as we pick points on the curve whose ordinates grow ever larger in numerical value, such points come ever closer to the line $x = 1$ (as the difference between their abscissas and $x = 1$ becomes ever smaller). We say that the

line $x = 1$ is an *asymptote* to the curve, in view of the following definition.

A line is said to be an asymptote to a curve if the distance from the line to a point on the curve approaches zero as a limit, as the point recedes indefinitely from the origin.*

In like manner from form (15) of the equation we conclude that the line $y = 0$ is an asymptote.

The student will do well to locate a number of points, particularly with values of x chosen close to 1, say $x = 1\frac{1}{2}$, $1\frac{1}{10}$, $\frac{1}{2}$, $\frac{9}{10}$, and with values of y chosen close to zero, say $y = \frac{1}{2}$, $\frac{1}{10}$, $-\frac{1}{2}$, $-\frac{1}{10}$. The inquiry into symmetry, intercepts, and extent will, of course, be of aid. The resulting graph is shown in Fig. 22.

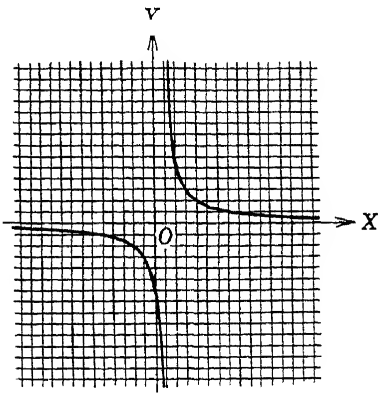


FIG. 22.

We may sum up the contents of the last two sections as embodying the following aids in plotting a curve:

I. Solve the given equation for y in terms of x, or for x in terms of y, or both, if convenient.

II. Examine the curve (*a*) for symmetry with respect to the coordinate axes and to the origin; (*b*) for values of the intercepts; (*c*) for extent; (*d*) for asymptotes.†

III. Establish a convenient number of points sufficiently distributed clearly to outline the curve.

IV. Draw the curve through the points established, with the information furnished by II as a guide.

As an illustration, consider the graph of the equation

$$x^2y + 4x^2 = 9.$$

* The last is another way of saying that the distances grow ever smaller and sink below any positive value, however small. A more precise definition of the term *limit* will be given later in the book.

† Note that we have spoken only about asymptotes represented by x = constant or y = constant, *i.e.*, asymptotes parallel to one or the other coordinate axis. A curve may possess asymptotes that are parallel to neither axis. We say nothing at present about the process of finding such.

Solving this for y and for x, we obtain

$$y = \frac{9 - 4x^2}{x^2} \tag{16}$$

$$x = \pm\frac{3}{\sqrt{y + 4}}. \tag{17}$$

The test for symmetry shows that the curve is symmetrical to the y-axis, but not to the x-axis or to the origin. Setting $x = 0$ in (16), we obtain no value for y (hence, there is no y-intercept); setting $y = 0$ in (17) we get $x = \pm 3/2$ (hence two intercepts on the x-axis). Equation (16) shows that y is real for all values of x. Equation (17) shows that x is real only when $y + 4$ is positive, *i.e.*, when $y > -4$. The curve, consequently extends only above the line $y = -4$. Equation (16) again shows that y becomes infinite for $x = 0$ (hence, the line $x = 0$ is an asymptote). Equation (17) shows that $y = -4$ is an asymptote.

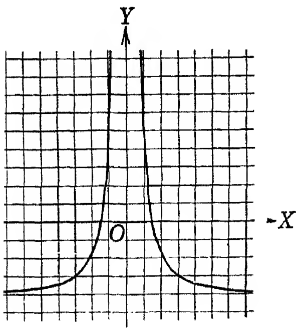


FIG. 23.

We establish a number of points on the curve, preferably by Eq. (16) which is free of radicals, and obtain the curve shown in Fig. 23.

As regards equations in the polar form, considerations of symmetry have already been taken up in Section 12. The question of what values of θ make r real and for what values of r θ exists, as well as the question of what values of θ make r infinite, are proper and helpful to raise when plotting a curve from its polar equation.

Problems

1. Discuss each equation as indicated in the text and plot the corresponding curve.

(a) $x^2 + y^2 = 16$.
(b) $2x^2 + y^2 = 8$.
(c) $2x^2 - y^2 = 8$.
(d) $y^2 - 3x = 3$.
(e) $x^2 + 3y + 6 = 0$.
(f) $y = 2x^3$.
(g) $x = 3(y - 2)^3$.
(h) $xy = 1$.
(i) $(x - 1)(y + 2) = 4$.
(j) $xy^2 = 4 - 9y^2$.
(k) $xy + 4y = x$.
(l) $xy + 4y = x^2$.

(m) $y = \frac{2}{x+1} - \frac{1}{x-3}$.
(n) $x^2 - 2x + y^2 + 4y = 4$.
(o) $x^2 - 4x + 4y^2 + 16y + 16 = 0$.
(p) $4x^2 - 24x - y^2 - 2y + 31 = 0$.
(q) $y^2 = x^2 - x$.
(r) $y^2 = x(x-1)(x+3)$.
(s) $x^2y^2 - x^2 = 4y^2$.
(t) $x^2y^2 - x^2 = -4y^2$.
(u) $xy^2 + 4y^2 = x^2(x-4)$.
(v) $xy^2 + 4y^2 = x(x-4)$.
(w) $y = (4-x)^2$.
(x) $y = (4-x^2)^2$.
(y) $x^2 = (4-y^2)^3$.
(z) $(x^2-1)(y-3) = 6$.

2. Discuss and plot the locus represented by the equation

$$(x-2)(y+3) = 0.$$

KEY: This equation is satisfied whenever either factor on the left is zero. Hence the corresponding locus consists of all the points for which $x = 2$, and all the points for which $y = -3$. The locus is, then, a pair of straight lines. Let the student draw them.

3. After the manner of Prob. 2, discuss and plot the locus represented by each of the following equations:

(a) $xy = 0$.
(b) $x^2(y-1) = 0$.
(c) $x(x-2) = 0$.
(d) $y^2 + 2y = 0$.
(e) $xy = 3x$.
(f) $xy + y = 0$.
(g) $xy - y + 2x - 2 = 0$.
(h) $(y - x^2)(x + 2y) = 0$.
(i) $2xy - 6y^3 - x^2 + 3xy^2 = 0$.
(j) $x^2 = y^2$.
(k) $x^2 = y^4$.

4. Discuss and plot the locus represented by each equation:

(a) $y = 2^x$.
(b) $y = 2^{-x}$.
(c) $y = \log_2 x$.
(d) $x = \log_2 y$.
(e) $y = 2^x + 1$.
(f) $y = 3^x$.
(g) $y = \log_3 x$.

5. Discuss and plot the locus represented by the equation $y = \sin x$, on the basis that x is measured in radians. HINT: A partial table of solutions of the equation is

x	0	$\pi/6$	$\pi/3$	$\pi/2$
y	0	$1/2$	$\sqrt{3}/2$	1

6. Discuss and plot the locus represented by each equation, on the basis that x is measured in radians.

(a) $y = \cos x$.
(b) $y = 2 \sin 2x$.
(c) $y = \cos \frac{x}{3}$.
(d) $y = \tan x$.
(e) $y = \sec x$.
(f) $y = \csc x$.

7. For each pair of equations, plot the two corresponding loci to one set of axes and determine the coordinates of the points of intersection graphically. Verify the values of those coordinates by solving each pair of equations simultaneously.

(a) $x + y = 1,$ $2x - y = 5.$

(b) $x - y = -1,$ $2x + 4y = 19.$

(c) $y = x^3,$ $y = x.$

(d) $x^2 + y^2 = 25,$ $x - y = 1.$

(e) $y - 2x = -2,$ $xy = 24.$

(f) $x^2 + y^2 = 5,$ $y - 2x = 5.$

(g) $y^2 - 2y - 4x + 9 = 0,$ $y^2 - 2y + x = 1.$

(h) $4x^2 + 9y^2 = 36,$ $x + y = 6.$

(i) $2x^2 + y^2 = 2,$ $xy - 2y - 3x + 6 = 0.$

8. Discuss the locus represented by each equation as to (I) symmetry; (II) real values of r and θ; (III) infinite values of r, and plot.

(a) $r = -3 \sin \theta.$

(b) $r + 4 \cos \theta = 0.$

(c) $r = \sin 2\theta.$

(d) $r = \cos 3\theta.$

(e) $r = \dfrac{2}{1 - \cos \theta}.$

(f) $r = \dfrac{2}{1 + \cos \theta}.$

(g) $r = \dfrac{3}{2 - \sin \theta}.$

(h) $2r + r \sin \theta = 3.$

(i) $r^2 = 4 \cos 2\theta.$

(j) $r^2 = \sin 2\theta.$

(k) $r^2 = -\cos 2\theta.$

(l) $\sec \theta = 3.$

(m) $r = 2\theta.$

(n) $r = 2 \cos \theta - 4 \sin \theta.$

(o) $\cos \theta = 2 \sin \theta.$

(p) $r = 2 \sin \dfrac{\theta}{2}.$

(q) $r = \tan \theta.$

(r) $r = 2(1 + 3 \sin \theta).$

(s) $r = 1 - 4 \cos \theta.$

(t) $r \sin \theta = 2.$

(u) $r = -\dfrac{3}{\cos \theta}.$

(v) $r = 2 \operatorname{ctn} \theta \csc \theta.$

(w) $r^2 - 3r + 2 = 0.$

9. Obtain the rectangular equation of the locus in (t) of Prob. 8; also in (u); also in (v). Plot each from its rectangular equation.

10. (a) Obtain the polar equation of the locus represented by

$$(x^2 + y^2)^2 = 4x^2 - 4y^2,$$

and plot the curve from that equation.

(b) Obtain the polar equation of the locus represented by

$$(x^2 + y^2)^2 = 4xy,$$

and plot the curve from that equation.

11. Describe and plot the locus for which

(a) $x > 2,$ $y = 1.$

(b) $x = 2,$ $y \leqq -1.$

(c) $x^2 + y^2 \geqq 4,$ $y = -x.$

(d) $x^2 + y^2 < 4,$ $y = -x.$

(e) $1 < x^2 + y^2 \leqq 4,$ $y = 2x.$

14. Parametric Equations. Frequently, a pair of equations is employed, instead of a single equation in x and y, to represent a locus. We begin with an illustration.

Suppose it be given that the abscissa and ordinate, respectively, of a variable point on the locus are described by:

$$\begin{aligned} x &= t + 2, \\ y &= 2t^2 - 1. \end{aligned} \tag{18}$$

We may select as many values as we please of x and y satisfying this pair of equations, by assigning values to t arbitrarily. A partial table of such values is

t	0	1	2	3	-1	-2	-3
x	2	3	4	5	1	0	-1
y	-1	1	7	17	1	7	17

This establishes the points $(2,-1)$, $(3,1)$, $(4,7)$, $(5,17)$, $(1,1)$, $(0,7)$, $(-1,17)$ as points on the locus represented by (18). These points plotted, the curve drawn through them will be an approximation to the locus in question.

Exercise 1. Draw the graph represented by (18). Amplify the above table if deemed desirable. Label each point plotted with the value of t which determines it.

It is clear that the pair of equations in (18) has, for the purpose of exhibiting the relation between x and y of any point on the locus, the same significance as the single equation

$$y = 2(x - 2)^2 - 1, \tag{19}$$

obtained by eliminating t between the two equations of (18).

Exercise 2. Draw the graph of (19).

We call, as is customary, the variable t, in terms of which x and y are expressed, a *parameter*, and the pair (18) *parametric* equations of the locus.

The value of parametric representation of a locus lies in the fact that it is often more convenient, as the student will find later, to obtain the value of x and the value of y each separately in terms of another quantity than to obtain directly the relation between x and y. That relation may be prohibitive to derive even after parametric equations have been found. Such, for

instance, would be the case with the locus represented by

$$x = t^4 - t + 1,$$
$$y = t^5 + t^2 + 2.$$

This pair of equations would still establish a relation between x and y, but the equation that describes that relation directly could not very easily be obtained, since t could not very readily be eliminated between the above pair.

Note that the parametric representation of a locus is not unique. If, in the pair (18) we introduce other parameters, say s, u, v, defined, respectively, by $t = 2s$, $t = u^2$, $t = e^v$, (18) assumes the forms:

$$x = 2s + 2, \qquad x = u^2 + 2, \qquad x = e^v + 2,$$
$$y = 8s^2 - 1; \qquad y = 2u^4 - 1; \qquad y = 2e^{2v} - 1,$$

each of them, when the parameter is eliminated, reducing to (19) and hence, representing the same locus.

Problems

1. Draw the locus represented by the pair of parametric equations. Whenever possible, eliminate the parameter and obtain the nonparametric equation of the locus.

(*a*) $x = t + 3$, $y = 2t - 1$.
(*b*) $x = t^2 + 1$, $y = 2t - 2$.
(*c*) $x = t - 1$, $y = t^3$.
(*d*) $x = t^2 - 3$, $y = 2$.
(*e*) $x = 2 \sin t$, $y = 2 \cos t$.
(*f*) $x = 1 - \cos t$, $y = 2 + \sin t$.

HINT: In (*e*), to eliminate the parameter write $x/2 = \sin t$, $y/2 = \cos t$, hence, by $\sin^2 t + \cos^2 t \equiv 1$, $x^2/4 + y^2/4 = 1$, or $x^2 + y^2 = 4$.

(*g*) $x = \log t$, $y = \log t^2$.
(*h*) $x = 3 \cos \theta$, $y = 2 \sin \theta$.
(*i*) $x + y = 2t + 2$, $2x - y = 3t$.
(*j*) $2x + y = s^2$, $x + 2y = s - 1$.
(*k*) $x = t^3 - t$, $y = t^3 + t$.
(*l*) $x = s + \log s$, $y = s + \log s^2$.
(*m*) $x = 2 \sec \theta$, $y = 4 \tan \theta$.
(*n*) $\theta = \sin^{-1} t$, $r = 1 + 2t$.

2. Find the intercepts on each coordinate axis for the locus represented by

(*a*) $x = t^2 - 1$, $y = 3t + 4$.
(*b*) $x = t^3 + t$, $y = t^2 + 1$.
(*c*) $x = 2 \cos t - 1$, $y = 2 \sin t - \sqrt{3}$.

3. Find the points where each locus intersects the line $x = 2$ and the line $y = -3$.

(*a*) $x = 2s - 1$, $y = 3s + \frac{1}{2}$.
(*b*) $x = u^2 + 1$, $y = u^3 + 5$.
(*c*) $x = v^4 + 4$, $y = v^2 - 4$.

15. The Equation of a Locus. Now that we have studied the procedure of examining and plotting a locus when its equation is given, we turn to the inverse problem of *deriving* the equation of a locus when the defining property of that locus is given. Bearing in mind the underlying definition, this means that the problem is to obtain the relation between the coordinates of a variable point on the locus, either in the form of an equation between x and y of such a point, if rectangular coordinates are used, or between its r and θ, when polar coordinates are used, or in the form of two equations that relate the coordinates to some parameter.

The following illustrations aim to display the manner in which a problem of this type is attacked.

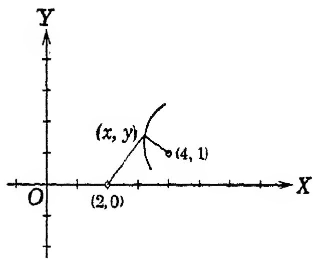


FIG. 24.

Example 1. Find the equation of the locus of a point for which the distance from (2,0) is equal to twice its distance from (4,1).

Solution. Let (x,y) be any point on the locus in question, as in Fig. 24. Its distance from (2,0) is, then, $\sqrt{(x-2)^2+y^2}$, and its distance from (4,1) is $\sqrt{(x-4)^2+(y-1)^2}$. The condition stated in the problem is, then, expressed by the equation

$$\sqrt{(x-2)^2+y^2} = 2\sqrt{(x-4)^2+(y-1)^2}.$$

This, indeed, is the equation sought. It expresses the geometric property stated for every point on the locus, and, conversely, every point whose coordinates, x and y, satisfy this equation possesses the property stated for the locus and hence lies on it.

To examine and plot the locus, we simplify its equation by rationalizing it, obtaining

$$x^2 - 4x + 4 + y^2 = 4(x^2 - 8x + 16 + y^2 - 2y + 1),$$

which reduces to

$$3x^2 + 3y^2 - 28x - 8y = -64.$$

Exercise 1. The point (4,0) is identified by inspection as a point possessing the property stated for the above locus. Verify that its coordinates satisfy the last equation.

Exercise 2. Plot the locus of Example 1 from its equation. Verify that it is a circle with center at $(\frac{14}{3}, \frac{4}{3})$ and radius equal to $\sqrt{20}/3$. KEY: Write the equation as

$$x^2 - \tfrac{28}{3}x + y^2 - \tfrac{8}{3}y = -\tfrac{64}{3}.$$

Complete the squares, to obtain:

$$x^2 - \frac{28}{3}x + \frac{196}{9} + y^2 - \frac{8}{3}y + \frac{16}{9} = -\frac{64}{3} + \frac{196}{9} + \frac{16}{9},$$

or,

$$(x - \tfrac{14}{3})^2 + (y - \tfrac{4}{3})^2 =$$

Example II. Find, in polar coordinates, the equation of a circle with center at (2,90°) and radius equal to 2.

Solution. Let (r,θ) be any point, P, on the circle. To obtain the relation between r and θ, draw the line PA and observe that the triangle OAP is a right triangle, with OA the hypotenuse. Hence, $OP = OA \cos AOP$. Now, $OP = r$; $OA = 4$, and angle $AOP = 90° - \theta$. The last equality is thus stated as

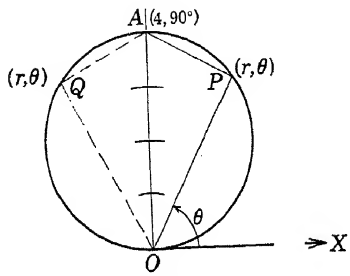


FIG. 25.

$$r = 4 \cos (90° - \theta),$$

$$r = 4 \sin \theta.$$

This, then, is the equation of the circle, since P was any point chosen at random on it.

Note that for a point (r,θ) in the position Q, the equation is

$$OQ = OA \cos (\theta - 90°),$$

or

$$r = 4 \cos (\theta - 90°),$$

and this also reduces to

$$r = 4 \sin \theta.$$

Example III. From A, the point (0,4), the line AP is drawn meeting the line $y = 2$ at P. AP is produced to Q, with $PQ = 2$. Find parametric equations of the locus of Q.

Solution. Let (x,y) designate the variable point Q. Choose the angle α from PA to the line $y = 2$ as parameter. Draw RS through Q, parallel to the y-axis. From the figure

$$x = OR = BS = BP + PS = BA \cot \alpha + PQ \cos \alpha,$$

$$y = RQ = RS - QS = RS - PQ \sin \alpha.$$

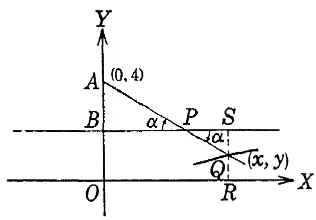


FIG. 26.

Note that $BA = PQ = RS = 2$. Hence, the parametric equations of the locus of Q are

$$x = 2 \operatorname{ctn} \alpha + 2 \cos \alpha,$$
$$y = 2 - 2 \sin \alpha.$$

Exercise 3. Draw a figure for Example III with Q to the left of OY and verify that the above solution holds in that case as well.

Exercise 4. The origin is identified by inspection as a point on this locus. What value of the parameter corresponds to it?

Exercise 5. Show that the line $y = 2$ is an asymptote of this locus. HINT: Find what value α must approach, for x to become numerically infinite and for $2 - y$ to approach zero.

Exercise 6. Plot the locus of this example.

The general procedure in deriving the equation of a locus may now be summed up somewhat as follows:

Choose a point in the plane assumed to satisfy the property stated for the locus.

Label it (x,y) or (r,θ), depending on the kind of coordinates employed.

Study the figure, completed by drawing extra lines if necessary, to find an equality between certain magnitudes (lines, angles, etc.) which arises from the property that defines the locus.

Embody that equality in an equation (or a pair of parametric equations) involving the coordinates of a variable point.

CAUTION: This process may yield an equation which is satisfied not only by the coordinates of the points on the contemplated locus but by the coordinates of other points as well. For example, if the problem were to find the equation of the segment OA, where O is the origin and A is $(1,1)$, the process would derive the equation $x = y$. This equation, however, is satisfied by the coordinates of all points collinear with O and A. The segment OA is not readily represented by a single equation but is defined by the pair of relations

$$\begin{cases} x = y, \\ 0 \leqq x \leqq 1. \end{cases}$$

Problems

1. Obtain the equation of each of the following loci:

(*a*) A line through $(2,4)$ parallel to the x-axis.

(*b*) A circle with center at $(3,-2)$ and radius equal to 4.

(*c*) A line through $(4,-1)$ whose slope equals 3. HINT: If (x,y) is any point on the line, the slope of the line is $\dfrac{y+1}{x-4}$.

(*d*) A line passing through $(5,0)$ and $(-4,2)$. *Ans.* $2x+9y = 10$.

(*e*) The perpendicular bisector of the segment joining $(-3,-1)$ and $(6,-3)$.

(*f*) A circle with center at $(2,4)$ and passing through $(3,7)$.
Ans. $x^2 - 4x + y^2 - 8y + 10 = 0$.

(*g*) A circle which has $(5,9)$ and $(-1,3)$ as the ends of a diameter.

(*h*) A line whose intercepts are -2 on OX and 5 on OY.
Ans. $2y - 5x = 10$.

2. Obtain the equation of each of the following loci:

(*a*) The locus of a point for which the distance from $(-3,2)$ is half the distance from $(4,0)$. *Ans.* $3x^2 + 3y^2 + 32x - 16y + 36 = 0$.

(*b*) The locus of a point for which the sum of the distances from $(4,0)$ and $(-4,0)$ equals 10. *Ans.* $9x^2 + 25y^2 = 225$.

(*c*) The locus of a point for which the difference of the distances from $(0,5)$ and $(0,-5)$ equals 6. *Ans.* $16y^2 - 9x^2 = 144$.

(*d*) The locus of a point for which the distance from $(6,0)$ equals the distance from the line $x = -6$. *Ans.* $y^2 = 24x$.

(*e*) The locus of a point for which the distance from $(-3,2)$ equals the distance from the line $y = 4$.

(*f*) The locus of a point for which the distance from $(3,0)$ equals twice the distance from the line $y = 5$.
Ans. $x^2 - 3y^2 - 6x + 40y - 91 = 0$.

(*g*) The locus of the centers of circles which pass through $(7,2)$ and $(3,0)$.

(*h*) The locus of the centers of circles that pass through $(-3,4)$ and are tangent to the y-axis. *Ans.* $y^2 - 8y + 6x + 25 = 0$.

(*i*) The locus of vertices of isosceles triangles whose bases extend from $(0,0)$ to $(5,-4)$.

(*j*) The locus of a point for which the sum of the squares of its distances from $(0,5)$ and $(0,-5)$ equals 100.

(*k*) The locus of a point which is the vertex of a right triangle having for hypotenuse the segment joining $(a,0)$ and $(0,b)$.

(*l*) The locus of vertices of triangles whose areas equal 10 and whose bases extend from $(-2,4)$ to $(3,4)$.

(*m*) The locus of vertices of triangles whose areas equal 12 and whose bases extend from $(5,3)$ to $(7,2)$.
Ans. $(x + 2y - 35)(x + 2y + 13) = 0$.

(*n*) The locus of a point such that the product of the slopes of the two lines joining it to $(4,0)$ and $(0,-3)$ equals 4.

(*o*) The locus of a point such that the slope of the line joining it to $(5,0)$ is twice the slope of the line joining it to $(1,0)$.

(*p*) The locus of the midpoints of the ordinates of the points on the curve $x^2 + y^2 = 20$.

(*q*) The locus of the centers of circles which are tangent to the line $x = -1$ and to the circle of radius 3, with center at $(4,0)$.
Ans. $(y^2 - 16x)(y^2 - 4x + 12) = 0$.

(*r*) The locus of the points of trisection, nearest the y-axis, of the abscissas of the points on the curve $2x^2 + y^2 = 18$.

(*s*) The locus of the centers of circles that are tangent to the line $y = 4$ and to the circle of radius 2, with center at $(0,-2)$.

(*t*) The locus of points such that the sum of the squares of their distances to the four points $(0,0)$, $(1,0)$, $(1,2)$, and $(0,2)$ equals 14.

(*u*) The locus of midpoints of the lines drawn from the origin to points on the curve $y^2 + x = 4$. *Ans.* $2y^2 + x = 2$.

(v) The locus of midpoints of the lines drawn from (2,1) to points on the curve $x^2 + y^2 = 9$. *Ans.* $x^2 + y^2 - 2x - y = 1$.

3. Find, in polar coordinates, the equations of the following loci:

(a) A circle with center at the pole and radius equal to 3.

(b) A circle with center at $(4,0°)$ and radius equal to 4.
Ans. $r = 8 \cos \theta$.

(c) A circle with center at $(4, 180°)$ and radius equal to 2. HINT: Use the cosine law in a triangle. *Ans.* $r^2 + 8r \cos \theta + 12 = 0$.

(d) A circle with radius equal to 2 and center at $(5,60°)$.

(e) A line through the pole and having an inclination of 120° to the polar axis.

(f) Each straight line whose inclination to the polar axis is 135° and whose distance from the pole is 3. *Ans.* $r \cos (\theta - 45°) = \pm 3$.

(g) A line perpendicular to the polar axis and passing through $(4,60°)$.

(h) Each line whose inclination to the polar axis is 60° and whose distance from the pole equals 5.

(i) A line parallel to the polar axis and passing through $(4,210°)$.
Ans. $r \sin \theta = -2$.

(j) A line through $(-4,150°)$ and perpendicular to the radius vector of that point.

4. Find, in polar coordinates, the equations of the following loci:

(a) The locus of a point whose distance from the pole equals its distance from the line perpendicular to the polar axis at $(3,180°)$.

$$\textit{Ans.} \quad r = \frac{3}{1 - \cos \theta}.$$

(b) The locus of a point whose distance from the pole is twice its distance from the line parallel to the polar axis and passing through $(4,270°)$.

$$\textit{Ans.} \quad r = \frac{8}{1 - 2 \sin \theta}.$$

(c) The locus of a point whose distance from the pole is twice its distance from the line perpendicular to the polar axis at $(5,0°)$.

(d) The locus of a point whose distance from the pole is $\frac{2}{3}$ its distance from the line parallel to the polar axis and passing through $(6,90°)$.

(e) The locus of midpoints of the chords passing through the pole, in the circle that passes through the pole and has its center at $(a,0)$.
Ans. $r = a \cos \theta$.

(f) The locus of points of trisection, nearest the pole, of the chords passing through the pole, in a circle that passes through the pole and has its center at $(b,90°)$.

5. In the circle with center at $(3,90°)$ and radius equal to 3, chords OQ are drawn from the pole O and produced past Q to P making $OQ \cdot OP$ equal to 48. Find the equation of the locus of P. *Ans.* $r = 8 \csc \theta$.

6. In the circle with center at $(a,0°)$ and radius equal to a, chords, OQ, are drawn from the pole O and produced past Q to P, making $OP = 2QP$. Find the equation of the locus of P.

7. At A, one end of the fixed diameter OA of a circle, a tangent is drawn to the circle. Through the other end, O, a chord is drawn meeting the circle

in Q and the tangent at R. On OQ a segment OP is laid off equal in length to QR. Find the equation of the locus of P.

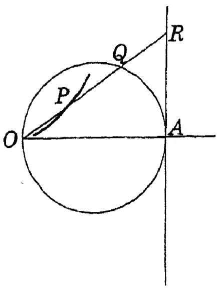


FIG. 27.

HINT: Take OA as the polar axis, and O as pole. Call the radius of the circle a.

NOTE: This locus is called the *cissoid*.

Ans. $r = 2a \sin \theta \tan \theta$.

8. Find parametric equations of the locus in Prob. 7 in rectangular coordinates, taking OA as the x-axis, and O as origin. Use the angle AOQ as parameter. *Ans.* $x = 2a \sin^2 \alpha$; $y = 2a \sin^2 \alpha \tan \alpha$.

9. Find the equation of the locus of a point P, such that the product of its distances from two fixed points, A and B, is equal to a^2, given that the length of AB is $2a$. HINT: Choose AB as the polar axis, and the point midway between A and B as the pole. NOTE: This is called the *lemniscate*. *Ans.* $r^2 = 2a^2 \cos 2\theta$.

10. Find the equation of the locus in Prob. 9 in rectangular coordinates, taking AB as the x-axis, and the point midway between A and B as the origin.

11. A fixed diameter, OA, of a circle is $2a$ in length. At A a tangent is drawn to the circle. A chord is drawn through O, intersecting the circle again in Q and the tangent at R. Through R a line is drawn parallel to OA and from Q a perpendicular is dropped upon that line, meeting it at P. The locus of P is a curve called the *witch*. Find parametric equations of the witch taking O as origin and OA as the y-axis and using angle QOA as parameter. (See Fig. 28.) *Ans.* $x = 2a \tan \alpha$; $y = 2a \cos^2 \alpha$.

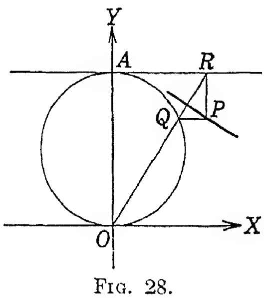


FIG. 28.

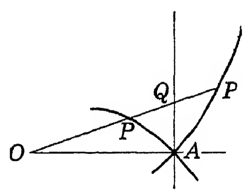


FIG. 29.

12. The line segment OA is of length a. At A a perpendicular is erected to OA, and a line is drawn through O, intersecting that perpendicular at Q. On OQ, QP is laid off on either side of Q, equal in length to AQ. Find the equation of the locus of P, choosing OA as the polar axis, and O as pole. This locus is called the *strophoid*. (See Fig. 29.)

Ans. $r^2 \cos \theta - 2ar + a^2 \cos \theta = 0$.

13. Find the equation of the locus of Prob. 12 in rectangular coordinates taking AO as the x-axis, and A as the origin.

14. From A, the point $(0, -3)$, lines AQ are drawn meeting the x-axis in Q and produced past Q to P, making $QP = \frac{1}{3}AQ$. Find the equation of the locus of P.

15. Find the equation of the locus of the midpoint of a line segment of length $2a$ which moves so that one end of it is on OX and the other end is on OY.

Ans. $x^2 + y^2 = a^2$.

16. In the preceding problem, find the equation of the locus of point of trisection, nearest OX, of the moving line segment.

17. O is the origin of coordinates, and A is (5,0). A perpendicular to OX is erected at A. A line is drawn through O, meeting that perpendicular at Q. AR is laid off on OX, in either direction from A, equal in length to half of AQ. At R a perpendicular is erected to OX, meeting OQ, or OQ produced, at P. Find the locus of P. HINT: Use angle AOQ as parameter (Fig. 30).

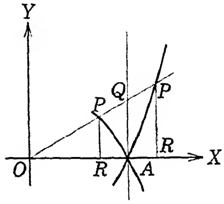


FIG. 30.

CHAPTER III

THE STRAIGHT LINE

16. The Straight Line as a Locus. Let a straight line, l, pass through the fixed point, P_1 (x_1,y_1), and have the slope m. The slope of the line P_1P, joining P_1 to any point P (x,y) in the plane is given by

$$\frac{y - y_1}{x - x_1}.$$

If, then, we wish to keep the point P upon the line l, we need only impose the condition that the slope of the line P_1P shall be equal to the slope of l. That is, we write

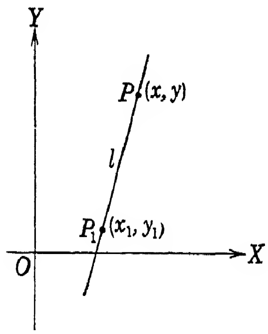


FIG. 31.

$$\frac{y - y_1}{x - x_1} = m$$

or

$$y - y_1 = m(x - x_1). \tag{20}$$

This equation is known as the *point-slope* form for the equation of a line. It enables one to find the equation of a line if he knows its slope, m, and the coordinates (x_1,y_1) of a point on it.

Exercise 1. Show that Eq. (20) can be put into the form

$$Ax + By + C = 0,$$

in which A, B, and C represent constants of which not both of A and B are zero.

Exercise 2. Prove that if $x_1 \neq x_2$ the line through the two fixed points (x_1,y_1) and (x_2,y_2) is represented by the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{21}$$

Formula (21) is known as the *two-point* form for the equation of a line.

Exercise 3. By substitution in the equation, prove that each of the two points (x_1,y_1) and (x_2,y_2) is on the graph of Eq. (21).

Exercise 4. Show that Eq. (21) can be put into the form

$$Ax + By + C = 0,$$

where A, B, and C are constants of which not both of A and B are zero.

Exercise 5. Show that the equation of the line passing through the distinct points (x_1,y_1) and (x_1,y_2) is $x = x_1$.

Exercise 6. Prove that the point $\left(\dfrac{r_1x_2 + r_2x_1}{r_1 + r_2}, \dfrac{r_1y_2 + r_2y_1}{r_1 + r_2}\right)$ satisfies Eq. (21) if $r_1 \neq -r_2$.

Exercise 7. Prove that the equation of the line whose x-intercept is $a \neq 0$, and whose y-intercept is $b \neq 0$ is

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{22}$$

HINT: The line passes through the distinct points $(a,0)$ and $(0,b)$.

This formula, (22), is called the *intercept* form for the equation of a line.

Exercise 8. Prove that the line whose y-intercept is b and whose slope is m has the equation

$$y = mx + b. \tag{23}$$

This equation is called the *slope-intercept* form for the equation of a line. It will be found to be very convenient for use in locating a straight line whose equation is given.

Problems

1. Find the equation of the line which passes through the given point and has the given direction in each case, and draw a figure.

(*a*) (3,1), slope $= -2$. *Ans.* $y + 2x = 7$.

(*b*) $(-1,5)$, slope $= \frac{1}{3}$.

(*c*) $(6,-2)$, perpendicular to the line whose slope is $-\frac{2}{5}$. *Ans.* $5x - 2y = 34$.

(*d*) $(-5,-7)$, parallel to the line joining (0,0) to $(-4,3)$.

(*e*) $(4,-3)$, parallel to the x-axis.

(*f*) (6,10), parallel to the y-axis.

(*g*) (0,0), bisecting the angle between the axes in the first and third quadrants.

(*h*) (0,0), bisecting the angle between the axes in the second and fourth quadrants.

(*i*) (3,5), having an inclination 60°.

(*j*) $(-2,8)$, having the inclination $\tan^{-1} (\frac{3}{7})$.

2. Find the equation of the line which passes through the two given points and draw a figure.

(*a*) (3,1), $(-2,5)$. *Ans.* $4x + 5y = 17$.
(*b*) $(-1,6)$, $(5,-9)$.
(*c*) $(-2,0)$, (3,3).
(*d*) (4,0), (0,5).
(*e*) $(-3,5)$, $(6,-7)$.
(*f*) $(3,-3)$, $(-5,5)$.
(*g*) $(-6,0)$, (0,7).
(*h*) (2,0), (0,0).
(*i*) (3,6), $(-7,6)$.
(*j*) $(-5,0)$, $(-5,8)$.
(*k*) (0,8), (0,0).
(*l*) $(1,-1)$, $(3,-7)$.

3. Find, in each case, the equation of the line satisfying the given condition and put the equation in the specified form.

(*a*) $(-2,7)$ on the line, and slope $= 3/8$. Form $Ax + By + C = 0$.

(*b*) $(-3,4)$ and $(2,-1)$ on the line. Form $y = mx + b$.

Ans. $y = -x + 1$.

(*c*) $(-1,6)$ and $(5,2)$ on the line. Form $\frac{x}{a} + \frac{y}{b} = 1$.

Ans. $\frac{x}{8} + \frac{y}{\frac{16}{3}} = 1$.

(*d*) $(-3,4)$ and $(6,-8)$ on the line. Form $y = mx + b$.

(*e*) $(2,6)$, and $(3,6)$ on the line. Form $Ax + By + C = 0$.

4. In Fig. 32 the line segment OP is 5 units long and angle XOP is 135°. If the line l is perpendicular to OP at P, find the equation of l.

Ans. $x\sqrt{2} - y\sqrt{2} + 10 = 0$.

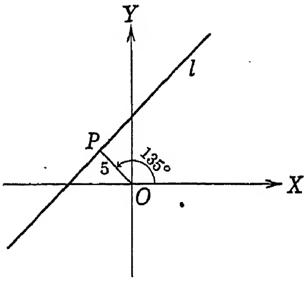


FIG. 32.

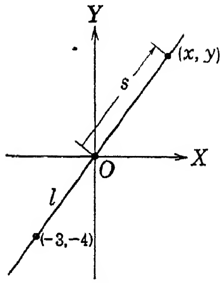


FIG. 33.

5. A line l passes through the origin and the point $(-3,-4)$. Express x and y parametrically in terms of the distance s from the origin to the point (x,y) on the line l.

6. A line l has slope equal to $-5/3$ and passes through the point $(3,1)$. Express x and y parametrically in terms of the x-intercept, a, of the line perpendicular to l at the point (x,y).

Ans. $x = \frac{9}{34}(a + 10)$, $y = \frac{3}{34}(18 - 5a)$.

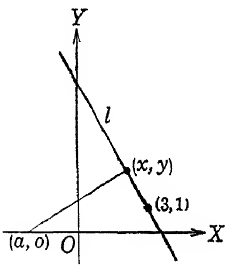


FIG. 34.

7. Find the equations of the three sides of the triangle whose vertices are $(3,1)$, $(-3,-2)$, and $(8,7)$.

8. Find the coordinates of the point of intersection of the diagonals of a quadrilateral whose consecutive vertices are $(-3,2)$, $(0,6)$, $(5,4)$, and $(3,-1)$.

Ans. $(39/31, 95/31)$.

9. The point $(-3,2)$ bisects that portion of a line which is cut off by the coordinate axes. Find the equation of the line.

Ans. $2x - 3y + 12 = 0$.

10. The point $(4,-2)$ divides the line segment from the point A on the x-axis to the point B on the y-axis in the ratio $1:2$. Find the equation of the line AB.

17. The Linear Equation. Let the constants A, B, and C in the equation

$$Ax + By + C = 0 \tag{24}$$

be subject to the one condition that A and B be not both zero. By working Exercises 1, 2, 3, and 4 below the student can show that this equation always represents a straight line.

Exercise 1. If $B \neq 0$, prove that the points $\left(0, -\frac{C}{B}\right)$ and $\left(1, -\frac{A+C}{B}\right)$ are on the locus of (24).

Exercise 2. Find the equation of the line which passes through the two points of Exercise 1 and show that it takes the form (24).

Exercise 3. If $B = 0$, prove that the points $(-C/A,0)$ and $(-C/A,1)$ are on the locus of (24).

Exercise 4. Find the equation of the line through the two points of Exercise 3 and show that the result is Eq. (24) with $B = 0$.

Problems

1. Put the given equation in the specified form, determine the constants requested, and draw a figure.

(a) $3x + 4y - 12 = 0$, Intercept form. Find a and b.
(b) $5x - 3y + 9 = 0$. Intercept form. Find a and b.
(c) $-8x + 6y + 4 = 0$. Intercept form. Find a and b.
(d) $5x - y + 2 = 0$. Slope-intercept form. Find the slope.
(e) $4x + 3y - 6 = 0$. Slope-intercept form. Find b.
(f) $7x + 5y + 9 = 0$. Slope-intercept form. Find m.
(g) $Ax + By + C = 0$. Slope-intercept form. Find m and b.

2. Prove that the lines $Ax + By + C = 0$ and $Ax + By + D = 0$ are parallel.

3. Prove that the lines $Ax + By + C = 0$ and $Bx - Ay + D = 0$ are perpendicular.

4. In each case below, find the equation of the line described.

(a) Through (0,0) parallel to the line $2x - 3y + 8 = 0$.
Ans. $2x - 3y = 0$.
(b) Through $(-2,5)$ parallel to the line $4x + y + 6 = 0$.
Ans. $4x + y + 3 = 0$.
(c) Through (1,2) parallel to $-6x + 5y + 12 = 0$.
(d) Through (0,0) perpendicular to the line $4x + 9y - 6 = 0$.
(e) Through $(0,-10)$ perpendicular to the line $4x + 3y = 0$.
Ans. $3x - 4y - 40 = 0$.
(f) Through $(5,-1)$ perpendicular to $x - y + 1 = 0$.

5. Determine k so that the line $3x - ky + 7 = 0$ shall be parallel to the line $6x - 8y + 9 = 0$.

6. Determine k so that the lines $5x + ky - 2 = 0$ and

$$10x + (6 - k)y = 0$$

shall be parallel.

7. Determine k,l, and m so that the lines $3x - 2ky - l = 0$ and

$$8x + (k + 1)y - m = 0$$

shall be perpendicular at the point (2,0).

Ans. $k = 3$ or -4, $l = 6$, $m = 16$.

8. Determine k so that the line $kx + (1 - k)y - k = 0$ shall be perpendicular to the line $\frac{3}{k}x + (1 + k)y + 6 = 0$.

9. Find the angle from the line $2x - 3y = 5$ to the line $3x - 4y + 7 = 0$.

Ans. $\tan^{-1} \frac{1}{18}$.

10. Find the point of intersection of the lines $5x - 12y = 3$ and $x + 2y = 5$.

11. Find the equation of the line which has equal intercepts on the coordinate axes and which passes through the point $(-6,8)$.

12. Find the equation of a line which passes through the point (6,2) and forms a triangle of area 32 with the coordinate axes.

13. Find the equation of the locus of a point which moves so as to remain equidistant from the points (1,5) and $(7,-3)$. Solve in two ways.

14. The hypotenuse of a right triangle coincides with the line

$$x - 2y + 6 = 0.$$

The vertex of the right angle is (5,3) and another vertex of the triangle is $(-2,2)$. Find the third vertex.

Ans. $(1\frac{4}{5}, 1\frac{6}{5})$.

15. Find the equation of the line passing through $(-3,4)$ and making an angle of 45° with the line $3x - y + 6 = 0$.

Ans. $x - 2y + 11 = 0$, $2x + y + 2 = 0$.

16. Find the equation of the line with x-intercept equal to 5 and making an angle $\theta = \tan^{-1} \frac{2}{3}$ with the line $x + 2y = 4$.

17. Determine k from the condition that the angle from the line $3x - 2y = 6$ to the line $2x - ky = 3$ is $\tan^{-1} 2$.

18. Show that if P_1 is (x_1,y_1) and P_2 is (x_2,y_2) then the equation

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

represents the straight line passing through P_1 and P_2.

18. The Straight Line in Polar Coordinates. Let any line l be given. Let a line ON be drawn from the pole, O, perpendicular to l at the point N. Choose the polar coordinates of N as (p,ω), subject to the understanding that p is positive or zero and ω is restricted as follows:

$$\text{Case I. } p > 0, \qquad 0 \leqq \omega < 360°.$$
$$\text{Case II. } p = 0, \qquad 0 \leqq \omega < 180°.$$

Considering first, Case I, let P, Fig. 35, be any other point on l, having the coordinates (r,θ). Let θ be temporarily restricted to differ from ω by an angle less than 90°. Then r is positive and equal to the length of OP, p equals ON and angle NOP is acute and equal to $\omega - \theta$ or $\theta - \omega$. From either of these cases it follows that

$$r \cos (\theta - \omega) - p = 0, \qquad (25)$$

the equation, in polar coordinates, of the line l.

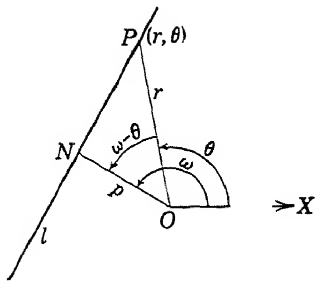


FIG. 35.

We shall refer to Eq. (25) as the *normal* form of the polar equation of a line, or the *polar normal form.* The angle ω, as restricted above, is called the *normal angle* for the line and the quantity p is called the *normal intercept* of the line, or, sometimes, the *normal* of the line.

Exercise 1. Prove that any set of coordinates (r,θ) of a point P on the line l will satisfy Eq. (25), even though the angle θ does not meet the condition temporarily imposed in the text. Note that if a point has polar coordinates (r,θ), it also has the polar coordinates $(r, \theta + 360°)$, $(r, \theta - 360°)$, and $(-r, \theta + 180°)$.

Equation (25) was derived for the case $p > 0$. If $p = 0$ the line l passes through the pole and has the inclination $\omega \pm 90°$. Hence, if a point P on the line has the coordinates (r,θ), the angle $\theta - \omega$ is an odd multiple of 90° and

$$\cos (\theta - \omega) = 0.$$

If this equation is satisfied, the equation

$$r \cos (\theta - \omega) = 0$$

is satisfied. But this last is identical with (25) when $p = 0$.

Problems

1. Draw the following lines and find for each its normal intercept, its normal angle, the angle from the polar axis to the line, and draw a figure.

(*a*) $r \cos (\theta - 25°) = 8$.

(*b*) $r \cos (\theta + 120°) = 5$.

Ans. $p = 5$, $\omega = 240°$; inclination $= 150°$.

(*c*) $r \cos \theta = -10$.

(*d*) $r \sin \theta = 7$.

(e) $r \sin (\theta - 50°) = 12.$

Ans. $p = 12$, $\omega = 140°$, inclination $= 50°$.

(f) $3r \cos \theta + 4r \sin \theta = 15.$ HINT: Divide by $\sqrt{3^2 + 4^2}$, or 5.

Ans. $\cos \omega = 3/5$, $\sin \omega = 4/5$.

(g) $r \cos \theta - \sqrt{3}r \sin \theta = 6.$

(h) $\sin (\theta - 20°) = 0.$

(i) $\cos (\theta - 150°) = 0.$

2. Write the polar equation of the line described in each case, and draw a figure.

(a) The initial line.

(b) A line through the pole perpendicular to the polar axis.

(c) A line tangent to a circle whose center is at the pole at the point whose rectangular coordinates are $(-3,4)$.

Ans. $r(4 \sin \theta - 3 \cos \theta) = 25.$

(d) A line whose inclination to the polar axis is 170° and whose shortest distance to the pole is 2.

Ans. $r \cos (\theta - 80°) = 2$, $r \cos (\theta - 260°) = 2.$

(e) Parallel to the x-axis, and y-intercept equals -2.

(f) Normal angle equals 80°, and the distance to the pole is 3.

3. Change each of the following to rectangular coordinates.

(a) $r = 3 \sec \theta.$

(b) $r \cos (\theta - 120°) = 0.$

(c) $r \sin (\theta + 45°) = 13.$

(d) $\sin \left(\theta - \frac{\pi}{8}\right) = 0.$

(e) $\cos (\theta + 150°) = 0.$

(f) $\theta = 100°.$

(g) $\theta = -\frac{3\pi}{2}.$

4. Change each of the following to polar coordinates and put it in the form (25). Draw a figure in each case.

(a) $x + y = 10.$

Ans. $r \cos (\theta - 45°) = 5\sqrt{2}.$

(b) $3x - 4y = 25.$

(c) $5x + 12y + 39 = 0.$

(d) $x = 6.$

(e) $y = -1.$

19. Normal Form in Rectangular Coordinates. Let us transform the polar normal form

$$r \cos (\theta - \omega) - p = 0$$

to rectangular coordinates. We have, first

$$r \cos \theta \cos \omega + r \sin \theta \sin \omega - p = 0,$$

whence, since $r \cos \theta = x$ and $r \sin \theta = y$, we have

$$x \cos \omega + y \sin \omega - p = 0. \tag{26}$$

This is known as the *normal form* for the equation of a line in rectangular coordinates. The significance of p and ω is pre-

cisely the same as in the polar normal form. Here, also, the normal intercept, p, and the normal angle, ω, are restricted by definition to satisfy the inequalities:

$$
\begin{aligned}
&p \geqq 0, \\
&0^\circ \leqq \omega < 360^\circ \qquad \text{if } p > 0, \\
&0^\circ \leqq \omega < 180^\circ \qquad \text{if } p = 0.
\end{aligned}
$$

To reduce an equation of the form

$$Ax + By + C = 0 \qquad (A, B \text{ not both zero}),$$

to the normal form, multiply through by k, obtaining

$$(Ak)x + (Bk)y + Ck = 0.$$

If this is the normal form we have

$$(Ak)^2 + (Bk)^2 = \cos^2 \omega + \sin^2 \omega = 1$$

or

$$k = \frac{1}{\pm\sqrt{A^2 + B^2}}.$$

Also, $(Ck) = -p$, and since p is positive the sign of $1/\pm\sqrt{A^2 + B^2} = k$ must be chosen opposite to the sign of C. If $C = 0$, then $p = 0$, and $0^\circ \leqq \omega < 180^\circ$, hence $Bk = \sin \omega \geqq 0$. Thus, if $C = 0$, the sign of k must be chosen the same as the sign of B. Finally if $C = B = 0$, we have $p = \omega = 0$ and $\cos \omega = 1$. Hence, in this case $k = 1/A$ and the normal form is $x = 0$.

Problems

1. Draw the line described in each case and write its equation in the rectangular normal form (26).

(*a*) Normal angle $= 300^\circ$, normal intercept $= 14$.

Ans. $\frac{x}{2} - y\frac{\sqrt{3}}{2} - 14 = 0.$

(*b*) Normal angle $= 210^\circ$, normal intercept $= 15$.

(*c*) Normal angle $= 135^\circ$, normal intercept $= 8$.

(*d*) Normal angle $= 90^\circ$, normal intercept $= 6$.

Ans. $y - 6 = 0.$

(*e*) Normal angle $= 0^\circ$, normal intercept $= 4$.

(*f*) Tangent, at $(-8, -6)$ to a circle with center at the origin.

Ans. $\frac{-4x}{5} - \frac{3y}{5} - 10 = 0.$

(g) Inclination $= 10°$, normal intercept $= 3$.

(h) Slope $= 1$, x-intercept $= 4\sqrt{2}$. *Ans.* $\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} - 4 = 0$.

(i) Parallel to the x-axis, y-intercept $= 7$.

(j) Parallel to the y-axis, x-intercept $= -9$.

(k) Passing through the origin, slope $= -4/3$.

Ans. $\frac{4}{5}x + \frac{3}{5}y = 0$.

(l) Passing through the origin, slope $= -5/12$.

2. Reduce each of the following to normal form and find its normal angle and normal intercept.

(a) $4y + 3x = 20$.

(b) $12x - 5y - 39 = 0$.

(c) $x + 2y + 12 = 0$.

Ans. $-\frac{x}{\sqrt{5}} - \frac{2y}{\sqrt{5}} - \frac{12}{\sqrt{5}} = 0$.

(d) $6y = 11$.

(e) $5x + 7 = 0$.

(f) $9x - 7y = 0$.

Ans. $-\frac{9x}{\sqrt{130}} + \frac{7y}{\quad} \quad 0$.

(g) $8y + 5x = 0$.

(h) $3y = 0$.

(i) $-5x = 2y$.

(j) $x \cos 20° + y \sin 20° = -4$.

(k) $x \cos 310° + y \sin 310° = 0$.

3. Prove, by illustration, that two parallel lines need not have the same normal angle.

4. Find the distance between the lines $3x + 4y = 10$, and $3x + 4y = 25$.

5. Find the distance between the lines $5x - 12y - 26 = 0$ and $12y - 5x = 39$.

6. Write the equation of the line parallel to the lines $4x + 7y - 8 = 0$ and $4x + 7y + 20 = 0$ and halfway between them.

Ans. $4x + 7y + 6 = 0$.

7. Write the equations of the two lines which are parallel to the line $3x - 2y = 2$ and 5 units from it.

8. Find the normal intercept of the line $Ax + By + C = 0$ by using it as the altitude of a triangle formed by the line and the coordinate axes.

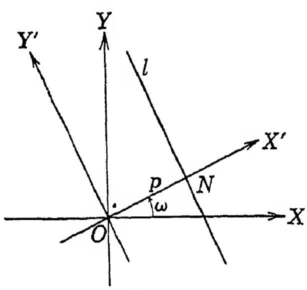


FIG. 36.

9. Derive Eq. (26) from the fact that, after the axes have been rotated through the angle ω the equation of the line is $x' - p = 0$.

10. Derive the rectangular normal form: (a) from its intercept form, by showing that $a = p/\cos \omega$ and $b = p/\sin \omega$; (b) from its slope-intercept form, by expressing m and b in terms of p and ω; (c) by finding the coordinates of the point where the normal meets the line, in terms of p and ω, and employing the point-slope form.

11. Prove by employing the normal form that if, for the lines $Ax + By + C = 0$ and $Dx + Ey + F = 0$, $A:B = D:E$, the lines are parallel or coincident.

12. Find for what value of k the line $y = 3x + k$ is at a distance $\sqrt{10}$ from the origin. *Ans.* $k = \pm 10$.

13. For what value of k is the line $2x - ky - 1 = 0$ at a distance of $\frac{1}{2}$ from the origin?

14. Find the equation of the line passing through (3,4) and whose distance from the origin equals 3. *Ans.* $x = 3$, $7x - 24y + 75 = 0$.

15. Find the equation of the line whose distance from the origin equals 3 and which makes an angle $\theta = \tan^{-1} 2$ with the line $3x - 2y = 5$.

20. Distance from a Line to a Point. A fundamental problem in the geometry of the plane is that of finding the shortest distance between a fixed point and a fixed line. It is customary to consider this distance as directed from the line to the point, with the positive direction indicated by the terminal side of ω, the unique normal angle of the line. Thus, for lines not through the origin, the distance from the line to the origin is negative, and the distance from the origin to the line is positive.

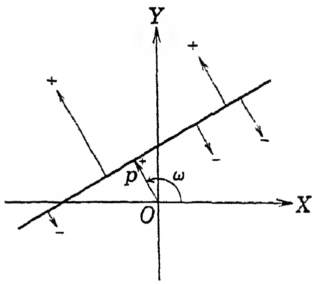


FIG. 37.

To find a formula for the directed distance d from the line l to the point P (x_1,y_1), let the normal form of the equation for l be

$$x \cos \omega + y \sin \omega - p = 0.$$

Drop a perpendicular from P to l meeting l in a point Q. Then

$$QP = d$$

and

$$PQ = -d.$$

Let P be taken as the pole of a system of polar coordinates with polar axis PX' parallel of OX and in the same sense. In this system the polar coordinates of Q are

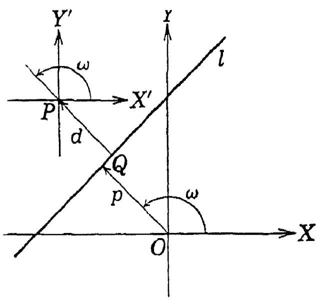


FIG. 38.

$$(-d,\omega).$$

The rectangular coordinates of Q, referred to P as origin, are

$$(-d \cos \omega, \ -d \sin \omega)$$

and, hence, referred to O as origin, are

$$(x_1 - d \cos \omega, \ y_1 - d \sin \omega)$$

Since Q is on l, these coordinates satisfy the equation of l, or

$$(x_1 - d \cos \omega) \cos \omega + (y_1 - d \sin \omega) \sin \omega - p = 0.$$

This gives

$$x_1 \cos \omega + y_1 \sin \omega - d (\cos^2 \omega + \sin^2 \omega) - p = 0,$$

or

$$d = x_1 \cos \omega + y_1 \sin \omega - p. \tag{27}$$

Note that, in view of the definition of the direction of d, its value, as obtained by formula (27), is positive when the distance from the line to the point extends in the direction of the terminal side of ω. The value is negative when that distance extends in the direction opposite to the terminal side of ω.

While Fig. (38) shows P in such a position that d is positive, there is no such assumption in the proof, and formula (27) gives the directed distance from the line whose equation, in normal form, is $x \cos \omega + y \sin \omega - p = 0$, to the point whose coordinates are (x_1,y_1), regardless of the positions of the line and point.

As an application of formula (27) we have the problem of finding the equations of the bisectors of the angles between the lines l_1 and l_2, whose equations are given. Let the normal forms of the equations be

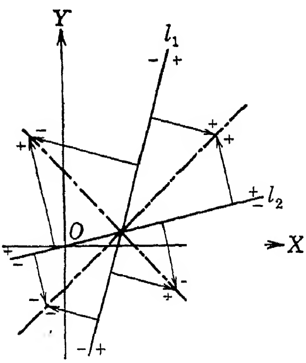


FIG. 39.

$$x \cos \omega_1 + y \sin \omega_1 - p_1 = 0$$

and

$$x \cos \omega_2 + y \sin \omega_2 - p_2 = 0,$$

respectively. If (x,y) is a point on that bisector which lies in the regions where the directed distances from l_1 and l_2 have like signs, the directed distances from these lines to (x,y) are equal. Hence the equation of that bisector is

$$(x \cos \omega_1 + y \sin \omega_1 - p_1) - (x \cos \omega_2 + y \sin \omega_2 - p_2) = 0. \tag{28}$$

The directed distances from l_1 and l_2 to a point (x,y) on the other bisector are numerically equal but opposite in sign. Their sum is, therefore, zero, and the equation of the bisector is

$$(x \cos \omega_1 + y \sin \omega_1 - p_1) + (x \cos \omega_2 + y \sin \omega_2 - p_2) = 0. \tag{29}$$

Problems

1. Find the directed distance from the line to the point in each of the following cases:

(*a*) $4x - 3y + 7 = 0$, $(2, -5)$. *Ans.* -6.
(*b*) $4x - 3y + 7 = 0$, $(-2, 5)$.
(*c*) $5x + 12y - 20 = 0$, $(3, 1)$. *Ans.* $7/13$.
(*d*) $5x + 12y - 20 = 0$, $(-8, 4)$.
(*e*) $6y - 11 = 0$, $(2, 5)$.
(*f*) $5x - 2y = 0$, $(4, 5)$. *Ans.* $-10/\sqrt{29}$.
(*g*) $5x - 2y = 0$, $(-2, 1)$.
(*h*) $x = 3$, $(4, 5)$.
(*i*) $x = -2$, $(-7, 3)$.

2. Find the equations of the bisectors of the two angles between the given pair of lines in each case, and draw a figure.

(*a*) $7x - 4y + 30 = 0$, $x + 8y - 10 = 0$.
Ans. $2x + y + 5 = 0$, $3x - 6y + 20 = 0$.
(*b*) $11x + 2y + 50 = 0$, $5x - 10y - 95 = 0$.
(*c*) $5x + 5y = 0$, $14x - 2y + 37 = 0$.
Ans. $24x + 8y + 37 = 0$, $4x - 12y + 37 = 0$.
(*d*) $6x - 7y + 15 = 0$, $-8x - 9y = 0$.
(*e*) $12x - y = 0$, $2x - 5y + 9 = 0$.
(*f*) $x + y + 3 = 0$, $2x - 5y + 9 = 0$.

3. Find the equations of the angle bisectors in Prob. 2 by employing the formula for the angle between two lines.

4. Find the locus of a point whose distance from the line $7x + 4y + 20 = 0$ is twice its distance from the line $8x - y - 16 = 0$.
Ans. $23x + 2y = 12$, $9x - 6y = 52$.

5. Find the locus of points whose distances from the line $7x - 9y - 100 = 0$ are three times their distances from the line $11x - 3y = 0$.

6. Prove that the directed distance from the line whose polar normal equation is $r \cos (\theta - \omega) - p = 0$ to the point whose polar coordinates are (r_1, θ_1) is given by

$$d = r_1 \cos (\theta_1 - \omega) - p.$$

What is the positive direction?

7. Find the directed distance from the line $r \cos (\theta - 75°) - 6 = 0$ to the point $(2, 15°)$. *Ans.* -5.

8. Find the directed distance from the line $r \cos (\theta - 170°) = 5$ to the point $(4, 50°)$.

9. Find the point of intersection of the three bisectors of the interior angles of the triangle determined by the lines $7x - 4y = 0$, $x = -8y$, and $8x + y + 16 = 0$.

10. Find the polar equation of the lines bisecting the angles between the lines $r \cos \theta - 5 = 0$ and $r \cos (\theta - 30°) = 8$.

11. Prove that the two lines represented by Eqs. (28) and (29) of the text are perpendicular.

12. Find the area of the triangle with vertices at (1,2), (4,5), and (2,8) by assuming one of its sides as a base and determining the length of the altitude upon it.

13. Find the area of a triangle with vertices at $(-3,2)$, $(-1,-2)$, and (1,4) by the method of Prob. 12.

14. Prove that the lines $Ax + By + C = 0$ and $Ax + By + D = 0$ are parallel, by showing that the distance from the first line to a point (u,v) on the second line is a constant.

15. Prove that according as the points (r,s) and (u,v) are on the same or opposite sides of the line $Ax + By + C = 0$, the quantities $Ar + Bs + C$ and $Au + Bv + C$ are of the same or opposite signs.

16. Find the equation of the locus of a point equidistant from the line $3x - 4y = 6$ and the point (0,1).

Ans. $16x^2 + 24xy + 9y^2 + 36x - 98y - 11 = 0.$

17. The two equal sides of an isosceles triangle coincide with the lines $y - 2x + 5 = 0$ and $y + 2x - 11 = 0$. The base coincides with a line whose y-intercept is -1. Find the length of each side of the triangle.

18. A line has a slope equal to m, and it passes through the point (x_1,y_1). Derive the point slope form of its equation in the following manner. Assume its equation to be $y = mx + k$, and determine k by imposing the condition that the distance from the line to the point (x_1,y_1) be zero.

21. Systems of Lines. Suppose the quantities A, B, and C in the equation

$$Ax + By + C = 0 \tag{30}$$

are not definite constants but depend upon some quantity k in such a way that, if k is assigned, A, B, and C become fixed. Under such circumstances, each value assigned to k will determine a line (if A and B do not both become zero). The totality of lines obtainable in this way from one such equation is called a *system*, or *family*, of lines, and the arbitrary quantity k is called the *parameter* of the system.

The foregoing definition is actually that of a *one-parameter* system of lines. Manifestly, we can have two-parameter systems also. Indeed the intercept form

$$\frac{x}{a} + \frac{y}{b} = 1,$$

the slope-intercept form

$$y = mx + b$$

and the normal form

$$x \cos \omega + y \sin \omega - p = 0$$

are illustrations of two-parameter systems.

Exercise 1. Prove that all the lines of the system

$$3x + ky - 6 = 0 \tag{31}$$

pass through the point (2,0). Give three distinct values to k and plot the corresponding lines.

Exercise 2. Prove that the slope of an individual line of the system (31) is $(-3/k)$. Show, from this, that the system includes all lines through the point (2,0) except the line $y = 0$.

Exercise 3. Prove that all lines of the family

$$2x - 3y + 4 - k = 0$$

have the same slope, $\frac{2}{3}$. Give three distinct values to k and plot the corresponding lines.

Exercise 4. Find the y-intercept of a line of the family of Exercise 3 in terms of the parameter k. Prove that the family includes all lines of slope $\frac{2}{3}$.

An especially interesting problem is that of finding the equation of the *system* of lines passing through the point of intersection of a given pair. Let the given lines be represented by the equations

$$A_1x + B_1y + C_1 = 0 \tag{32}$$

and

$$A_2x + B_2y + C_2 = 0. \tag{33}$$

Consider the equation

$$A_1x + B_1y + C_1 + k(A_2x + B_2y + C_2) = 0. \tag{34}$$

From its degree in x and y it is evident that it represents a system of lines. Now, if (u,v) is a point common to lines (32) and (33), it follows that

$$A_1u + B_1v + C_1 = 0$$

and

$$A_2u + B_2v + C_2 = 0.$$

As a result,

$$A_1u + B_1v + C_1 + k(A_2u + B_2v + C_2) = 0$$

regardless of the value of k, and this establishes the fact that every line of (34) passes through (u,v).

Conversely, consider any line through (u,v) and through any other point (r,s), not on the line (33), *i.e.*, subject to the condition $A_2r + B_2s + C_2 \neq 0$. It is a matter of direct verification that the equation

$$A_1x + B_1y + C_1 - \frac{A_1r + B_1s + C_1}{A_2r + B_2s + C_2}(A_2x + B_2y + C_2) = 0$$

represents that line. Indeed, it is satisfied if we put $x = u$, $y = v$ in it, as well as $x = r$, $y = s$. But the last equation is obtained from (34) by choosing k as equal to $-\dfrac{A_1r + B_1s + C_1}{A_2r + B_2s + C_2}$. Thus, we have proved that every member of the system (34) passes through the point of intersection of lines (32) and (33), and, conversely, every line of the system [with the exception of the line (33) itself] may be represented by Eq. (34) with an appropriate choice of k.

Exercise 5. If the lines (32) and (33) are parallel and distinct, prove that the system (34) represents every line parallel to the two, except the line (33) itself.

Exercise 6. If the lines (32) and (33) are coincident, prove that every line in the system (34) coincides with them.

Exercise 7. Prove that the equation

$$k(A_1x + B_1y + C_1) + A_2x + B_2y + C_2 = 0$$

also represents the system of lines through the intersection of (32) and (33), or a system of lines parallel to both of them, as the case may be. What line of the system does this equation fail to represent?

Exercise 8. Prove that the equation

$$h(A_1x + B_1y + C_1) + k(A_2x + B_2y + C_2) = 0,$$

where h and k are arbitrary constants, represents the same system as (34). Are the lines (32) and (33) included in this representation?

Problems

1. Write the equations of the following systems of lines. Point out in each case the line that is not represented.

(a) All lines passing through the origin.

Ans. $y = kx$; $x = ky$; $hx + ky = 0$.

(b) All lines passing through the point $(3, -5)$.

(c) All lines passing through the point $(-2, -4)$.

(d) All lines having -4 as x-intercept.

Ans. $y = kx + 4k$; $h(x + 4) + ky = 0$.

(e) All lines having 5 as y-intercept.

(f) All lines having slope equal to 2.

(g) All lines having the inclination 45°.

(h) All lines parallel to the line $3x - 5y + 11 = 0$.

(i) All lines perpendicular to the line $4x + 2y = 0$.

(j) All lines tangent to the circle whose radius is 1 and whose center is at the origin.

2. Describe the following systems of lines and draw a figure showing three distinct lines of each system:

(*a*) $(2 - k)x + y - 5 = 0$.
(*b*) $x + 4y + 1 + k = 0$.
(*c*) $y = mx - m$.
(*d*) $x \cos \omega + y \sin \omega - 3 = 0$.

3. Write the equation of the system of lines passing through (1,2) and find a line in the system satisfying the condition stated below for each case:
(*a*) Passing through $(5, -2)$.
(*b*) Parallel to the line $2x - 4y + 11 = 0$.
(*c*) Perpendicular to the line $x + y = 0$.
(*d*) Passing 2 units from the origin.
Ans. $y = 2$, $4x + 3y = 10$.

4. Write the equation of the system of lines having the slope $\frac{3}{4}$ and find a line of the system for each condition stated below.
(*a*) Passing through (1,3).
(*b*) Passing 3 units from the origin.
Ans. $3x - 4y \pm 15 = 0$.
(*c*) Having y-intercept equal to 4.

5. Write the equation of the family of lines having their normal intercepts equal to 4, and find a line satisfying the condition stated below for each case.
(*a*) Passing through (4,2). *Ans.* $x = 4$, $3x + 4y = 20$.
(*b*) Parallel to the line $5x - 12y + 1 = 0$.
Ans. $5x - 12y \pm 52 = 0$.
(*c*) Perpendicular to $12x + 35y = 0$.
(*d*) Having x-intercept equal to -8. *Ans.* $\pm y\sqrt{3} = x + 8$.

6. Write the equation of the system of lines passing through the point of intersection of the lines $2x + y - 5 = 0$ and $x + 3y + 8 = 0$. Find the line of the system having the property:
(*a*) Passing through $(-1, -1)$.
(*b*) Having slope equal to $-\frac{6}{7}$. *Ans.* $30x + 35y + 9 = 0$.
(*c*) Parallel to the y-axis.
(*d*) Parallel to the x-axis.
(*e*) Having normal equal to $\frac{3}{5}$.

7. Write the equation of the system of lines passing through the point of intersection of the lines $3x + 4y + 5 = 0$ and $4x - 3y + 8 = 0$. Find the line of the system satisfying the condition:
(*a*) Passing through (2,1).
(*b*) Passing through $(-1, -\frac{8}{5})$.
(*c*) Passing 1 unit from the origin.
Ans. $3x + 4y + 5 = 0$, $203x - 396y + 445 = 0$.
(*d*) Having slope equal to -1.
(*e*) Parallel to $8x + 9y = 5$.

8. Write the equation of the system of lines passing through the point of intersection of the lines $4x - 10y - 25 = 0$, and $6x - 14y - 53 = 0$. Find the line of the system satisfying the property:
(*a*) Passing through $(\frac{3}{2}, \frac{1}{2})$.
(*b*) Passing through $(-\frac{15}{2}, \frac{3}{2})$.
(*c*) Having the normal angle equal to $\tan^{-1}(-\frac{12}{5})$.
Ans. $5x - 12y - 39 = 0$.

9. Prove that if the three lines

$$(l_1) \qquad A_1x + B_1y + C_1 = 0$$
$$(l_2) \qquad A_2x + B_2y + C_2 = 0$$
$$(l_3) \qquad A_3x + B_3y + C_3 = 0$$

have a common point or are all parallel, then

$$\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = 0.$$

10. Prove that if the determinant appearing in Prob. 9 has the value zero, the lines (l_1), (l_2), and (l_3) either all pass through one point or are all parallel.

CHAPTER IV

EQUATIONS OF THE SECOND DEGREE

22. Rectangular Equation of a Circle. Let a circle of radius r have the center $Q\ (h,k)$. If $P\ (x,y)$ is any point on the circumference we have, from the distance formula

$$(x - h)^2 + (y - k)^2 = r^2.$$

If the squared quantities are expanded, this equation can be written as

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0,$$

an equation of the form

$$x^2 + y^2 + Dx + Ey + F = 0 \quad (35)$$

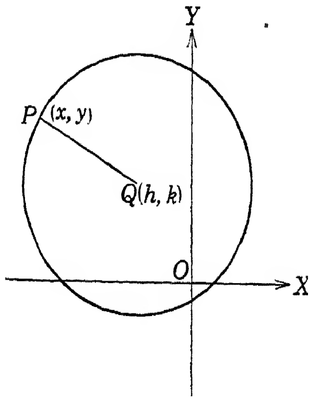


FIG. 40.

in which

$$\begin{aligned} D &= -2h, \\ E &= -2k, \\ F &= h^2 + k^2 - r^2. \end{aligned} \qquad (36)$$

It is natural to ask whether or not every equation of the form (35) represents a circle. The answer is found by solving equations (36) for h, k, and r in terms of D, E, and F. We obtain

$$h = -\frac{D}{2}, \qquad k = -\frac{E}{2},$$

and

$$= h^2 + k^2 - F = D^2 + E^2 - 4F$$

If $D^2 + E^2 - 4F$ is positive, we have $r = \frac{1}{2}\sqrt{D^2 + E^2 - 4F}$ and (35) represents a circle with center at $(-D/2, -E/2)$. If $D^2 + E^2 - 4F = 0$, $r = 0$ and (35) represents the point $(-D/2, -E/2)$. If $D^2 + E^2 - 4F$ is negative, there is no real value of r which satisfies the system (36), and (35) represents no locus.

Exercise 1. Complete the squares for the terms in x and the terms in y of Eq. (35) and thus obtain

$$\left(x+\frac{D}{2}\right)^2+\left(y+\frac{E}{2}\right)^2=\frac{D^2+E^2-4F}{4}. \tag{37}$$

Illustration 1. Find the equation of a circle passing through $P\ (6,-1)$ and $Q\ (7,-8)$ and whose center lies on the line $5x+3y=0$.

Solution. The center of the desired circle evidently lies on the perpendicular bisector of the line segment PQ, that is, on the line $x-7y=38$. Since we are given, further, that the center lies on the line $5x+3y=0$, its coordinates are found by solving simultaneously the two equations

$$\begin{aligned} x-7y&=38,\\ 5x+3y&=0. \end{aligned}$$

The solution is $x=3$, $y=-5$, so the center has the coordinates $(3,-5)$. The radius is clearly the distance from $(3,-5)$ to either P or Q and is found to equal 5.

We can now display the equation of the given circle as

$$(x-3)^2+(y+5)^2=25$$

or

$$x^2+y^2-6x+10y+9=0.$$

Illustration 2. Find the equation of the circle tangent to the lines $y-3x=20$ and $x+3y=10$ and passing through the origin.

Solution. Let the center have the coordinates (h,k). The distance from each of the two given lines to (h,k) and from the origin to (h,k) must all be equal to the radius and, hence, equal to each other. These distances are obtained as

$$\frac{k-3h-20}{-\sqrt{10}}\qquad\frac{h+3k-10}{-\sqrt{10}},\qquad\text{and}$$

Equating the first two of these, we obtain $2h+k+5=0$. Equating the last two of the expressions, we obtain

$$-6hk+k^2+20h+60k-100=0.$$

Solving these equations simultaneously for h and k, we obtain $h=-3$, $k=1$ and $h=5$, $k=-15$. The center of the required circle may have either of the sets of coordinates $(-3,1)$ and $(5,-15)$, that is, there are two possible circles. Since they are required to pass through the origin the radii are readily found to be $\sqrt{10}$ and $5\sqrt{10}$, respectively. The equation of the former is, then, $(x+3)^2+(y-1)^2=10$, or $x^2+y^2+6x-2y=0$, while the other circle has the equation $(x-5)^2+(y+15)^2=250$, or $x^2+y^2-10x+30y=0$.

Problems

1. Find the equations of the circles which meet the conditions stated below and draw a figure in each case.

(*a*) Center at (0,0), radius 7.

(*b*) Center at (2, −3), radius 4.

Ans. $x^2 + y^2 - 4x + 6y - 3 = 0.$

(*c*) Center at (−5,2), radius 2.

(*d*) Center at (4,5), passing through (−2,1).

Ans. $x^2 + y^2 - 8x - 10y - 11 = 0.$

(*e*) Having the points (4,2) and (−3,6) as the ends of a diameter.

(*f*) Center at (5,3), tangent to the line $x + 2y = 6$.

Ans. $x^2 + y^2 - 10x - 6y + 29 = 0.$

(*g*) Passing through (5,1) and (−1,3), center on the line $y = -3x$.

Ans. $3x^2 + 3y^2 - 4x + 12y - 70 = 0.$

(*h*) Tangent to both axes and center on the line $y + x = 5$.

(*i*) Tangent to the line $y - 3x = 7$ at (−1,4) and having center on $x + 2y = 6$. *Ans.* $x^2 + y^2 + 8x - 10y + 31 = 0.$

(*j*) Tangent to the line $2x + 3y = 10$ at (2,2) and passing through (5, −1). *Ans.* $x^2 + y^2 + 8x + 14y - 52 = 0.$

(*k*) Tangent to the line $x - y + 3 = 0$ and passing through the two points (1,2) and (−1,0). *Ans.* $x^2 + y^2 - 2y - 1 = 0.$

(*l*) Tangent to the line $x + 2y = 6$ at (2,2) and tangent to the line $y = 2x + 3$.

(*m*) Tangent to the lines $x + y = 4$ and $x - y = 8$ and passing through the origin.

(*n*) Inscribed in the triangle whose sides have the equations $8x + y = 8$, $x + 8y = 4$, $7x + 4y = 14$.

Ans. $65[(70x - 87)^2 + (70y - 47)^2] = 33{,}489.$

(*o*) Passing through the points (2,0), (−5,4), (1, −3).

(*p*) Passing through the points (1,6), (4, −2), and (−1, −1).

2. Find the center and radius of each of the following circles and draw a figure.

(*a*) $x^2 + y^2 - 6x - 10y + 18 = 0.$ *Ans.* (3,5), $r = 4$.

(*b*) $x^2 + y^2 + 10x - 2y - 10 = 0.$

(*c*) $x^2 + y^2 - 5x + 6y + 10 = 0.$

(*d*) $x^2 + y^2 + 3x - 7y + 8 = 0.$

(*e*) $2x^2 + 2y^2 - 18x - 2y + 33 = 0.$ *Ans.* $(\frac{9}{2}, \frac{1}{2})$, $r = 2$.

(*f*) $4x^2 + 4y^2 - 24x + 12y + 45 = 0.$

3. Show that the circles $x^2 + y^2 - 4x - 10y + 27 = 0$ and

$$x^2 + y^2 - 10x - 4y + 21 = 0$$

are tangent to each other at the point (3,4).

4. Find the equation of the line tangent to the circle

$$x^2 + y^2 - 2x - 4y - 20 = 0$$

at the point (4,6). *Ans.* $3x + 4y = 36.$

5. Find the equations of the lines tangent to the circle

$$x^2 + y^2 + 6x + 8y - 1 = 0$$

having the slope $-\frac{1}{5}$. *Ans.* $x + 5y - 3 = 0$, $x + 5y + 49 = 0$.

6. Find the equations of the lines tangent to the circle

$$x^2 + y^2 - 4x - 21 = 0$$

and passing through the point $(-3,10)$. *Ans.* $x = -3$, $3x + 4y = 31$.

7. Prove analytically that an angle inscribed in a semicircle is a right angle.

8. Find the locus of the third vertex of a triangle having vertices at $(3,0)$ and $(-3,0)$ and $45°$ as the internal angle at the moving vertex. Name the curve and draw a figure.

9. Find the locus of a point which moves so that the sum of the squares of its distances from the points $(0,4)$ and $(0,-4)$ is 100. Name the curve and draw a figure.

10. Chords of the circle $x^2 + y^2 + 4x = 0$ are drawn from the point $(0,0)$. Prove that the locus of their midpoints is a circle.

11. Prove that the circles $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ intersect at right angles. NOTE: By the angle between two circles at a point of intersection is meant the angle between their tangents at that point.

12. Find the angle between the circle $x^2 + y^2 - 6x - 4y + 8 = 0$ and the line $x + 2y = 3$ at each of their points of intersection. NOTE: By the angle between a line and a circle, at a point of their intersection, is meant the angle between the line and the tangent to the circle at that point.

Ans. $\tan^{-1} \frac{3}{4}$.

13. A straight line through the origin intersects the line $x = 1$ in the point Q. On the line OQ is a point P so situated that $(OP)(OQ) = 1$. Find the equation of the locus of the point P. Name the curve. *Ans.* $x^2 + y^2 - x = 0$.

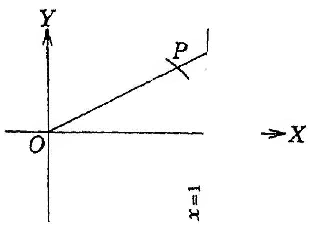


FIG. 41.

14. Prove that the square of the length of the line tangent to the circle

$$x^2 + y^2 + Dx + Ey + F = 0$$

and drawn from the point (x_1,y_1), outside the circle, is given by

$$T^2 = x_1^2 + y_1^2 + Dx_1 + Ey_1 + F.$$

15. If the point (x_1,y_1) is inside the circle

$$x^2 + y^2 + Dx + Ey + F = 0$$

prove that the quantity

$$x_1^2 + y_1^2 + Dx_1 + Ey_1 + F$$

is negative.

16. Prove that the equations

$$\begin{aligned} x &= h + r\cos\theta \\ y &= k + r\sin\theta \end{aligned}$$

are the parametric equations of a circle with center at (h,k) and radius equal to $|r|$ if h, k, and r are constants and θ is a parameter. HINT: Solve for $\sin\theta$ and $\cos\theta$, square, and add. Draw a figure showing the geometrical significance of the parameter θ.

17. Write parametric equations for the circle $x^2 + y^2 = 25$, using as parameter the angle from the y-axis to a variable radius.

18. Draw the graph defined parametrically by

$$\begin{aligned} x &= 10\cos\theta, \\ y &= 10\sin\theta. \end{aligned}$$

19. If the fixed points (x_1,y_1), (x_2,y_2), and (x_3,y_3) are not all on the same straight line prove that the equation

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

represents a circle passing through the three points.

20. Prove that if tangents are drawn to the circle $x^2 + y^2 = a^2$ from the point (x_1,y_1), the equation of the line joining the points of tangency (the chord of contact) is

$$x_1x + y_1y = a^2.$$

Show also that this is the equation of the tangent at (x_1,y_1) if that point is on the circle.

21. Prove that if tangents are drawn from the point (x_1,y_1) to the circle $x^2 + y^2 + Dx + Ey + F = 0$, the equation of the chord of contact is

$$x_1x + y_1y + \frac{D}{2}(x + x_1) + \frac{E}{2}(y + y_1) + F = 0.$$

Show also that this is the equation of the tangent at (x_1,y_1) if that point is on the circle.

23. The Circle in Polar Coordinates. Let a circle of radius a have its center at the point Q whose polar coordinates are (p,ω), where p is positive. Let any other point P on the circle have the polar coordinates (r,θ) where, for the present, θ is subject to the restriction $\omega - 180° \leqq \theta \leqq \omega + 180°$, and r is positive or zero. Then, if O is the pole, the angle POQ is either $\omega - \theta$ or $\theta - \omega$. The side of the

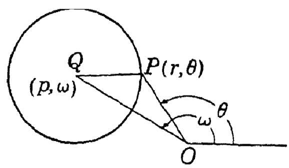


FIG. 42.

triangle POQ opposite O is of length a (see Fig. 42) and by the law of cosines

$$a^2 = p^2 + r^2 - 2rp \cos (\theta - \omega), \tag{38}$$

the equation of the circle in polar coordinates.

Conversely, if an equation in r and θ can be put into the form (38) where a, p, and ω are real constants, and $a > 0$, it represents a circle with center at (p,ω) and radius equal to a.

Exercise 1. Prove the converse statement just made.

Exercise 2. If a point (r,θ) satisfies Eq. (38), prove that its coordinates $(-r, \theta + 180°)$ also satisfy and, hence, that the restrictions on r and θ made in the text above are not necessary.

Exercise 3. Prove that an equation of the circle whose center is the pole and whose radius is a is $r = a$.

Exercise 4. If the center of a circle is on the polar axis and the circle passes through the pole, prove from (38) that the circle is represented by one or the other of the equations

$$r = 2a \cos \theta,$$
$$r = -2a \cos \theta.$$

Exercise 5. If the center of a circle is $(a,\pi/2)$ and the circle passes through the pole, prove from (38), or otherwise, that the equation is

$$r = 2a \sin \theta.$$

Problems

1. Determine the polar coordinates of the center and find the radius of the following circles. Draw a figure in each case.

(a) $r = 6$.
(b) $r = 8 \cos \theta$.
(c) $r = -5 \sin \theta$.
(d) $r + 4 \cos \theta = 0$.
(e) $r - 10 \sin \theta = 0$.
(f) $r = 2\sqrt{3} \cos \theta + 2 \sin \theta$. *Ans.* $(2,30°)$; rad. $= 2$.
(g) $r = 6 \cos \theta - 4 \sin \theta$.
(h) $r^2 - 5r + 6 = 0$.
(i) $r^2 + 16 - 10r \cos (\theta - 160°) = 0$.
Ans. $(5, 160°)$; rad. $= 3$.
(j) $r^2 - 11 + 6r \cos (\theta + 10°) = 0$.
(k) $r^2 + 3 - 4r \sin (\theta - 20°) = 0$.
(l) $r^2 - 7 - 6r \sin (\theta + 40°) = 0$. *Ans.* $(3,50°)$; rad. $= 4$.

2. Write the polar equation of the circle

(a) whose radius is 5 and whose center is $(8, -\pi/4)$.
Ans. $r^2 + 39 - 16r \cos (\theta + 45°) = 0$.
(b) whose radius is 3 and whose center is $(3,0)$.
(c) whose radius is 9 and whose center is $(-9,90°)$.

(*d*) whose radius is 5 and whose center is $(5,50°)$.
(*e*) whose radius is 12 and whose center is $(-6,116°)$.

3. Find the distance between the centers of the circles

$$r^2 + 24 - 14r \cos (\theta - 155°) = 0$$

and

$$r^2 - 15 - 14r \cos (\theta - 95°) = 0. \qquad \textit{Ans.}\ 7.$$

4. Find the polar equation of the circle passing through $(2,0°)$ and $(1,90°)$ and with center on the line whose equation is $\theta = 60°$.

5. Change (38) to rectangular coordinates. Verify that the result is the equation of a circle with center at $(p \cos \omega, p \sin \omega)$ and radius a.

6. Find the polar equation of the locus of a point for which the square of the distance from the pole is twice its distance from the line $r \cos \theta + 4 = 0$. Plot the locus. *Ans.* $r^2 - 2r \cos \theta = 8$.

7. From the point O a tangent is drawn to a given circle, with A as the point of tangency. Any secant is drawn from O, cutting the circle in B and C. Prove that $\overline{OA}^2 = \overline{OB} \cdot \overline{OC}$. HINT: Choose O as pole, OA as polar axis. Call the fixed point A $(a,0°)$, and the moving points B and C, respectively, (r_1,θ) and (r_2,θ). Prove $r_1r_2 = a^2$.

24. Systems of Circles. Let the quantities h, k, and r depend upon a quantity a in such a way that they all become fixed when a is assigned. Then, for each value of a there is determined a unique circle whose center is (h,k) and whose radius is r. The totality of circles obtainable in this way is called a *system* of circles. An equation containing a which becomes the equation of a circle whenever a is assigned a value, is called the equation of the system of circles thus determined.

Exercise 1. Describe the system of circles defined by the equation $(x - a)^2 + y^2 = a^2$. What are the coordinates of the center of any individual circle? To what line are all these circles tangent?

Exercise 2. If the circles of a system are all tangent to the coordinate axes and have their centers in the first and third quadrants prove that the abscissa and ordinate of the center of any one circle are equal. What is the radius? What is the equation of the system?

Exercise 3. If the two fixed circles C_1 and C_2 are

$$x^2 + y^2 + D_1x + E_1y + F_1 = 0$$

and

$$x^2 + y^2 + D_2x + E_2y + F_2 = 0,$$

prove that the equation

$$x^2 + y^2 + D_1x + E_1y + F_1 + k(x^2 + y^2 + D_2x + E_2y + F_2) = 0 \quad (39)$$

(*a*) represents a straight line if $k = -1$ and C_1 and C_2 are not concentric;
(*b*) is reducible to the form $x^2 + y^2 + Dx + Ey + F = 0$ if $k \neq -1$;
(*c*) represents a circle, a point, or has no graph if $k \neq -1$.

The straight line of Exercise 3(*a*) is called the *radical axis* of the two given circles C_1 and C_2. It can be shown to be the radical axis of any two distinct circles of the system (39) and might very well be spoken of as the radical axis of the system. The appearance of the system of circles and their radical axis is

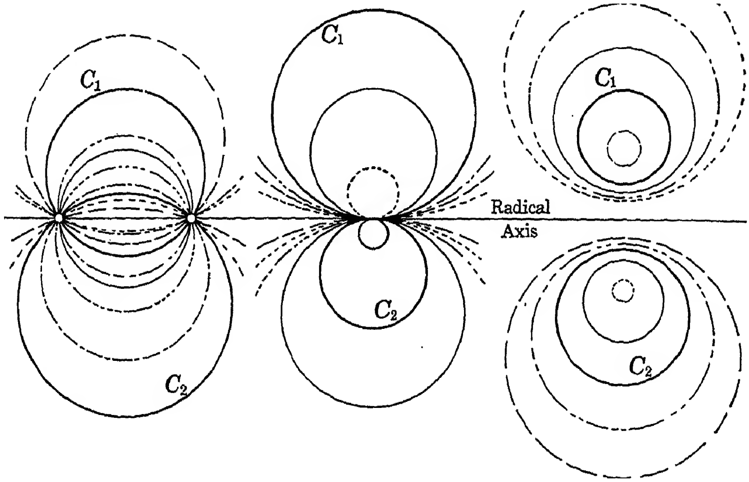


FIG. 43.

shown in Fig. 43 for the three cases in which the two given circles are intersecting, tangent, or have no point in common.

Problems

1. Write the equations of the system of circles described and draw a figure in each case.

(*a*) Center on the x-axis and radius equal to 1.

(*b*) Center on the curve $y = x^2$, radius equal to half the ordinate of the center. *Ans.* $4x^2 + 4y^2 - 8ax - 8a^2y + 4a^2 + 3a^4 = 0$.

(*c*) Tangent to the two lines $6x - 7y + 15 = 0$, $9x + 2y - 6 = 0$ and lying in the acute angle between them.

$$\textit{Ans.}\quad (x - a)^2 + \left(y - \frac{15a + 9}{5}\right)^2 = \frac{(75a - 12)^2}{2125}.$$

2. Find the center and radius of the general circle of the system, describe the system and draw a few of the circles for each of the following cases:

(*a*) $(x - 4)^2 + (y - 2k)^2 = k^2$, where k is a parameter.

(*b*) $(x - 2a)^2 + (y - a^2)^2 = a^2$, where a is a parameter.

(*c*) $(x + a)^2 + (y - 3a)^2 = a^2/9$, where a is a parameter.

3. Prove that the center of any circle of the system (39) lies on the line of centers of the circles C_1 and C_2.

4. If the circles C_1 and C_2 of Exercise 3 of the text are concentric, what can you say about the center of the circles of (39)?

5. Prove that the radical axis of a pair of circles is perpendicular to their line of centers.

6. If the circles C_1 and C_2 of Exercise 3 of the text intersect in two distinct points, prove that every curve of the system (39) is real and passes through those points.

7. Prove that the same system of circles is obtained as (39) if we use any two distinct circles in it in place of C_1 and C_2.

8. If C_1 and C_2 of Exercise 3 of the text are tangent to each other, prove that all circles of (39) are tangent, at a point, to the radical axis of C_1 and C_2.

9. If C_1 and C_2 of Exercise 3 of the text have no point in common prove that no two distinct circles of the system (39) can have points in common. HINT: This is most easily proved by using Prob. 7.

10. Prove that the radical axis of any two circles of the system (39) is the same as the radical axis of C_1 and C_2.

11. Write the equation of the systems of circles described in each case and draw a figure.

(a) Passing through the points of intersection of the circles

$$+ y^2 - 4x = 0 \qquad \text{and} \qquad x^2 + y^2 + 6y = 0.$$

Ans. $(1 + k)x^2 + (1 + k)y^2 - 4x + 6ky = 0,\ k \neq -1.$

(b) Tangent to both the circles $x^2 + y^2 + 4x - 6y - 12 = 0$ and $x^2 + y^2 + 10x - 14y - 26 = 0$ at their point of tangency.

(c) Having, in pairs, the same radical axis as the circles $x^2 + y^2 + 4x - 6y + 4 = 0$ and $x^2 + y^2 - 8x + 6y + 16 = 0$. HINT: See Prob. 10.

Ans. $x^2(1 + k) + y^2(1 + k) + x(4 - 8k) + y(-6 + 6k) + (4 + 16k) = 0,\ k \neq -1.$

(d) Concentric with the circle $x^2 + y^2 + 10x - 4y + 35 = 0$.

(e) Such that any circle of the system and the circle

$$x^2 + y^2 + 2x - 7y + 11 = 0$$

have the radical axis $4x - 3y + 12 = 0$.

12. Find the center and radius of the general circle of the system, describe the system and draw a few circles for each case.

(a) $x^2 + y^2 - 8x + 4y + 16 + k(x^2 + y^2 + 6x + 2y + 1) = 0$, where k is a parameter.

(b) $x^2 + y^2 + 18x - 24y + 17 + k(x^2 + y^2 - 18x + 30y - 19) = 0$, where k is a parameter.

13. Write the equation of the system of circles passing through the points of intersection of the circles

$$+ y^2 + 2x - 3 = 0 \qquad \text{and} \qquad + y^2 - 2x - 3 = 0.$$

Find the radical axis of the system and that circle of the system which passes through the point (4,2). *Ans.* Circle, $4x^2 + 4y^2 - 17x - 12 = 0.$

14. Write the equation of the system of circles which pass through the points of intersection of the circles

$$x^2 + y^2 - 4x + 2y - 6 = 0 \quad \text{and} \quad x^2 + y^2 + 2x + 2y - 7 = 0.$$

Find the radical axis of the system and the circle of the system which passes through the point $(3, -3)$.

15. Write the equation of the system of circles passing through the points of intersection of the circles

$$x^2 + y^2 - 2x + y + 1 = 0 \quad \text{and} \quad x^2 + y^2 - 3x + 2y + 3 = 0.$$

Find the circle of the system which

(a) passes through the point $(2, -\frac{1}{2})$;

(b) has its center on the x-axis; *Ans.* $x^2 + y^2 - x - 1 = 0$.

(c) has its center on the line $x - y = 0$;
Ans. $2x^2 + 2y^2 - x - y - 4 = 0$.

(d) has its radius equal to $\frac{1}{2}\sqrt{5}$;

(e) is tangent to the x-axis.

16. Prove that the three radical axes of three circles taken two at a time, meet in a common point or are parallel. The point of intersection of the three radical axes of three circles is called the *radical center* of the three circles.

17. Find the radical center of the three given circles.

(a) $x^2 + y^2 - 2x + 4y = 3$, $x^2 + y^2 - 4x - 3y + 1 = 0$, $x^2 + y^2 + x + 2y - 2 = 0$. *Ans.* $(\frac{1}{25}, \frac{14}{25})$.

(b) $x^2 + y^2 + x - 5y - 4 = 0$, $x^2 + y^2 - 3x + y + 2 = 0$, $4x^2 + 4y^2 - 4x - 4y + 1 = 0$.

25. Conic Sections. Let a point P move in such a way that its distance from a fixed point F divided by its distance from a fixed line d, not passing through F, is a constant e, where the distances are considered as positive. That is, in Fig. 44, the point P moves in such a way that

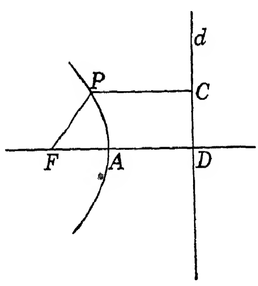


FIG. 44.

$$\frac{FP}{PC} = e.$$

The locus of P is called a *conic section* or, more simply, a *conic*. The fixed point F is called a *focus*, and the line d is called a *directrix*. The constant e is called the *eccentricity*.

From the definition it follows that a conic is symmetrical with respect to a line through the focus perpendicular to the directrix. This axis of symmetry is called the *transverse axis* of the conic. Let the point of intersection of the transverse axis and the

directrix (Fig. 44) be labeled D. Between points F and D there is one point A such that

$$\frac{FA}{AD} = e,$$

viz., the point which divides the line segment FD (internally) in the ratio $e:1$. If we call a point at which a conic crosses its transverse axis a *vertex*, then a conic has one vertex A between its focus and its directrix. Now if $e \neq 1$ we can also find a vertex A' such that

$$\frac{FA'}{DA'} = e \tag{40}$$

viz., the point which divides FD (externally) in the ratio $-e:1$. If $e = 1$ the vertex A is the midpoint of FD but there is no second vertex A'. Why?

We conclude that:

If $e = 1$, the conic has one vertex midway between its focus and directrix.

If $e \neq 1$ the conic has two vertices, one of which is between the focus and the directrix.

Looking again at (40), we see that if $e < 1$; then

$$FA' < DA'$$

or A' is nearer to F than to D. On the other hand, if $e > 1$, then

$$FA' > DA'$$

or, A' is nearer to D than to F.

We call the conic a *parabola* if $e = 1$, an *ellipse* if $e < 1$ and a *hyperbola* if $e > 1$. In this nomenclature, we may summarize the above conclusions as follows:

A parabola has only one vertex.

An ellipse has two vertices, both on the same side of its directrix.

A hyperbola has two vertices which lie on opposite sides of its directrix.

A point on the transverse axis of an ellipse or hyperbola and midway between the two vertices is called the *center* of the curve, and those curves are called *central conics*, in contrast with the parabola which has but one vertex and, hence, no center. The line perpendicular to the transverse axis of a central conic at its

center is called the *conjugate axis*. We shall show later that it is an axis of symmetry.

26. Equations of Central Conics. Let the coordinate axes be chosen so that the vertex A lying between the focus and directrix has the coordinates $(a,0)$, $(a > 0)$, while the other vertex A' has coordinates $(-a,0)$. The x-axis is then the transverse axis, the y-axis is the conjugate axis, and the origin is the center.

We have seen that the vertices of an ellipse lie on the same side of the directrix. The positions of the focus F $(c,0)$ and the directrix $x = d$ are those shown in Fig. 45,

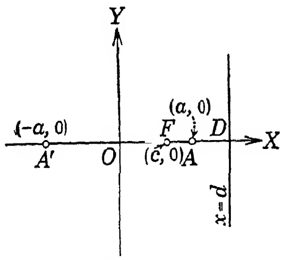


FIG. 45.

with c positive, as will be shown presently. Since A and A' are points of the ellipse we have

$$\frac{FA}{AD} = e, \qquad \frac{A'F}{A'D} = e.$$

These may be written

$$\frac{a - c}{d - a} = e, \qquad \frac{c + a}{a + d} = e.$$

We have seen that the vertices of a hyperbola lie on opposite sides of the directrix. The positions of the focus F $(c,0)$ and the directrix $x = d$ are those shown in Fig. 46,

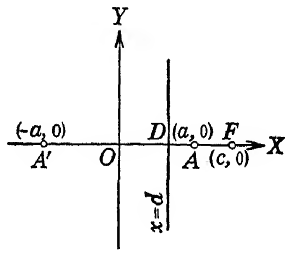


FIG. 46.

with d positive, as will be shown presently. Since A and A' are points of the hyperbola we have

$$\frac{AF}{DA} = e, \qquad \frac{A'F}{A'D} = e.$$

These may be written

$$\frac{c - a}{a - d} = e, \qquad \frac{c + a}{a + d} = e.$$

Clearing of fractions, we have, in both cases,

$$a + c = de + ae,$$
$$a - c = de - ae.$$

Adding and subtracting, we obtain

$$2a = 2de,$$
$$2c = 2ae,$$

and hence,

$$c = ae, \qquad d = \frac{a}{e}. \tag{41}$$

We have now established the signs of c and d assumed in Figs. 45 and 46.

The distance from P (x,y) to the focus may now be written in the form

$$FP = \sqrt{(x - ae)^2 + y^2},$$

while the distance from P to the directrix is given by

$$\pm\left(x - \frac{a}{e}\right),$$

where that sign is chosen which makes the expression positive. The equation of the conic is, then,

$$\frac{\sqrt{(x - ae)^2 + y^2}}{\pm\left(x - \frac{a}{e}\right)} = e.$$

Simplifying, we find that this can be written as

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

Since, for the ellipse, $e^2 < 1$, the quantity

$$a^2(1 - e^2)$$

is positive and, for simplicity, we may write

$$a^2(1 - e^2) = b^2. \tag{42}$$

In this notation the equation of the ellipse becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{44}$$

Note that, in view of Eqs. (41) and (42) we have $c^2 = a^2 - b^2$.

Since, for the hyperbola, $e^2 > 1$, the quantity

$$a^2(1 - e^2)$$

is negative and, for simplicity, we may write

$$a^2(1 - e^2) = -b^2. \tag{43}$$

In this notation the equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{45}$$

Note that, in view of Eqs. (41) and (43) we have $c^2 = a^2 + b^2$.

Exercise 1. Prove that a central conic is symmetric with respect to its conjugate axis.

Exercise 2. Prove, from symmetry considerations, that a central conic has a second combination of focus and directrix.

Exercise 3. Prove that $x^2 \leqq a^2$ and $y^2 \leqq b^2$ for the above ellipse.

Exercise 4. Prove that $x^2 \geqq a^2$ for the above hyperbola.

Exercise 5. Prove that all points of an ellipse lie between the two directrices.

Exercise 6. Prove that all points of a hyperbola lie outside the strip bounded by its two directrices.

Exercise 7. Show that the equation

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1,$$

which represents any central conic, becomes

$$r^2 = \frac{a^2(1 - e^2)}{1 - e^2 \cos^2 \theta} \tag{46}$$

when it is transformed to polar coordinates.

Exercise 8. Prove, from Eq. (46), that any line through the pole intersects an ellipse in two real points. Find the points farthest from the pole and those nearest. HINT: Use $e < 1$.

Exercise 9. Prove, from Eq. (46), with $e > 1$, that

(*a*) r is real and finite if $\cos^2 \theta > 1/e^2$.

(*b*) $r = \infty$ if $\cos^2 \theta = 1/e^2$.

(*c*) r is imaginary if $\cos^2 \theta < 1/e^2$.

(*d*) r^2 is least when $\cos^2 \theta = 1$.

From Exercise 8 we conclude that an ellipse must be a closed oval, something as shown in Fig. 47.

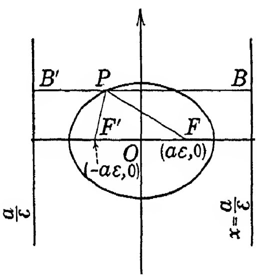


FIG. 47.

Since F' $(-ae,0)$ and the line $x = -a/e$ are a focus and directrix, we have

$$\frac{PF'}{B'P} = e.$$

Also

$$\frac{PF}{PB} = e.$$

From Exercise 9 we conclude that a hyperbola must be two branched, something as shown in Fig. 48.

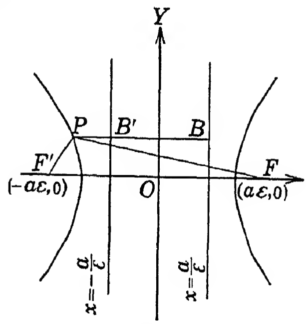


FIG. 48.

Since F' $(-ae,0)$ and the line $x = -a/e$ are a focus and directrix, we have

$$\frac{PF'}{PB'} \quad e.$$

Also

$$\frac{PF}{PB}$$

Clearing of fractions and adding, we have

$$\begin{aligned} PF + PF' &= e(B'P + PB) \\ &= e(B'B) \\ &= e\left(\frac{2a}{e}\right) \\ &= 2a. \end{aligned}$$

This result shows that for an ellipse the sum of the two *focal radii*, PF and PF', is a constant and equal to the distance between the vertices.

We shall call, in keeping with common usage, the distance a from the center to each vertex, the *transverse semiaxis* (or *semimajor axis*), and the distance b, laid off on the conjugate axis from the center, the *conjugate semiaxis* (or *semiminor axis*). Thus the transverse and conjugate axes are unlimited in extent, while segments of them of lengths $2a$ and $2b$, bisected at their point of intersection, are called *major* and *minor* axes, respectively.

Clearing of fractions and subtracting, we have

$$\begin{aligned} PF - PF' &= e(PB - PB') \\ &= e(B'B) \\ &= e\left(\frac{2a}{e}\right) \\ &= 2a. \end{aligned}$$

This result shows that for a hyperbola the difference of the two *focal radii* PF and PF' is a constant and equal to the distance between the vertices.

We shall call, in keeping with common usage, the distance a from the center to each vertex, the *transverse semiaxis* (or *semireal axis*), and the distance b, laid off on the conjugate axis from the center, the *conjugate semiaxis* (or *semiimaginary axis*). Thus the transverse and conjugate axes are unlimited in extent, while segments of them of lengths $2a$ and $2b$, bisected at their point of intersection, are called *real* and *imaginary* axes, respectively.

The properties of the ellipse and hyperbola studied thus far have been either identical or closely analogous. We come now to a property of a hyperbola which has no real equivalent in the ellipse.

Exercise 10. Prove in the light of (43) that if $\cos^2\theta = 1/e^2$, as in Exercise 9(b), then $\tan\theta = \pm b/a$, and the lines through the center with that inclination are $\frac{x}{a} \pm \frac{y}{b} = 0$.

Exercise 11. Prove that the point whose coordinates are

$$\left(x, -\frac{b}{a}\sqrt{x^2 - a^2}\right)$$

satisfies Eq. (45), of the hyperbola, if $x > a$.

Exercise 12. Prove that the distance δ from the line $\frac{x}{a} + \frac{y}{b} = 0$ to the point given in Exercise 11 can be written as

$$a^2b \qquad \frac{1}{+\sqrt{x^2 - a^2}}\Bigg).$$

Exercise 13. Show that the distance δ of Exercise 12 can be made as small as we please by taking x sufficiently large, *i.e.*, by making the point of the hyperbola recede sufficiently far from the origin.

These exercises, 10 to 13, prove that the line $\frac{x}{a} + \frac{y}{b} = 0$ is an asymptote to the hyperbola. From the symmetry of the figure, the line $\frac{x}{a} - \frac{y}{b} = 0$ is another asymptote. The two asymptotes can be simultaneously represented by the factorable equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

obtained from (45) by changing 1, in the right-hand member, to 0.

Summary of Formulas and Notation

Ellipse

Major axis $= 2a$
Minor axis $= 2b$
Distance from center to a vertex $= a$
Distance from center to a focus $= c$
Distance from center to a directrix $= d$
Eccentricity $= e < 1$.

$$a^2 - b^2 = c^2.$$

$$e = \frac{c}{a}.$$

$$d = \frac{a}{e}.$$

Sum of focal radii $2a$.

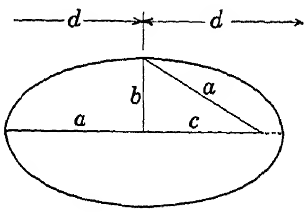


Fig. 49.

Hyperbola

Real axis $= 2a$
Imaginary axis $= 2b$
Distance from center to a vertex $= a$
Distance from center to a focus $= c$
Distance from center to a directrix $= d$
Eccentricity $= e > 1$.

$$a^2 + b^2 = c^2.$$

$$e = \frac{c}{a}.$$

$$d = \frac{a}{e}.$$

Difference of focal radii $= 2a$.

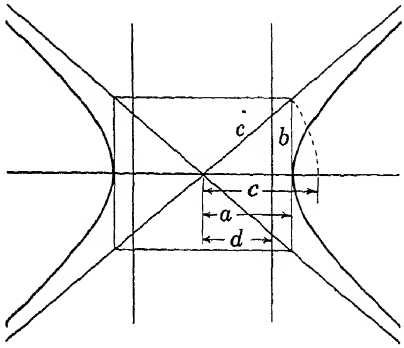


Fig. 50.

To find the equations of central conics with other choices of the coordinate system, the student has merely to translate and rotate the axes. Rotation through 90° and translations are included in Exercises 14 to 19 below.

Exercise 14. By translating the axes, show that

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

is the equation of an ellipse whose center is the point (h,k) and whose transverse axis is parallel to OX.

Exercise 15. By translating the axes, show that

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

is the equation of a hyperbola whose center is the point (h,k) and whose transverse axis is parallel to OX.

Exercise 16. By rotating the axes, show that

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1$$

is the equation of an ellipse whose center is the origin and whose transverse axis is OY.

Exercise 17. By rotating the axes, show that

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$$

is the equation of a hyperbola whose center is the origin and whose transverse axis is OY.

Exercise 18. By translating the axes, show that

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

is the equation of an ellipse whose center is the point (h,k) and whose transverse axis is parallel to OY.

Exercise 19. By translating the axes, show that

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

is the equation of a hyperbola whose center is the point (h,k) and whose transverse axis is parallel to OY.

Exercise 20. Prove that, if A, C, and F are constants such that the product AC is not zero, the equation

$$Ax^2 + Cy^2 + F = 0$$

represents a central conic, a circle, a point, a pair of intersecting straight lines, or has no locus. Hint: Establish separately the cases of the table

AC	AF	Remarks	Curve
$+$	$-$	$A = C$	Circle
$+$	$-$	$A \neq C$	Ellipse
$+$	0		Point
$+$	$+$		No locus
$-$	$\neq 0$		Hyperbola
$-$	0		2 lines

Problems

1. Write the equation of the central conic described in each case and draw a figure.

(*a*) Ellipse with major axis from $(-3,0)$ to $(3,0)$ and minor axis equal to 4. *Ans.* $4x^2 + 9y^2 = 36$.

(*b*) Ellipse with minor axis from $(-5,0)$ to $(5,0)$ and major axis equal to 14. *Ans.* $49x^2 + 25y^2 = 1225$.

(*c*) Hyperbola with real axis from $(-2,0)$ to $(2,0)$ and imaginary axis equal to 6. *Ans.* $9x^2 - 4y^2 = 36$.

(*d*) Hyperbola with real axis from $(0,-3)$ to $(0,3)$ and imaginary axis equal to 4. *Ans.* $4y^2 - 9x^2 = 36$.

(*e*) Ellipse with foci at $(5,0)$ and $(-5,0)$ and one directrix the line $x = -20$. *Ans.* $3x^2 + 4y^2 = 300$.

(*f*) Ellipse with vertices at $(0,5)$ and $(0,-5)$ and passing through the point $(2,-5\sqrt{5}/3)$. *Ans.* $25x^2 + 9y^2 = 225$.

(*g*) Ellipse with foci at $(3,0)$ and $(-3,0)$ and passing through the point $(-4,\sqrt{15})$.

(*h*) Hyperbola with center at the origin, each vertex midway between the center and the corresponding focus, and one directrix the line $x = 2$. *Ans.* $\dfrac{x^2}{16} - \dfrac{y^2}{48} = 1$.

(*i*) Hyperbola with one focus at $(26,0)$ and whose asymptotes are the lines $12y = \pm 5x$. *Ans.* $25x^2 - 144y^2 = 14{,}400$.

(*j*) Hyperbola whose vertices coincide with those of the ellipse having foci at $(0, \pm 4)$ and minor axis equal to 6, the foci of the hyperbola to lie on the directrices of the ellipse.

Ans. $9y^2 - 16x^2 = 225$.

(*k*) Ellipse with center at $(4,-2)$, a vertex at $(9,-2)$, and a focus at $(0,-2)$. *Ans.* $\dfrac{(x-4)^2}{25} + \dfrac{(y+2)^2}{9} = 1$.

(*l*) Ellipse with directrices $y = 3 \pm 169/12$, and a focus $(-2,15)$.

(*m*) Hyperbola with center at $(3,-5)$, a vertex at $(7,-5)$ and a focus at $(8,-5)$. *Ans.* $\dfrac{(x-3)^2}{16} - \dfrac{(y+5)^2}{9} = 1$.

(*n*) Hyperbola with center at $(-2,-1)$, a focus at $(-2,14)$ and a directrix in the line $5y = -53$. *Ans.* $-\dfrac{(x+2)^2}{81} + \dfrac{(y+1)^2}{144} = 1$.

(*o*) Ellipse with one focus at $(5,2)$, nearest vertex at $(7,2)$ and center on the line $x = 1$.

(*p*) Ellipse, with transverse axis parallel to OY, in which the focus is the point of trisection of the semimajor axis, nearest the vertex; the distance from the vertex to the nearest directrix is 4; the center lies on the line $y = 2$, and the line joining the origin and the center is of slope 2.

Ans. $9(x-1)^2 + 5(y-2)^2 = 320$.

(*q*) Hyperbola with foci at $(3,6)$ and $(3,0)$ and passing through the point $\left(5, 3 + \dfrac{6}{\sqrt{5}}\right)$. *Ans.* $5(y-3)^2 - 4(x-3)^2 = 20$.

(r) Hyperbola, with asymptotes meeting at $(-3,1)$, one of the asymptotes passing through $(1,7)$, and one focus at $(-3 - 3\sqrt{13}, 1)$.

Ans. $9(x+3)^2 - 4(y-1)^2 = 324$.

2. Prove that the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ has for asymptotes the lines $y = \pm ax/b$.

3. Find the coordinates of the center, vertices, and foci, the eccentricity, and the equation of the directrices, in each of the following cases. Draw a figure and, if $e > 1$, find the equations of the asymptotes.

(a) $\frac{x^2}{25} + \frac{y^2}{9} = 1$. (e) $\frac{(x-3)^2}{25} + \frac{(y+4)^2}{16} = 1$.

(b) $\frac{x^2}{25} + \frac{y^2}{144} = 1$. (f) $\frac{(x+1)^2}{25} + \frac{(y+2)^2}{144} = 1$.

(c) $\frac{x^2}{16} - \frac{y^2}{9} = 1$. (g) $\frac{(x+2)^2}{9} - \frac{(y+5)^2}{16} = 1$.

(d) $\frac{y^2}{25} - \frac{x^2}{25} = 1$. (h) $\frac{(x-8)^2}{49} - \frac{(y-8)^2}{49} = -1$.

(i) $6x^2 + 9y^2 - 24x - 54y + 51 = 0$. HINT: Write it as $6(x^2 - 4x) + 9(y^2 - 6y) = -51$ and complete the squares.

(j) $9x^2 + 4y^2 - 18x + 16y - 11 = 0$.

(k) $9x^2 - 18y^2 + 54x - 36y + 79 = 0$.

(l) $x^2 - y^2 + 6x + 10y - 4 = 0$.

(m) $3y^2 - 4x^2 - 8x - 24y - 40 = 0$.

(n) $2x^2 + 3y^2 - 3y - 12 = 0$.

(o) $4x^2 + 4y^2 + 20x - 32y + 89 = 0$.

(p) $2x^2 + 2y^2 - 2x + 18y + 33 = 0$.

(q) $3x^2 + 4y^2 - 30x + 16y + 100 = 0$.

(r) $4x^2 - y^2 + 56x + 2y + 191 = 0$.

4. If the real axis and the imaginary axis of a hyperbola are equal, the hyperbola is called *equilateral.* Prove that the eccentricity of an equilateral hyperbola is equal to $\sqrt{2}$.

5. Prove that the equations $\begin{cases} x = a\cos\theta \\ y = b\sin\theta \end{cases}$ are parametric equations for an ellipse with center at the origin and that $\begin{cases} x = h + a\cos\theta \\ y = k + b\sin\theta \end{cases}$ are parametric equations for an ellipse with center at (h,k).

6. Draw the ellipse $\begin{cases} x = 3 + 6\cos\theta \\ y = -2 + 8\sin\theta \end{cases}$. Locate its foci and directrices.

7. Write parametric equations for the ellipse $\frac{x^2}{81} + \frac{(y-7)^2}{49} = 1$.

8. Prove that the equations $\begin{cases} x = a\sec\theta \\ y = b\tan\theta \end{cases}$, as well as $\begin{cases} x = h + a\sec\theta \\ y = k + b\tan\theta \end{cases}$ represent hyperbolas.

9. Find parametric equations for the hyperbola $\frac{(x+2)^2}{25} - \frac{y^2}{36} = -1$.

10. Draw the hyperbola represented by $\begin{cases} x = 9\sec\theta \\ y - 2 = 8\tan\theta \end{cases}$. Locate its foci and directrices. State the equation of each asymptote.

11. Prove that if the semiaxes of one ellipse are proportional to those of another, they have the same eccentricity.

12. Prove that the real and imaginary axes of two hyperbolas are proportional if, and only if, they have the same eccentricity.

13. Prove that the *latus rectum*, or chord of a conic through a focus and perpendicular to the transverse axis, is equal to $2b^2/a$ for central conics.

14. Find the equation of the locus of a point which moves so that the product of its distances from the two lines $4x = 3y$ and $4x + 3y = 0$ is equal to 1. Plot the locus.

15. Prove that the product of the distances from a point (x,y) on the hyperbola $\frac{x^-}{a^2} - \frac{y^2}{b^2} = 1$ to the two asymptotes of the hyperbola is equal to a^2b^2/c^2.

16. If the real and imaginary axes of a hyperbola coincide, respectively, with the imaginary and real axes of another hyperbola, each hyperbola is said to be the *conjugate* of the other. Prove that the two eccentricities e_1 and e_2 of a pair of conjugate hyperbolas are related by the equation $e_1^2 + e_2^2 = e_1^2e_2^2$.

17. Prove that the locus of a point, for which the sum of its distances from two fixed points equals $2a$, is an ellipse having the two fixed points for its foci and $2a$ for the length of its major axis.

18. Prove that the locus of a point, for which the difference of its distances from two fixed points equals $2a$, is a hyperbola having the two fixed points for its foci and $2a$ for the length of its real axis.

19. Find the equation of the locus of a point which moves so that

(*a*) the sum of its distances from $(0,5)$ and $(0,-5)$ is 12;

(*b*) the sum of its distances from $(3,1)$ and $(-5,1)$ equals 10;

(*c*) the difference of its distances from $(-4,0)$ and $(-4,9)$ equals 4;

(*d*) the difference of its distances from $(5,1)$ and $(-3,1)$ equals 6.

20. Find the equation of the hyperbola conjugate to the hyperbola $4x^2 - 9y^2 - 16x - 18y = 29$. Find the eccentricity of each hyperbola and verify the relation between them stated in Prob. 16.

Ans. $9(y + 1)^2 - 4(x - 2)^2 = 36$.

21. An arch in the form of a semi-ellipse is 48 ft. wide at the base, and its height is 20 ft. How wide is the arch at the height of 10 ft. above the base?

Ans. $24\sqrt{3}$ ft.

22. Construct a circle with radius equal to a and center at the origin. On the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, produce the ordinate of any point P (x,y) until it meets that circle at Q (x,y'). Prove that $y/y' = b/a$. From that derive the parametric equations of the ellipse, using as parameter, the angle θ from the positive x-axis to the line OQ. (Check with Prob. 5.) The angle θ is called the *eccentric angle* of the point P which it determines.

23. (*a*) On the ellipse $\frac{x^2}{12} + \frac{y^2}{9} = 1$, locate the points with the following eccentric angles: 0°, 30°, 240°, 270°.

(*b*) Find the eccentric angle for each of the points $(2,3)$, $(-2\sqrt{3},\sqrt{3})$, and $(2\sqrt{2},-\sqrt{6})$ on the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$.

24. (*a*) Describe the family of conics represented by the equation

$$\frac{x^2}{k^2} + \frac{y^2}{k^2 - c^2} = 1,$$

where c is a fixed number and k is the parameter of the family.

(*b*) Find the equation of that ellipse in (*a*) for which the difference of the major and minor axes equals c. *Ans.* $144x^2 + 400y^2 = 225c^2$.

25. Chords of an ellipse are drawn through one end of the major axis. Prove that the locus of the midpoints of the chords is an ellipse.

26. Prove the following properties of the hyperbola:

(*a*) The distance from a focus to either asymptote equals the imaginary semiaxis, *i.e.*, half the imaginary axis.

(*b*) If the asymptotes are perpendicular, the hyperbola (*rectangular*) is *equilateral* (*i.e.*, $a = b$), and conversely.

(*c*) For any point on an equilateral hyperbola, the product of the two focal radii equals the square of the distance of that point from the center.

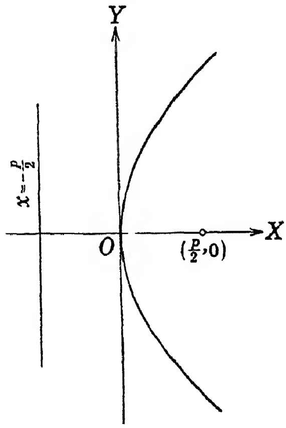


FIG. 51.

27. The Parabola. As we have already seen, a parabola cuts its transverse axis at only one point (its vertex) which is midway between the focus and the directrix. Let the coordinates of the focus be $(p/2,0)$, $(p > 0)$ and the directrix be the line $x = -p/2$. If P (x,y) is any point on the parabola, we have, since the eccentricity equals unity, the equation

$$\frac{\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2}}{x + \frac{p}{2}} = 1.$$

or

$$x^2 - px + \frac{p^2}{4} + y^2 = x^2 + px + \frac{p^2}{4}.$$

The simplified equation becomes

$$y^2 = 2px. \tag{47}$$

Exercise 1. Prove that the equation of the parabola whose vertex is at the origin and whose focus is at the point $(-p/2,0)$, $p > 0$, is

$$y^2 = -2px \tag{48}$$

Exercise 2. Prove that the equation of the parabola whose vertex is the origin and whose focus is the point $(0,p/2)$, $p > 0$, is

$$x^2 = 2py. \tag{49}$$

Exercise 3. Prove that the equation of the parabola whose vertex is the origin and whose focus is the point $(0, -p/2)$, $p > 0$, is

$$x^2 = -2py. \tag{50}$$

Exercise 4. By assuming Eqs. (47), (48), (49), and (50) in their respective cases and translating the axes, show that the equation of the parabola with vertex at (h,k), transverse axis parallel to one of the coordinate axes, and focus p units from the directrix is

(a) $(y - k)^2 = 2p(x - h)$ if the focus is to the right of the vertex.
(b) $(y - k)^2 = -2p(x - h)$ if the focus is to the left of the vertex.
(c) $(x - h)^2 = 2p(y - k)$ if the focus is upward from the vertex.
(d) $(x - h)^2 = -2p(y - k)$ if the focus is downward from the vertex.

Exercise 5. Prove that an equation of the form

$$x^2 + Dx + Ey + F = 0$$

represents a parabola if E is not zero and, if $E = 0$, it represents a pair of parallel straight lines, one straight line, or no locus, according as $D^2 - 4F$ is greater than, equal to, or less than, zero.

Exercise 6. Prove that an equation of the form

$$y^2 + Dx + Ey + F = 0$$

represents a parabola if D is not zero and, if $D = 0$, it represents two parallel straight lines, one straight line, or no locus, according as $E^2 - 4F$ is greater than, equal to, or less than, zero.

Problems

1. Find the equation of the parabola satisfying the stated condition in each case, and draw a figure:

(a) Vertex at (0,0) and focus at (5,0). *Ans.* $y^2 = 20x$.
(b) Vertex at (0,0), focus at $(0,-1)$. *Ans.* $x^2 = -4y$.
(c) Vertex at (0,0), directrix the line $y = -2$.
(d) Focus at $(-3,0)$, directrix the line $x = 3$. *Ans.* $y^2 = -12x$.
(e) Vertex at (0,0), focus on y-axis, parabola passing through $(-3,2)$. *Ans.* $2x^2 = 9y$.
(f) Vertex at origin, focus on the x-axis and on the line through $(-1,2)$ and $(-3,-4)$. Does the curve pass through either of the latter two points? *Ans.* $3y^2 = -20x$.
(g) Vertex at (2,1), focus at $(-4,1)$. *Ans.* $y^2 - 2y + 24x = 47$.
(h) Focus at $(-3,-2)$, directrix the line $y = 6$. *Ans.* $x^2 + 6x + 16y = 23$.

(*i*) Vertex at $(-1,5)$, directrix the line $x = 6$.

(*j*) Transverse axis the line $y = 3$, parabola passing through the points $(6,4)$ and $(4,-2)$. *Ans.* $y^2 - 6y + 12x = 64$.

(*k*) Focus at $(0,0)$, vertex on x-axis, parabola passing through $(-3,4)$. *Ans.* $y^2 + 4x = 4$, $y^2 - 16x = 64$.

(*l*) Focus at $(1,5)$, directrix the line $3x - 4y = 0$.

(*m*) Focus at $(-1,2)$, directrix the line $x + 2y = 6$.

2. Find the coordinates of the vertex and focus and the equation of the directrix of the parabola whose equation is given and draw a figure in each case.

(*a*) $y^2 = 12x$.
(*b*) $x^2 = 10y$.
(*c*) $x^2 = -6y$.
(*d*) $y^2 = -5x$.
(*e*) $(y - 2)^2 = -6x$.
(*f*) $(x + 1)^2 = -8(y - 5)$.
(*g*) $(y + 3)^2 = 20(x + 2)$.
(*h*) $x^2 = -10(y - 1)$.
(*i*) $(x + 3)^2 = 5y$.
(*j*) $x^2 + 2x - 3y - 5 = 0$. HINT: Write this as

$$(x + 1)^2 = 3y + 6 = 3(y + 2).$$

(*k*) $y^2 + 5x + 4y + 19 = 0$.
(*l*) $x^2 - 3x - 5y = 0$.
(*m*) $y^2 - 2x + 6y - 3 = 0$.

3. Prove that the equations $\begin{cases} x = 3t - 5 \\ y = t^2 + 2t + 8 \end{cases}$ represent a parabola. Locate its vertex, focus, and directrix. *Ans.* Vertex $(-8, 7)$.

4. Prove that in any two parabolas corresponding distances are proportional (*i.e.*, such distances as focus to vertex, focus to directrix, etc.).

5. Prove that any parabola with its transverse axis parallel to the y-axis can be represented by the equation

$$y = Ax^2 + Bx + C \tag{51}$$

where A, B, and C are constants with $A \neq 0$.

6. Prove that any parabola with its transverse axis parallel to the x-axis can be represented by the equation

$$x = Ay^2 + By + C \tag{52}$$

where A, B, and C are constants with $A \neq 0$.

7. Prove that any equation of the form of either (51) or (52) represents a parabola if $A \neq 0$.

8. By the *latus rectum* of a parabola is meant the chord perpendicular to the transverse axis at the focus. Prove that the latus rectum of a parabola is equal to twice the distance from the focus to the directrix.

9. The ends of the latus rectum of a parabola are the points $(3,7)$ and $(3,1)$. Find its equation. (See Prob. 8.)

Ans. $y^2 + 6x - 8y - 11 = 0$; $y^2 - 6x - 8y + 25 = 0$.

10. Find the equation of the locus of the center of a moving circle which always touches the circle $x^2 + y^2 - 10x + 4y + 20 = 0$ and the line $x = 10$. Plot the locus. *Ans.* $y^2 + 16x + 4y - 140 = 0$.

11. A parabolic arch is 36 ft. wide at the base and 20 ft. high. At what height above the base is it 18 ft. wide? *Ans.* 15 ft.

12. The two points $(-4,5)$ and $(-4,-3)$ are on a parabola and are each at a distance of 5 units from its focus. Find the equation of the parabola.

Partial Ans. $y^2 - 4x - 2y = 31$; $y^2 - 2y + 4x + 1 = 0$.

13. The vertex of an ellipse is at the focus of a parabola; the corresponding focus of the ellipse is at the vertex of the parabola and the directrix of the parabola passes through the center of the ellipse. Find the eccentricity of the ellipse. *Ans.* $\frac{1}{2}$.

14. Write the parametric equations of the parabola $y^2 = 2px$, taking as the parameter the slope of the line through a variable point of the parabola and the origin. *Ans.* $x = 2p/m^2$, $y = 2p/m$.

15. Write the equation of the parabola with focus at origin and directrix the line $x = -p$. Change the equation to polar coordinates.

16. If through a fixed point on the axis of the parabola $y^2 = 2px$ chords are drawn, show that the product of the abscissas of the end points of such chords is constant. HINT: Recall the formula for the product of the roots of a quadratic equation.

17. The directrix of a parabola is the line $y = 4$, and its axis is the line $x = 4$. Find the equation, given that it passes through $(7,9)$.

Ans. $x^2 - 8x - 18y + 169 = 0$,
$x^2 - 8x - 2y + 25 = 0$.

28. Conics in Polar Coordinates. Let the pole be taken as a focus of a conic and let the corresponding directrix of a conic be a line whose normal intercept is $p \neq 0$, and whose normal angle is ω. If the polar coordinates of a point P on the conic are (r,θ), the distance from P to the focus is

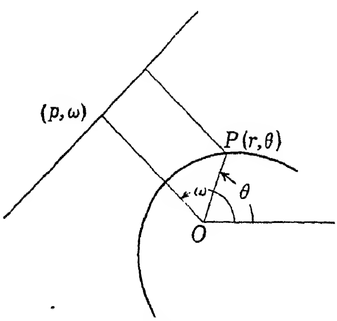


FIG. 52.

$$r$$

and the distance from P to the directrix is

$$-r \cos (\omega - \theta) + p.$$

The equation of the conic is, therefore,

$$\frac{r}{-r \cos (\omega - \theta) + p} = e.$$

This can be solved for r to obtain

$$r = \frac{ep}{1 + e \cos (\omega - \theta)} \tag{53}$$

Exercise 1. Show that in the special cases $\omega = 0°$, $\omega = 90°$, $\omega = 180°$, and $\omega = 270°$, the following special equations and special figures result:

(a) $\omega = 0°$

$$r = \frac{ep}{1 + e \cos \theta}$$

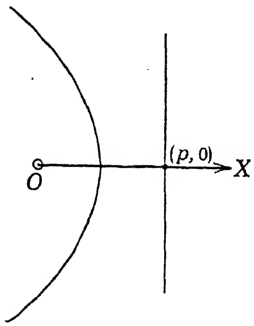


FIG. 53.

(c) $\omega = 180°$

$$r = \frac{ep}{1 - e \cos \theta}$$

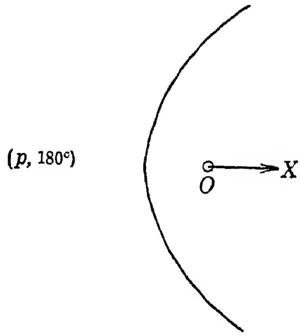


FIG. 55.

(b) $\omega = 90°$

$$r = \frac{ep}{1 + e \sin \theta}$$

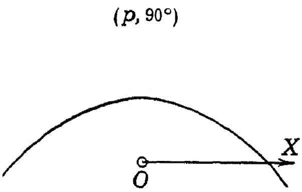


FIG. 54.

(d) $\omega = 270°$

$$\frac{ep}{1 - e \sin \theta}$$

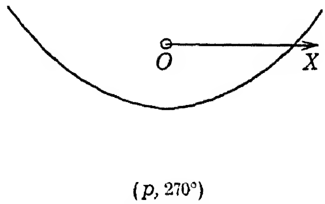


FIG. 56.

Exercise 2. If $e = 1$, show that the r of Eq. (53) is never negative. Find the value of θ for which $r = \infty$.

Exercise 3. If $e < 1$, show that the r of Eq. (53) is never negative. Show also that r is never infinite. Find the two vertices.

Exercise 4. If $e > 1$, show that the r of Eq. (53) can be either positive or negative. Find the values of θ for which $r = \infty$. Find the coordinates of the two vertices.

Problems

1. In each case, below, find the eccentricity e, the polar coordinates (p, ω) of the point of intersection of the transverse axis with the directrix, the coordinates of the vertex or vertices, and plot.

(a) $r = \dfrac{20}{2 + 5 \cos \theta}$.

(b) $r = \dfrac{12}{3 - 2 \sin \theta}$.

(c) $r = \dfrac{8}{1 + \cos \left(\theta - \dfrac{\pi}{4} \right)}$.

(d) $r = \dfrac{5}{3 + 4 \cos (\theta - 100°)}$.

(e) $r = \dfrac{60}{4 + 3 \cos \theta - 4 \sin \theta}$.

Ans. $e = 5/4$, $p = 12$, $\cos \omega = 3/5$, $\sin \omega = -4/5$

(f) $r = \dfrac{24}{2 - \cos \theta}$.

(g) $r = \dfrac{39}{13 - 5 \cos \theta - 12 \sin \theta}$.

2. Write the equation of the conic having a focus at the pole and satisfying the condition stated in each case.

(a) $p = 5$, $\omega = 0°$, $e = 2$.
(b) $p = 8$, $\omega = 270°$, $e = 3/4$.
(c) $p = 6$, $\omega = 180°$, $e = 5/3$.
(d) $p = 12$, $\omega = 80°$, $e = 2/3$.
(e) $p = 10$, $\omega = 225°$, $e = 3/2$.
(f) $p = 18$, $\omega = -\pi/6$, $e = 1$.

Ans. $r = \dfrac{36}{2 - \sin \theta + \sqrt{3} \cos \theta}$.

(g) Directrix is the line $r \cos (\theta - \pi) - 5 = 0$ and $e = 1$.
(h) Directrix is the line $r = 3 \csc \theta$ and $e = 1/2$.

3. Derive the polar equation of the parabola whose vertex is at the pole and whose directrix is a line with the normal intercept equal to $p/2 \neq 0$ and normal angle 180°.

4. From the definition of a conic, derive the polar equation of a central conic with center at the pole, transverse axis on the initial line and distance from the pole to directrix equal to a/e. Compare with Exercise 7, page 76.

5. Prove that the focus of a conic divides chords through it into two parts such that the sum of their reciprocals is a constant.

6. Prove that if two chords through the focus of a parabola are perpendicular, the sum of the reciprocals of their lengths is a constant.

7. If PFP' and QFQ' are two perpendicular chords through the focus F of a conic, show that $\dfrac{1}{FP \cdot FP'} + \dfrac{1}{FQ \cdot FQ'}$ is a constant.

29. The General Second-degree Equation. We have seen, in the foregoing sections, that the rectangular equation of a conic is always of the second degree. It is natural, now, to inquire what may be expected as the locus defined by a second-degree equation, *i.e.*, an equation of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{54}$$

in which A, B, and C are not all zero. In particular, if $B = 0$ and $A = C$, we have seen that this equation represents a circle, a point, or no locus, the equation, in the last case, being satisfied by no pair of real numbers (x,y). Also, in particular, if $B = 0$

and $A \neq C$, we have seen that the locus is sometimes a conic and that, in that case, the transverse axis is parallel to one of the coordinate axes.

To see what sort of equation represents a conic in case the axis is not parallel to either of the coordinate axes, let the directrix be the line

$$x \cos \omega + y \sin \omega - p = 0.$$

If the focus is the fixed point (x_1,y_1) and the eccentricity is e, the equation of the conic is

$$\frac{\sqrt{(x - x_1)^2 + (y - y_1)^2}}{|x \cos \omega + y \sin \omega - p|} = e.$$

Squaring and clearing of fractions, we obtain

$$(x - x_1)^2 + (y - y_1)^2 = e^2(x \cos \omega + y \sin \omega - p)^2.$$

This, in turn, may be written as

$$\begin{aligned}(1 - e^2 \cos^2 \omega)x^2 - 2e^2 \sin \omega \cos \omega xy + (1 - e^2 \sin^2 \omega)y^2 \\ - 2(x_1 - pe^2 \cos \omega)x - 2(y_1 - pe^2 \sin \omega)y \\ + x_1^2 + y_1^2 - e^2p^2 = 0.\end{aligned}$$

If this equation is multiplied through by a constant k which is not zero, it gives

$$\mathcal{A}x^2 + \mathcal{B}xy + \mathcal{C}y^2 + \mathcal{D}x + \mathcal{E}y + \mathcal{F} = 0 \tag{55}$$

where
$$\begin{aligned}\mathcal{A} &= k(1 - e^2 \cos^2 \omega),\\ \mathcal{B} &= -2ke^2 \sin \omega \cos \omega,\\ \mathcal{C} &= k(1 - e^2 \sin^2 \omega),\\ \mathcal{D} &= -2k(x_1 - pe^2 \cos \omega),\\ \mathcal{E} &= -2k(y_1 - pe^2 \sin \omega),\\ \mathcal{F} &= k(x_1^2 + y_1^2 - e^2p^2).\end{aligned}$$

Evidently, $\mathcal{B} = 0$ if, and only if, $\sin \omega \cos \omega = 0$, that is, if, and only if,

$$\omega = 0° + n \cdot 90°$$

where n is an integer. Thus we conclude that if an equation of the form (55) represents a conic, $\mathcal{B} = 0$ or $\mathcal{B} \neq 0$ according as the transverse axis is parallel to one of the coordinate axes or is oblique to both of them. Therefore we may expect to remove the xy term by rotating the axes through the angle ω. To determine

ω from Eq. (55), note that

$$\begin{aligned}\frac{\mathcal{B}}{\mathcal{A} - \mathcal{C}} &= \frac{-2ke^2 \sin \omega \cos \omega}{k(1 - e^2 \cos^2 \omega) - k(1 - e^2 \sin^2 \omega)} \\ &= \frac{2 \sin \omega \cos \omega}{\cos^2 \omega - \sin^2 \omega} \\ &= \frac{\sin 2\omega}{\cos 2\omega} \\ &= \tan 2\omega.\end{aligned}$$

Let us try this method on Eq. (54), which is of the same form as (55) but not known to represent a conic. That is, let an angle ω be determined by the equation

$$\tan 2\omega = \frac{B}{A - C} \tag{56}$$

and rotate the axes through the angle ω by substituting

$$\begin{aligned} x &= x' \cos \omega - y' \sin \omega \\ y &= x' \sin \omega + y' \cos \omega. \end{aligned} \tag{8}$$

We have, from (54),

$$\begin{aligned} &A[x'^2 \cos^2 \omega - 2x'y' \sin \omega \cos \omega + y'^2 \sin^2 \omega] \\ &\quad + B[x'^2 \sin \omega \cos \omega + x'y'(\cos^2 \omega - \sin^2 \omega) - y'^2 \sin \omega \cos \omega] \\ &\qquad + C[x'^2 \sin^2 \omega + 2x'y' \sin \omega \cos \omega + y'^2 \cos^2 \omega] \\ &\quad + D[x' \cos \omega - y' \sin \omega] + E[x' \sin \omega + y' \cos \omega] + F = 0. \end{aligned}$$

This can be collected into the form

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0 \tag{57}$$

where

$$\begin{aligned} A' &= A \cos^2 \omega + B \sin \omega \cos \omega + C \sin^2 \omega, \\ B' &= -2A \sin \omega \cos \omega + B(\cos^2 \omega - \sin^2 \omega) \\ &\qquad + 2C \sin \omega \cos \omega., \\ C' &= A \sin^2 \omega - B \sin \omega \cos \omega + C \cos^2 \omega. \end{aligned} \tag{58}$$

Exercise 1. Prove: $B'^2 - 4A'C' = B^2 - 4AC$.

Since we are rotating the axes in order to remove the xy term, we have tacitly assumed that $B \neq 0$. Hence we may express

B'/B as

$$\begin{aligned}\frac{B'}{B} &= \frac{-2(A - C)}{B}\sin\omega\cos\omega + (\cos^2\omega - \sin^2\omega)\\ &= -\left(\frac{A - C}{B}\right)\sin 2\omega + \cos 2\omega\\ &= -\operatorname{ctn} 2\omega \sin 2\omega + \cos 2\omega\\ &= -\cos 2\omega + \cos 2\omega\\ &= 0.\end{aligned}$$

The result may be summarized in the form of a theorem:

If $B \neq 0$ the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

may be transformed into the equation

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

by rotating the axes through an angle ω defined by $\tan 2\omega = \dfrac{B}{A - C}$. *In the resulting equation not both of A' and C' are zero.*

It remains to be proved that A' and C' cannot both vanish. To establish this, we recognize that Eq. (57) is obtained from (54) by rotating the axes through the angle ω and that, in turn, (54) can be recovered from (57) by rotating the axes back through the angle $-\omega$. But if A' and C' were both zero the transformation

$$\begin{aligned}x' &= x\cos\omega + y\sin\omega,\\ y' &= -x\sin\omega + y\cos\omega\end{aligned}$$

would give the result

$$D'(x\cos\omega + y\sin\omega) + E'(-x\sin\omega + y\cos\omega) + F' = 0,$$

an equation different from

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \qquad (54)$$

in which at least B is different from zero.

Treatment of the equation

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \qquad (59)$$

must vary according as one of A' and C' is zero or neither is zero. In the latter case we may collect the terms in x' and those

in y' and complete their squares, obtaining

$$A'\left(x'^2+\frac{D'}{A'}x'+\frac{D'^2}{4A'^2}\right)+C'\left(y'^2+\frac{E'}{C'}y'+\frac{E'^2}{4C'^2}\right)=\frac{D'^2}{4A'}+\frac{E'^2}{4C'}-F',$$

an equation of the form

$$A'(x'-h)^2+C'(y'-k)^2+F''=0$$

in which

$$h=-\frac{D'}{2A'},\qquad k=-\frac{E'}{2C'},\qquad F''=F'-\frac{D'^2}{4A'}-\frac{E'^2}{4C'}.$$

By translating the axes to a new origin whose coordinates in the $x'y'$ system are (h,k), we have

$$x''=x'-h$$
$$y''=y'-k,$$

which give

$$A'x''^2+C'y''^2+F''=0.$$

The student's work on Exercise 20, page 79, has shown that this equation represents a circle, an ellipse, a point, a hyperbola, a pair of intersecting straight lines, or no real locus. Recall that we are still working with the case in which neither A' nor C' is zero. Note, also that, by the equality in Exercise 1 of this section and the fact that $B'=0$, the case $A'C'>0$ obtains if B^2-4AC is negative, while the case $A'C'<0$ follows when B^2-4AC is positive. If, now, we consider a circle as the limiting case of an ellipse as its foci approach its center, the point the limiting case of a circle or ellipse with shrinking dimensions, and a pair of intersecting lines as the limiting case of a hyperbola as the vertices approach the center, we may state that:

If $B^2-4AC<0$, *Eq.* (54) *represents an ellipse or no real locus.*
If $B^2-4AC>0$, *Eq.* (54) *represents a hyperbola.*

If A' in Eq. (59) is zero, then $C'\neq 0$, and the equation can be written as

$$C'y'^2+D'x'+E'y'+F'=0,$$

an equation which has been shown to represent a parabola or two parallel or coincident straight lines if it has a real locus. (See Exercise 6, page 84.)

Similarly, if, in Eq. (59) $C' = 0$, then $A' \neq 0$ and the locus, if real, is a parabola or two straight lines, parallel or coincident.

If we consider two parallel lines, distinct or coincident, as the limiting case of a parabola, and recall that the two cases just discussed arise, in view of Exercise 1 and $B' = 0$, from the original case $B^2 - 4AC = 0$, we may state that:

If $B^2 - 4AC = 0$, Eq. (54) represents a parabola or no real locus.

In any case, therefore, we conclude that if the equation of second degree has a real locus, it is a conic or one of the limiting forms of a conic included in the list: circle, point, two intersecting straight lines, two parallel straight lines, one straight line.

It is customary to speak of the cases of one or two straight lines as *degenerate conics.*

Problems

1. Classify each of the following as ellipse, hyperbola, or parabola, given that they are all nondegenerate conics.

(*a*) $36x^2 + 24xy + 29y^2 + 24x + 238y - 55 = 0.$
(*b*) $12x^2 + 7xy - 12y^2 + 3x + 4y + 24 = 0.$
(*c*) $4x^2 + 4xy + y^2 - 7x + 5y - 4 = 0.$
(*d*) $xy - y^2 - x - 2 = 0.$
(*e*) $6x^2 - 7xy - 3y^2 + 2x - 7y + 3 = 0.$
(*f*) $3x^2 - 4xy + 2y^2 - 13x + 5y + 20 = 0.$
(*g*) $x^2 - 2xy + y^2 + 18x - 10y + 13 = 0.$
(*h*) $5x^2 - 14xy - 20x + 2y^2 + 8y + 14 = 0.$

2. Find an angle through which the axes may be rotated in order to remove the xy term from the equation

$$5x^2 - 4xy + 2y^2 - 10x + 8y - 11 = 0.$$

KEY: From Eq. (56) we have

$$\tan 2\omega = \frac{B}{A - C} = -\frac{4}{3},$$

or

$$\cos 2\omega = -\frac{3}{5} \text{ (taking } 2\omega \text{ in the second quadrant)},$$

whence

$$\sin \omega = \sqrt{\frac{1 - \cos 2\omega}{2}} = \sqrt{\frac{1 + \frac{3}{5}}{2}} = \frac{2}{\sqrt{5}},$$

$$\cos \omega = \sqrt{\frac{1 + \cos 2\omega}{2}} = \sqrt{\frac{1 - \frac{3}{5}}{2}} = \frac{1}{\sqrt{5}}.$$

3. Find an angle through which the axes may be rotated in order to remove the xy term from each equation in Prob. 1 above.

4. For each case, below, remove the xy term by rotation of axes and plot the locus, showing both sets of axes. Insofar as they exist, find the eccentricity, vertices, foci, center, asymptotes, and directrices, all coordinates and equations referring to the original axes.

(a) $x^2 + xy + y^2 - 2x - y - 7 = 0$.

Ans. $a^2 = 16$, $b^2 = 16/3$, $e = \sqrt{2/3}$.

(b) $41x^2 - 24xy + 34y^2 - 82x + 24y + 16 = 0$.

(c) $6x^2 - 4xy + 9y^2 - 12x + 4y + 6 = 0$.

(d) $34x^2 + 24xy + 41y^2 + 48x + 164y + 189 = 0$.

(e) $12xy - 5y^2 + 3x + 2y = 0$.

Ans. $a = \sqrt{13}/8$, $b = \sqrt{13}/12$, $e = \sqrt{13}/3$.

(f) $x^2 - 12xy - 4y^2 - 3x - 2y = 0$.

(g) $6x^2 + 7xy - 18y^2 + 7x - y + 144 = 0$.

(h) $x^2 - 2xy + y^2 + x - 1 = 0$. *Ans.* $e = 1$, $2p = \sqrt{2}/4$.

(i) $x^2 + 4xy + 4y^2 - 6x - 8y - 3 = 0$.

(j) $x^2 + 6xy + 9y^2 - 8x - 24y + 15 = 0$.

(k) $x^2 - 10xy + 25y^2 - 20x + 100y + 100 = 0$.

(l) $4x^2 - 4xy + y^2 + 2x - y + 1 = 0$.

5. Prove that the equation $xy = a$ represents an equilateral hyperbola whose asymptotes are the coordinate axes.

6. Determine k so that the equation $3xy + 2x - 9y + k = 0$ may represent two straight lines, and plot. *Ans.* $k = -6$.

7. Determine k so that the equation $9x^2 - 6xy + y^2 - 18x + ky = 0$ may represent two straight lines, and plot. *Ans.* $k = 6$.

8. Determine the value of B in order that the equation

$$9x^2 + Bxy + 4y^2 - 5x + 8y - 1 = 0$$

may represent a parabola.

9. Determine the range of B in order that the equation

$$5x^2 + Bxy + 3y^2 - 4x + y - 3 = 0$$

may represent (a) an ellipse, (b) a hyperbola.

10. Find the equation of the parabola whose directrix is the line

$$3x - 4y - 12 = 0$$

and whose focus is the point $(6, -1)$.

Ans. $16x^2 + 24xy + 9y^2 - 228x - 46y + 781 = 0$.

11. Find the equation of the ellipse whose eccentricity is $\frac{1}{2}$, whose focus is the origin, and whose corresponding directrix is the line

$$5x - 12y + 39 = 0.$$

12. Prove that the equation $(x^2 + y^2 - 25) + k(xy - 12) = 0$ represents a system of conics, each passing through the points (3,4), (4,3), $(-3, -4)$, and $(-4, -3)$, the four points of intersection of the circle $x^2 + y^2 - 25 = 0$ and the hyperbola $xy - 12 = 0$. Find those values of k for which the conic is (a) an ellipse. (b) a parabola. (c) a hyperbola.

13. Write the equation of the system of conics passing through the points of intersection of the parabola $y^2 - x = 0$ and the ellipse

$$16x^2 + 9y^2 - 128x + 112 = 0.$$

Find the conic in the system which passes through the point $(-1,1)$.

Ans. $32x^2 - 247y^2 + 9x + 224 = 0.$

14. Find the equation of the conic passing through the five points $P\ (0,-2)$, $Q\ (1,-2)$, $R\ (0,0)$, $S\ (-1,0)$, and $T\ (3,-1)$. KEY: first method: The two lines PQ and RS form a degenerate conic passing through the four points P, Q, R, and S. Its equation is $y(y + 2) = 0$ or

$$y^2 + 2y = 0.$$

The two lines PR and QS form a degenerate conic, also passing through these points. Its equation is $x(x + y + 1) = 0$ or

$$x^2 + xy + x = 0.$$

Hence, the conics of the system

$$(y^2 + 2y) + k(x^2 + xy + x) = 0$$

all pass through the points P,Q,R, and S. The one which passes through T is given by

$$(1 - 2) + k(9 - 3 + 3) = 0,$$

whence $k = 1/9$. The desired equation is, therefore,

$$(y^2 + 2y) + 1/9(x^2 + xy + x) = 0,$$

or

$$x^2 + xy + 9y^2 + x + 18y = 0.$$

Second method: We know that the equation is of the form

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

Substituting, in turn, the coordinates of each of the five given points, we have

$$\begin{array}{lrl}
P: & 4C - 2E + F &= 0,\\
Q: & A - 2B + 4C + D - 2E + F &= 0,\\
R: & F &= 0,\\
S: & A - D + F &= 0,\\
T: & 9A - 3B + C + 3D - E + F &= 0.
\end{array}$$

Solving simultaneously, in terms of D, we obtain

$$\begin{array}{ll}
A = D, & E = 18D,\\
B = D, & F = 0.\\
C = 9D, &
\end{array}$$

Setting $D = 1$, we have the equation

$$x^2 + xy + 9y^2 + x + 18y = 0$$

15. Write the equation of the conic which passes through the points P (0,0), Q (0,3), R (−2,0), S (2,1), and T (4,3). Name the curve.

Ans. $x^2 - 2xy + 2y^2 + 2x - 6y = 0.$

16. Write the equation of the conic through the five points (3,4), (−3,4), (−5,0), (0,5), and (−4, −3). Name the curve.

17. Write the equation of the conic which passes through the three points (0,0), (1,1), and (2,4) and is symmetrical to the y-axis. *Ans.* $x^2 = y.$

18. If $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are as given for Eq. (55) prove that the sign of the quantity $\mathcal{B}^2 - 4\mathcal{A}\mathcal{C}$ is dependent only upon the value of e.

19. It is proved in the theory of equations that a system of three homogeneous linear equations in three variables, as

$$\begin{aligned} L_1u + M_1v + N_1w &= 0, \\ L_2u + M_2v + N_2w &= 0, \\ L_3u + M_3v + N_3w &= 0, \end{aligned}$$

has solutions other that $u = 0$, $v = 0$, and $w = 0$ if, and only if, the determinant of the coefficients,

$$\begin{vmatrix} L_1 & M_1 & N_1 \\ L_2 & M_2 & N_2 \\ L_3 & M_3 & N_3 \end{vmatrix}$$

is equal to zero. Use this statement to prove that A', B', and C' in (58) cannot all be zero unless A, B, and C are all zero.

20. Prove that the value of the quantity $A + C$ in the equation of the conic $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ remains unchanged under translation and rotation of axes; also that $B^2 - 4AC$ is unchanged under translation, as well as under rotation, of axes. (See Exercise 1 above.)

21. Explain why a circle is determined by three points, whereas the general conic is determined by five points.

22. Prove that if $A + C = 0$ in the equation of a conic, the conic is an equilateral hyperbola or two perpendicular lines. HINT: See Prob. 20.

23. Find the equation of the ellipse whose eccentricity is $2/3$ and which has the point (5,−2) and line $4x + 3y = 0$ as focus and corresponding directrix.

24. Find the equation of the hyperbola whose eccentricity is 2 and which has the point (−1,6) and line $3x - 4y = 0$ as focus and corresponding directrix.

CHAPTER V

SPECIAL PLANE CURVES

30. Introduction. In Chaps. III and IV we have treated extensively the straight line and the conic curves whose rectangular equations are of the first and second degree. In the brief treatment to which we are limited in this book, no such thorough discussion is possible for loci defined by third-, fourth-, and higher degree equations. We shall therefore be content, in this concluding chapter on plane analytic geometry, to call attention to a collection of curves of historical or scientific interest. They have been selected for their variety, making classification difficult and exposition lacking in unity.

31. Curves of Historical Interest. *A. Ovals of Cassini.* The locus of a point which moves in such a way that the product of

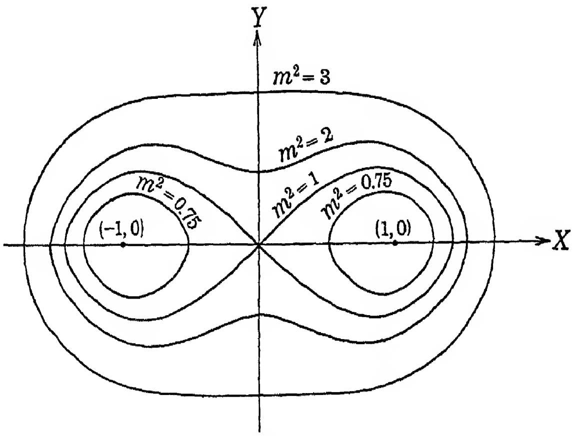


FIG. 57.

its distances from two fixed points is a constant was proposed by Cassini* as the possible shape of the orbits of the planets.

The curves are known as *Cassinian ovals* and have the possible variety in shape shown in Fig. 57, depending on the relation between the fixed product and the distance between the fixed points.

* Giovanni Domenico Cassini (1625–1712), first of four generations of the Cassini family to fill the post of director of the observatory of Paris.

Exercise 1. If the two fixed points are $(a,0)$ and $(-a,0)$ and the product of the distances is m^2, derive the equation of the Cassinian oval.

Ans. $x^4 + y^4 + 2x^2y^2 - 2a^2x^2 + 2a^2y^2 + a^4 - m^4 = 0.$

Exercise 2. Plot the Cassinian oval of Exercise 1 (*a*) with $m = 2$, $a = 4$; (*b*) with $m = 8$, $a = 4$; (*c*) with $m = a = 4$. (The locus of this case is also called the *lemniscate.*)

Exercise 3. Transform the equation of Exercise 1 to polar coordinates.

Ans. $r^4 + a^4 - 2a^2r^2 \cos 2\theta = m^4.$

Exercise 4. Find the equation of the lemniscate in polar coordinates by setting $m = a$ in the equation of Exercise 3. *Ans.* $r^2 = 2a^2 \cos 2\theta.$

B. Cissoid. A circle of radius a has its center at the point $(a,0)$. A line is drawn tangent to this circle at the point C $(2a,0)$. An arbitrary line OB through the origin cuts the circle at a point A and meets the line $x = 2a$ in the point B. The line segment OB is divided internally at P so that $OA = PB$. As the line OB turns about O, the locus of the point P is called a *cissoid.* It was studied by Diocles about 180 B.C.

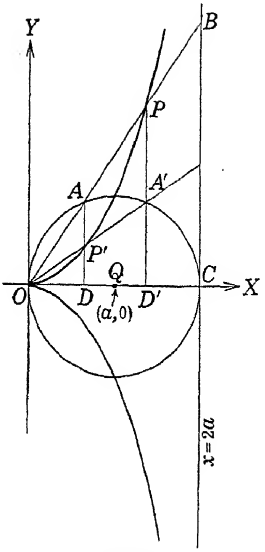


FIG. 58.

Exercise 5. Taking the angle θ = angle COB as parameter, show that $OA = 2a \cos \theta$, $OB = 2a \sec \theta$, and, hence, that $OP = 2a(\sec \theta - \cos \theta)$. Express the coordinates (x,y) of the point P in terms of θ.

Ans. $x = 2a \sin^2 \theta$, $y = 2a \sin^3 \theta \sec \theta.$

Exercise 6. Eliminate θ between the equations of Exercise 5. *Ans.* $x^3 + xy^2 = 2ay^2.$

Exercise 7. In Fig. 58, consider the line segment DD' as any line segment on OX bisected at the center Q of the circle. Consider, also, that perpendiculars at D and D' are drawn to OX, meeting the circle in the points A and A'. Let the line OA cut $D'A'$ at the point P and the line OA' cut DA at P'. Find parametric equations of the locus of the points P and P'. HINT: Take either of the lines DA, $D'A'$ as the line $x = a - t$. Then the other is $x = a + t$. Express the coordinates (x,y) of a point on the locus, parametrically in terms of t. *Ans.* $x = a + t$, $y = (a + t)^{3/2}(a - t)^{-1/2}.$

Exercise 8. By eliminating t from the equations of Exercise 7, prove that the locus is the cissoid.

Exercise 9. Transform the equation of the cissoid to polar coordinates and plot the curve. *Ans.* $r = 2a \sec \theta \sin^2 \theta.$

Exercise 10. Prove that the line through the points $(2a,0)$ and $(a,2a)$ cuts the cissoid in a point given by the parameter value θ such that $\tan^3 \theta = 2.$

Exercise 11. Using the result of Exercise 10, devise a method of constructing the edge of a cube whose volume is twice the volume of a given cube, by use of a cissoid.

C. Conchoid of Nicomedes. A fixed point A has the coordinates $(-a,0)$. A variable line l passes through A, meeting the y-axis in a point B. On l are established two points P and P' at the constant distance b from B. The locus of the two points P and P' is called a *conchoid of Nicomedes* after its inventor, a Greek who lived about 225 B.C.

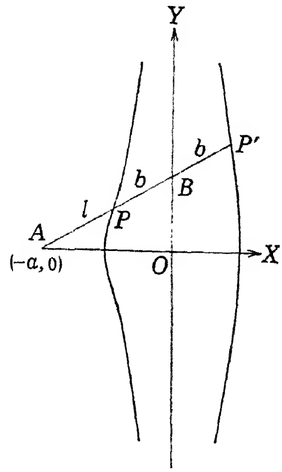


FIG. 59.

Exercise 12. Using the inclination α of the line l as parameter, derive the parametric equations of the locus of the point P and show that, for $\alpha' = 180° + \alpha$, the point P' is determined by the same equations.

Ans. $x = b \cos \alpha,\ y = a \tan \alpha + b \sin \alpha.$

Exercise 13. Eliminate α between the equations of Exercise 12.

Ans. $x^2[y^2 + (x + a)^2] - b^2(x + a)^2 = 0.$

Exercise 14. Find the polar equation of the conchoid with A as pole and AX as polar axis. *Ans.* $r = a \sec \theta + b.$

Exercise 15. Plot the conchoid for the cases $a = 4, b = 2$; $a = 4, b = 4$; $a = 4, b = 8$.

Problems

1. A line through the origin intersects the circle $x^2 + y^2 - 2ay = 0$ in the point R and the line $y = 2a$ in the point Q. Lines through R and Q, parallel to OX and OY, respectively, intersect in a point P. The locus of the point P is known as the *witch of Agnesi*. Find its equation and draw the curve. HINT: Use the angle θ from the y-axis to the line OR as parameter.

$$\textit{Ans.}\quad \begin{cases} x = 2a \tan \theta \\ y = 2a \cos^2 \theta \end{cases}, \text{ or } x^2y = 4a^2(2a - y).$$

2. A line through the point $A(a,0)$ meets OY at D. On AD is laid off a distance DP in either direction, equal to DO. The locus of the points P thus obtained is known as a *strophoid*. Draw the curve and find its equation.

$$\textit{Ans.}\quad \begin{cases} x = a \sin \\ y = a(1 - \sin \theta) \tan \theta, \end{cases} \text{or, } y^2(a + x) = x^2(a - x).$$

3. A line through the pole meets the circle $r = a \cos \theta$ in a point R. A distance RP, equal to a constant b, is laid off on this line in either direction from R. The locus of points P thus obtained is known as a *limaçon*. Find its equation and draw the curve for the two cases $b = 2a$, and $b = \frac{1}{2}a$.

Ans. $r = a \cos \theta + b.$

4. A limaçon for which $b = a$ is called a *cardioid*. Draw the cardioid and display its equation. *Ans.* $r = a(1 + \cos \theta).$

5. Obtain the equation of the *folium of Descartes* from its parametric representation $x = \frac{3at}{1 + t^3}$, $y = \frac{3at^2}{1 + t^3}$. Plot the curve and prove that the line $x + y + a = 0$ is an asymptote.

6. Plot the following spirals.

(a) *Spiral of Archimedes:* $r = a\theta$.
(b) *Logarithmic spiral:* $\log r = a\theta$.
(c) *Parabolic spiral:* $(r - a)^2 = 4a^2\theta$.
(d) *Lituus:* $r^2\theta = a^2$.
(e) *Hyperbolic spiral:* $r\theta = a$.

7. If a thread is held taut and kept in a plane as it is unwound from a stationary spool, the locus of any fixed point on the thread is called an *involute* of the circle. Find parametric equations of that involute of the circle $x^2 + y^2 = a^2$ which contains the point $(a,0)$.

Ans. $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$.

32. Cycloids. If a circle rolls on a line, without slipping, remaining in the same plane, the curve traced by a point of the circumference is called a *cycloid*. To obtain its equations, let the circle have the radius a and let Fig. 60 represent the

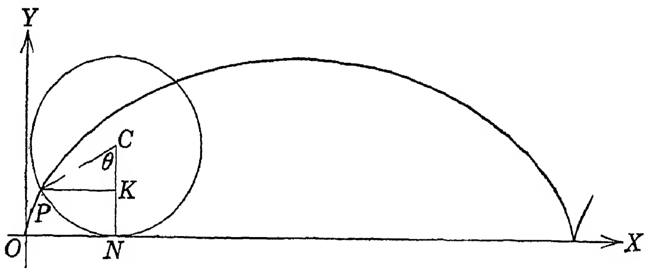


FIG. 60.

situation, the circle having started with the fixed point resting at the origin, and having rolled on the x-axis through an angle θ. Since it rolls without slipping we have $ON = \text{arc } PN = a\theta$. Evidently, $x = ON - PK = a\theta - a\sin\theta$ and

$$y = NC - KC = a - a\cos\theta.$$

Hence we may represent the cycloid parametrically by

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Problems

1. Display a table of the four least positive values of θ and the corresponding values of x and y for which (a) y is maximum, (b) y is minimum, where (x,y) are the coordinates of a point on the above cycloid.

2. Instead of a point on the circumference of the circle consider a point P that lies on a fixed radius, or radius produced, at a distance b from the center. Let the circle roll on OX and find parametric equations of the locus of P.

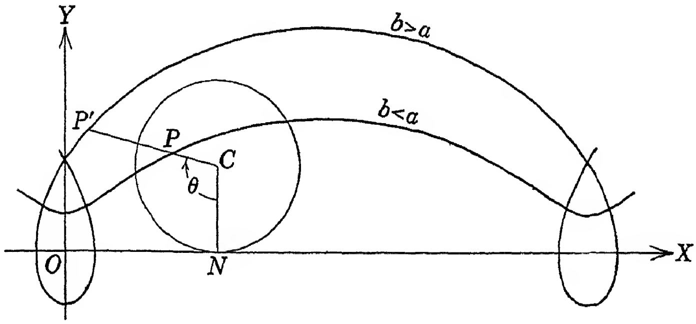


FIG. 61.

Plot the curve with $b = \frac{3}{4}a$ (*curtate cycloid* when $b < a$) and with $b = \frac{5}{4}a$ (*prolate cycloid* when $b > a$). These curves are also called *trochoids*.

3. A circle of radius b rolls, without slipping, on the outside of a circle of radius a, remaining always in the plane of the latter circle, which is stationary. A point P on the circumference of the first circle traces a curve called an *epicycloid*. Find its parametric equations. HINT: Note, in Fig. 62, that $a\theta = b\varphi$. Express ON and NC in terms of a, b, and θ. Express angle LCP, and then LC and LP in terms of a, b, and θ.

Ans. $x = (a + b)\cos\theta - b\cos\dfrac{a+b}{b}\theta;\ y = (a + b)\sin\theta - b\sin\dfrac{a+b}{b}\theta.$

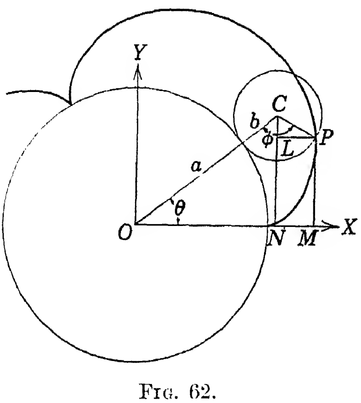


FIG. 62.

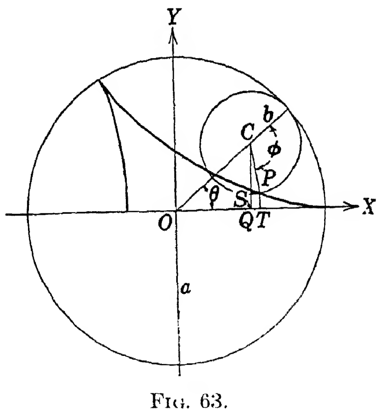


FIG. 63.

4. A curve generated as the epicycloid of Prob. 3, except that the circle of radius b is rolling on the inside of the stationary circle of radius a, is called a *hypocycloid*. In this case, of course, $b < a$. Find its equations, referring to Fig. 63 and following the hint of the above problem.

Ans. $x = (a - b)\cos\theta + b\cos\dfrac{a-b}{b}\theta;\ y = (a - b)\sin\theta - b\sin\dfrac{a-b}{b}\theta.$

5. Simplify the equations of the epicycloid (Prob. 3) as much as possible for the following cases, and draw the figure in each case. (*a*) $b = a$, (*b*) $b = \frac{1}{2}a$, (*c*) $b = \frac{1}{3}a$.

6. Simplify the equations of the hypocycloid (Prob. 4) as much as possible for the following cases and draw the figure in each case: (*a*) $b = \frac{1}{2}a$, (*b*) $b = \frac{1}{3}a$; (*c*) $b = \frac{1}{4}a$.

7. Eliminate the parameter θ from the result of Prob. 6 (*c*).

Ans. $x^{2/3} + y^{2/3} = a^{2/3}$.

33. Logarithmic and Exponential Curves. Curves whose equations involve logarithms of variables or constants with variable exponents occur frequently in scientific applications of mathematics. In such cases use is often made of *natural logarithms* whose base is e, approximately equal to 2.71828. The advantage of this base becomes apparent only after the student has been introduced to the Calculus. However, tables of values of $\log_e x$ and e^x are to be found in any good numerical table. In using this base the student should bear in mind that, as for any other base, $\log_e x$ is defined only for positive values of x and that $\log_e 1 = 0$. Change of base may be effected by the formula $\log_e x = \log_{10} x \cdot \log_e 10$.

Problems

1. Plot the curves

(*a*) $y = \log_e x$.
(*b*) $y = \log_{10} x$.
(*c*) $y = x \log_e x$.
(*d*) $y = x^2 \log_{10} x$.
(*e*) $y = \log_e (1 + x)$.
(*f*) $y = \log_{10} (1 + x^2)$.
(*g*) $y = \log_{10} \sqrt{1 + x}$.
(*h*) $y = 10^x$.
(*i*) $y = e^x$.
(*j*) $y = e^{x+3}$.
(*k*) $y = e^{\sqrt{x+1}}$.
(*l*) $y = x10^{-2x}$.
(*m*) $y = x^2e^{-x}$.

2. Plot the curve $y = \frac{b}{\pi}e^{-h^2x^2}$ for $h = 1$. This curve is known as the *probability curve.*

3. Plot the curve $y = \frac{a}{2}(e^{x/a} + e^{-x/a})$. This curve is called a *catenary.* It is the curve assumed by a hanging chain or cord of uniform cross section. The expression $\dfrac{e^{x/a} + e^{-x/a}}{2}$ is called the *hyperbolic cosine* of x/a and written $\cosh x/a$.

34. Trigonometric Curves. Curves whose equations involve trigonometric functions and inverse trigonometric functions occur very frequently in applied mathematics, especially in connection with vibration and periodic phenomena. In all

cases the angle is understood to be measured in radians when plotted. As an example, the graph of the equation $y = \cos x$ is

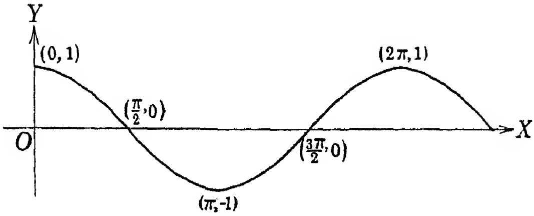


FIG. 64.

shown in Fig. 64, where the coordinates of several points are marked. From the reduction formula

$$\cos (x + 2\pi n) = \cos x,$$

where n is an integer, we see that if two values of x differ by a multiple of 2π the corresponding points have the same ordinate. Such a condition is known as *periodicity*. The least number p whose multiples may be added to any value of x without affecting y is called the *period*. Thus, in the above case the period is 2π.

The plotting of trigonometric curves is frequently expedited by the use of the geometric representation of the functions involved, as given in any text in trigonometry. Also, the practice of adding the ordinates of several curves is often an aid. For example, consider the equation

$$y = \frac{x}{4} + \frac{1}{2} \sin 2x.$$

The ordinate of this curve, for any value of x, is the sum of the

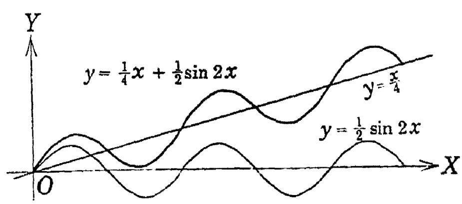


FIG. 65.

ordinates of the curves

$$y = \frac{x}{4}, \qquad y = \frac{1}{2} \sin 2x.$$

We first plot the latter two curves on the same system of axes, as shown in Fig. 65, and then add the corresponding ordinates, algebraically.

Problems

Plot the following curves. If periodic, find the period.

1. $y = \sin x.$
2. $y = \sin 3x.$
3. $y = 2 \sin \frac{x}{2}.$
4. $y = \tan x.$
5. $4y = \operatorname{ctn} x.$
6. $y = 5 \cos x.$
7. $y = 3 \cos \frac{x}{3}.$
8. $y = \sec x.$
9. $y = \csc x.$
10. $y = \tan^2 2x.$
11. $y = \sin x + \cos x.$
12. $y = \frac{x}{10} - \cos x + \sin x.$
13. $y = 2 \sin (2\pi x - \pi).$
14. $y = e^{-\frac{x^2}{100}} \sin (\pi x).$
15. $y = \sin^{-1} x.$
16. $y = \sin^{-1}\sqrt{1 - x^2}.$
17. $y = \cos^{-1} x.$
18. $y = \cos^{-1} (x + 1).$
19. $y = \tan^{-1} x.$
20. $y = \operatorname{ctn}^{-1} x.$
21. $y = \sec^{-1} x.$
22. $y = \csc^{-1} x.$
23. $y = 3 \sin x + 4 \cos x.$
24. $xy = \sin x.$
25. $y = x \sin x.$
26. $y = x \sin \frac{1}{x}.$
27. $y = e^{-x} \cos 3x.$

CHAPTER VI

SOLID ANALYTIC GEOMETRY

35. Space Coordinates. *A. Rectangular Systems.* Let three mutually perpendicular lines, OX, OY, and OZ, be graduated with a common unit so that each represents a one-dimensional coordinate system with O as origin. Let any point P, in space, be projected upon these lines by planes perpendicular to the lines, meeting them in P_x, P_y, and P_z, whose coordinates, in the respective systems, are (x), (y), and (z). We say, then, that the point P has the *rectangular coordinates* (x,y,z). It may be noticed, at once, that this system of coordinates is an extension to three dimensions, of the system of rectangular coordinates in a plane, already so familiar to the student.

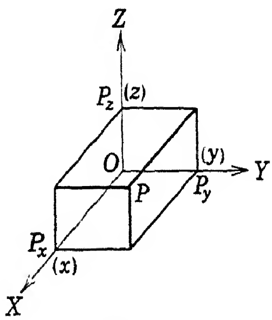


FIG. 66.

Certain of the formulas previously met with in one- and two-dimensional systems of coordinates are immediately adaptable to space, as set forth in the following three exercises.

Exercise 1. If a directed line segment extends from P_1 (x_1,y_1,z_1) to P_2 (x_2,y_2,z_2), prove that the projections of this segment upon the coordinate axes are equal to $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$, in the obvious order.

Exercise 2. If a line segment is directed from P_1 (x_1,y_1,z_1) to P_2 (x_2,y_2,z_2), prove that the coordinates (x,y,z) of the point P which divides P_1P_2 in the ratio $r_1:r_2$, are given by

$$x = \frac{r_1x_2 + r_2x_1}{r_1 + r_2}, \qquad y = \frac{r_1y_2 + r_2y_1}{r_1 + r_2}, \qquad z = \frac{r_1z_2 + r_2z_1}{r_1 + r_2}.$$

Exercise 3. Prove that the distance between two points P_1 (x_1,y_1,z_1) and P_2 (x_2,y_2,z_2) is given by the formula,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

HINT: Pass planes through both points, parallel to the coordinate planes and forming a rectangular box. The result follows from right triangles in the figure.

The methods of designating the direction of a line, l, used in plane analytic geometry, are only slightly more difficult to extend to space. Recall that, in a plane, the angle, α, from the positive x-axis to the line, has been called the inclination of the line. If the angle between the line and the y-axis were called β then we would have $\beta = \pm(90° - \alpha)$, and, in either case,

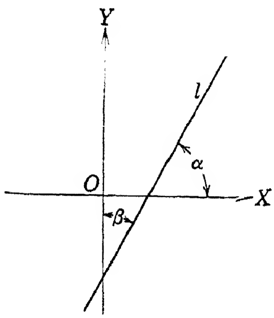


FIG. 67.

$$\cos^2 \alpha + \cos^2 \beta = 1.$$

Also, the slope of the line, which has been defined as $\tan \alpha$, could be defined as $\cos \beta/\cos \alpha$. These observations furnish us with a clue to the methods of designating the direction of a line in space. Indeed, we shall call the three angles α, β, and γ which a line makes with OX, OY, and OZ, respectively, the *direction angles* of the line and their cosines, *viz.*, $\cos \alpha$, $\cos \beta$, $\cos \gamma$, the *direction cosines* of the line. These direction angles are always considered as positive angles, unless zero, and not greater than 180° but are, nevertheless, ambiguous since each of the three axes makes two such angles with any given line. While each of the three direction angles, α, β, γ, of a line has two possible values, it is customary to restrict them to two sets (α,β,γ) and (α',β',γ') defined as follows: From a point Q on the line l draw QR, QS, and QT parallel to and directed as OX, OY, and OZ. Let P and P' be any two points on l such that Q is between them. Then the three angles of the set (α,β,γ) are defined as

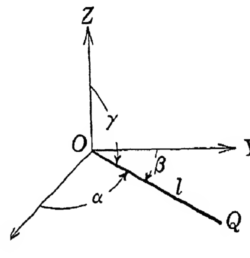


FIG. 68.

$$\alpha = \angle RQP,$$
$$\beta = \angle SQP,$$
$$\gamma = \angle TQP,$$

as shown in Fig. 69, while the angles of the set (α',β',γ') are defined as

$$\alpha' = \angle RQP',$$
$$\beta' = \angle SQP',$$
$$\gamma' = \angle TQP'.$$

It is obvious that the angles of the set (α',β',γ') are supplements of the angles of the set (α,β,γ) and that

$$\cos\alpha + \cos\alpha' = \cos\beta + \cos\beta' = \cos\gamma + \cos\gamma' = 0.$$

We have noticed above that the slope of a line, in plane analytic geometry, could be defined as the ratio of $\cos\beta$ to $\cos\alpha$ and, hence, as the ratio of any two numbers (not both zero) proportional to $\cos\beta$ and $\cos\alpha$. Just as in plane analytics we often find it more convenient to use the slope of a line rather than its inclination, so in solid analytics we will often find it convenient to use the ratios

$$\cos\alpha : \cos\beta : \cos\gamma$$

or the ratios of any three numbers proportional to these cosines.

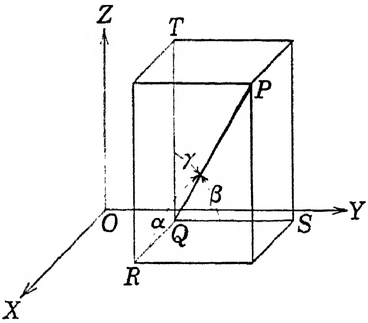


FIG. 69.

Those ratios are known as the *direction ratios* of the line having the direction angles α, β, and γ (and the direction cosines $\cos\alpha$, $\cos\beta$, and $\cos\gamma$). It is customary to employ three numbers l, m, and n satisfying the equations

$$l = p\cos\alpha, \qquad m = p\cos\beta, \qquad n = p\cos\gamma, \tag{60}$$

where p is any number not zero, and to speak of the set (l,m,n) as *direction numbers* of the line. Obviously, they are not unique, their ratios only are significant, and the ratios $l:m:n$ form the direction ratios of the line in question.

Since a line is determined completely by two points, it must be possible, given the coordinates of two distinct points in space, to determine, from those coordinates, the direction angles, of a line joining them, its direction cosines, its direction ratios, and a set of its direction numbers. The formulas giving these relations are displayed in the exercises below to be proved by the student.

Exercise 4. Prove that the direction cosines of the line joining Q (x_1,y_1,z_1) and P (x_2,y_2,z_2) and directed from Q to P are given by the formulas

$$\cos\alpha = \frac{x_2 - x_1}{r},$$

$$\cos \beta = \frac{y_2 - y_1}{d}.$$

$$\cos \gamma = \frac{z_2 - z_1}{d},$$

where d is the distance between the two points, *i.e.*,

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

HINT: Use Fig. 69 and show that $QR = x_2 - x_1$, etc., and note that QPR, QPS, QPT are right triangles, each having PQ as hypotenuse.

Show also that if the line is directed from P to Q, the values of $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ are the negatives of the above.

Exercise 5. Prove, from the values of $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ displayed in Exercise 4, that $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

Exercise 6. Using the results of Exercise 4, show that a set of direction numbers of the line through the points Q (x_1,y_1,z_1) and P (x_2,y_2,z_2) is $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$.

Exercise 7. If a line has direction numbers (l,m,n), show, from (60) and the formula of Exercise 5, that one of its two sets of direction cosines is

$$\left(\frac{l}{\sqrt{l^2 + m^2 + n^2}}, \quad \frac{m}{\sqrt{l^2 + m^2 + n^2}}, \quad \frac{n}{\sqrt{l^2 + m^2 + n^2}}\right).$$

Exercise 8. If two lines have the direction cosines (λ_1,μ_1,ν_1) and (λ_2,μ_2,ν_2), prove that θ, one of the angles between the lines, is given by the formula

$$\cos \theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2.$$

HINT: Note that if the set $(\lambda_1,\mu_1\nu_1,)$ is taken as the coordinates of a point P_1 and the set (λ_2,μ_2,ν_2) is taken as the coordinates of a point P_2, then the line segment OP_1 is of length unity and has the direction cosines (λ_1,μ_1,ν_1), while the line segment OP_2 is also of length unity and has the direction cosines (λ_2,μ_2,ν_2). Call the angle P_1OP_2, the desired angle, θ, and find it by the law of cosines.

Exercise 9. Using the notation and results of Exercise 8, prove that if the two lines are parallel then the set (λ_1,μ_1,ν_1) is identical either with the set (λ_2,μ_2,ν_2) or with the set $(-\lambda_2, -\mu_2, -\nu_2)$.

Exercise 10. Using the notation and results of Exercise 8, prove that if the two lines are perpendicular then

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0,$$

and conversely.

Exercise 11. Using the result of Exercise 10, prove that if a line L_1 has direction numbers (l_1,m_1,n_1) and L_2 has direction numbers (l_2,m_2,n_2) then L_1 and L_2 are perpendicular if, and only if,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

Exercise 12. If the positive directions of two directed lines make an angle θ with each other, prove that a directed line segment of length l on one line projects into a directed line segment of length $l \cos \theta$ upon the other line.

Problems

1. Plot the following points:

(a) (1,2,3). (d) (0,0,−7). (g) (−2,−5,4).
(b) (−2,4,3). (e) (4,−2,−3). (h) (0,−2,3).
(c) (5,0,8). (f) (−6,9,−7). (i) (−3,−4,−5).

2. If a directed line segment extends from P_1 to P_2, find its projections upon the three coordinate axes in each case.

(a) P_1 (4,−5,6), P_2 (−2,−3,−4). (c) P_1 (5,5,−5), P_2 (1,2,−5).
(b) P_1 (−6,−2,4), P_2 (−3,4,3). (d) P_1 (−1,2,2), P_2 (−1,2,8).

3. Using the points P_1 and P_2 of Prob. 2, find the coordinates of the point P which divides the line segment P_1P_2 in the given ratio $r_1:r_2$ in each case.

(a) $r_1:r_2 = 3:1$. *Ans.* $(-\frac{1}{2}, -\frac{7}{2}, -\frac{3}{2})$. (c) $r_1:r_2 = 1:-2$.
(b) $r_1:r_2 = 1:1$. *Ans.* $(-\frac{9}{2}, 1, \frac{7}{2})$. (d) $r_1:r_2 = 5:-3$.

4. Find the distance between P_1 and P_2, using the points of Prob. 2.

Ans. (a) $\sqrt{140}$.

5. Each of the following sets of three points determines a triangle. Determine whether or not the triangle has any simple special properties such as being isosceles, equilateral, or right-angled.

(a) (1,−5,6), (2,−3,−1), (3,−6,3).
(b) (3,7,−4), (0,1,−6), (5,4,2).
(c) (1,0,1), (1,4,−1), (9,−1,−6).
(d) (−5,−5,7), (−1,−2,3), (3,1,7).
(e) (5,2,−3), (6,−5,−3), (2,−2,2).

6. Show that the following points are equidistant from the x-axis: (6,5,0), (−3,4,−3), (2,0,−5), (9,−3,−4), (0,0,5).

7. Which of the following sets can be the direction cosines of a line? Which can be direction numbers? $(\frac{1}{8}, 1, -\frac{1}{8})$, $(\frac{2}{7}, -\frac{3}{7}, \frac{6}{7})$, (0,0,0), $(\frac{1}{9}, -\frac{4}{9}, \frac{8}{9})$.

8. Find direction numbers of the line joining the point (1,2,3) to each of the following points:

(a) (0,4,5). *Ans.* (−1,2,2). (c) (4,6,−9). (e) (1,2,7).
(b) (1,4,6). *Ans.* (0,2,3). (d) (1,−7,11). (f) (0,−6,−11).

9. Find direction cosines for each of the lines of Prob. 8.

Ans. (a) $(-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$.

10. Find all possible sets of direction cosines, given that

(a) $\cos \beta = -\frac{2}{7}$, $\cos \alpha = -2 \cos \gamma$.

Ans. $(\frac{6}{7}, -\frac{2}{7}, -\frac{3}{7})$, $(-\frac{6}{7}, -\frac{2}{7}, \frac{3}{7})$.

(b) $\gamma = 45°$, $\beta = 120°$.
(c) $\alpha = 150°$, $\beta = \gamma$.
(d) $\alpha = \beta = 180° - \gamma$.

11. Which of the following sets are direction numbers of a line perpendicular to the line having direction numbers (4,1,−3)?

(*a*) $(-3,3,5)$. (*c*) $(4,-1,5)$. (*e*) $(0,3,1)$.
(*b*) $(-6,5,7)$. (*d*) $(0,1,0)$. (*f*) $(1,-4,0)$.

12. In each case below find whether the line joining P_1 and P_2 is parallel to the line having direction numbers $(2,-3,4)$:
(*a*) P_1 $(2,4,-1)$, P_2 $(6,-2,7)$. (*c*) P_1 $(-3,4,0)$, P_2 $(-5,7,-4)$.
(*b*) P_1 $(3,-1,-3)$, P_2 $(5,2,7)$. (*d*) P_1 $(7,0,0)$, P_2 $(1,9,-12)$.

13. Show that the points P_1 $(2,-1,-5)$, P_2 $(5,2,-2)$, and P_3 $(6,3,-1)$ lie on a straight line. In what ratio does the point P_2 divide the line segment from P_1 to P_3?

14. Find one of the angles between the two lines having the given direction numbers, in each case:
(*a*) $(3,0,4)$, $(9,20,12)$. (*c*) $(5,3\sqrt{3},4\sqrt{3})$, $(1,0,0)$.
(*b*) $(3,-4,12)$, $(-14,-5,2)$. (*d*) $(1,-2,1)$, $(-1,2,1)$.

15. Find direction numbers for the line joining the points $(3,1,7)$ and $(6,7,5)$ and direction numbers of the projections of the given line upon the planes of reference.

16. Given the points P_1 $(1,-3,4)$, P_2 $(4,2,-2)$, P_3 $(0,1,5)$, and P_4 $(6,5,-3)$, find the sum of the projections, upon the x-axis, of the directed line segments P_1P_2, P_2P_3, P_3P_4, and P_4P_1, also upon the y-axis and upon the z-axis.

17. Prove that the sum of the projections, upon any one of the coordinate axes, of the directed line segments P_1P_2, P_2P_3, and P_3P_4 depends only upon the positions of P_1 and P_4.

18. What can be said about the location of a point P if its coordinates (x,y,z) satisfy the equations
(*a*) $x = 0$?
(*b*) $y = 5$?
(*c*) $\quad = -2$?
(*d*) $\quad = z$?
(*e*) $\quad = 2x$?
(*f*) $x^2 + y^2 + z^2 = 25$?
(*g*) $y^2 + z^2 = 16$?
(*h*) $\left\{\begin{matrix} x = 0 \\ y = 2 \end{matrix}\right\}$?
(*i*) $x = y = z$?
(*j*) $\left\{\begin{matrix} x^2 + y^2 = 25 \\ z = 9 \end{matrix}\right\}$?
(*k*) $y^2 = z^2$?

19. What equation or equations, involving the coordinates (x,y,z) of a point, hold if the point is anywhere on the (*a*) zx-plane? (*b*) plane passing through $(4,-2,-3)$ parallel to the xy-plane? (*c*) z-axis? (*d*) line passing through $(-5,3,7)$ parallel to OY? (*e*) sphere with center at $(1,2,3)$ and radius 10?

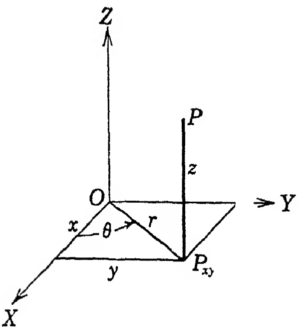


FIG. 70.

B. Other Systems of Space Coordinates. I. CYLINDRICAL COORDINATES. If a system of polar coordinates is set up in the xy-plane, as shown in Fig. 70, so that the point P_{xy}, whose coordinates in the xy-plane are (x,y), has the polar coordinates (r,θ), then the point P, with rectangular space

coordinates (x,y,z), is said to have the *cylindrical coordinates* (r,θ,z).

II. SPHERICAL COORDINATES. In this system we use the same angle θ as in cylindrical coordinates. Instead of giving z and r, however, we give the angle φ between the positive z-axis and the line segment OP, and the length ρ of that segment. The *spherical coordinates* of the point P are written as a set in the form (ρ,θ,φ).

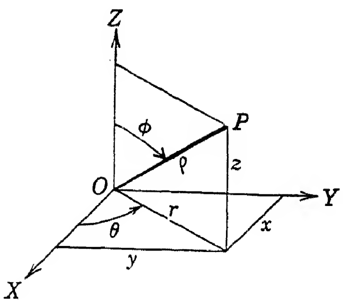


FIG. 71.

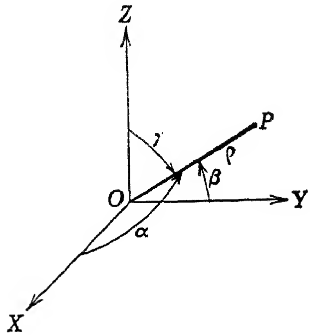


FIG. 72.

III. POLAR SPACE COORDINATES. Another system of coordinates, sometimes used for a point P in space, known as *polar space coordinates*, is that in which the coordinates $(\rho,\alpha,\beta,\gamma)$ are the distance $OP = \rho$, and the direction angles, α, β, and γ of the segment OP.

The expressions for these various coordinates of a point P in terms of its rectangular coordinates are readily deduced from the figures. They are displayed in the following table.

Cylindrical	Spherical	Polar space
$r = \sqrt{x^2 + y^2}$	$\rho = \sqrt{x^2 + y^2 + z^2}$	$\rho = \sqrt{x^2 + y^2 + z^2}$
$\theta = \tan^{-1}(y/x)$	$\theta = \tan^{-1}(y/x)$	$\alpha = \cos^{-1}(x/\sqrt{x^2 + y^2 + z^2})$
$z = z$	$\varphi = \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2})$	$\beta = \cos^{-1}(y/\sqrt{x^2 + y^2 + z^2})$
		$\gamma = \cos^{-1}(z/\sqrt{x^2 + y^2 + z^2})$

The expressions for the rectangular coordinates (x,y,z) of a point P in terms of its cylindrical coordinates (r,θ,z), its spherical coordinates (ρ,θ,φ) and its polar space coordinates $(\rho,\alpha,\beta,\gamma)$ are given by the table

Cylindrical	Spherical	Polar space
$x = r \cos \theta$	$x = \rho \sin \varphi \cos \theta$	$x = \rho \cos \alpha$
$y = r \sin \theta$	$y = \rho \sin \varphi \sin \theta$	$y = \rho \cos \beta$
$z = z$	$z = \rho \cos \varphi$	$z = \rho \cos \gamma$

Problems

1. What can be said concerning the location of a point if it is known only that

(a) $\rho = 10$? (c) $\alpha = 20°$? (e) $\varphi = 25°$? (g) $\rho = 0$?
(b) $r = 10$? (d) $\theta = 40°$? (f) $r = 0$? (h) $\beta = 0$?

2. Express, in as simple a manner as possible, in terms of some of the various coordinates, x, y, z, r, ρ, θ, φ, of a point P, a condition imposed by placing the point P

(a) In a plane parallel to the xy-plane and 3 units above it. *Ans.* $z = 3$.

(b) In the yz-plane. *Ans.* $x = 0$, $\alpha = 90°$, or $\theta = \pm 90°$.

(c) On the y-axis.

(d) On the z-axis.

(e) On the surface of a sphere of radius 5 with its center at the origin.

(f) On a cylinder of radius 5 with OZ as the axis of revolution.

(g) On a cone with vertex at the origin, OZ as axis of revolution and with semivertical angle equal to 30°.

3. Transform to rectangular coordinates the point whose cylindrical coordinates are (a) $(3,2\pi/3,1)$. (b) $(6,90°,-5)$. (c) $(4,\frac{5}{4}\pi,2)$. (d) $(6,0,-1)$.

Ans. (a) $(-\frac{3}{2},3\sqrt{3}/2,1)$.

4. Transform to rectangular coordinates the point whose spherical coordinates are (a) $(6,30°,120°)$; (b) $(10,45°,90°)$; (c) $(2,135°,60°)$; (d) $(8,315°,45°)$. *Ans.* (a) $(\frac{9}{2},3\sqrt{3}/2,-3)$.

5. Transform to rectangular coordinates the point whose polar space coordinates are (a) $(18,135°,120°,120°)$; (b) $(6,120°,60°,135°)$; (c) $[9, \cos^{-1}(\frac{1}{3}), \cos^{-1}(-\frac{2}{3}), \cos^{-1}(\frac{2}{3})]$. *Ans.* (a) $(-9\sqrt{2}, -9, -9)$.

6. Find the cylindrical, polar space, and spherical coordinates of the point whose rectangular coordinates are (a) $(1,2,-2)$; (b) $(-1,\sqrt{3},1)$; (c) $(8,-9,-12)$.

7. Derive a formula for the distance between two points whose polar space coordinates are $(\rho_1,\alpha_1,\beta_1,\gamma_1)$ and $(\rho_2,\alpha_2,\beta_2,\gamma_2)$.

8. Transform the equation $x^2 + y^2 + z^2 = 9$ to (a) cylindrical coordinates; (b) spherical coordinates; (c) polar space coordinates.

36. Transformation of Rectangular Coordinates. *I. Translation.* We define a *translation* of axes for rectangular coordinates in space in exactly the same way as it was defined for the plane. The verification of the formulas is left to the student in

Exercise 1. Draw a figure and prove that if the axes are translated so that the new origin has the old coordinates (h,k,l) then the old coordinates (x,y,z) and new coordinates (x',y',z') are connected by the relations

$$x = x' + h, \qquad y' + k, \qquad z = z' + l.$$

II. Rotation. Suppose the axes have been rotated to new positions, as shown in Fig. 73, so that the direction cosines of OX', OY', and OZ', referred to the old axes, are, respectively, λ_1,μ_1,ν_1; λ_2,μ_2,ν_2; and λ_3,μ_3,ν_3. Since the three new axes are mutually perpendicular, we have

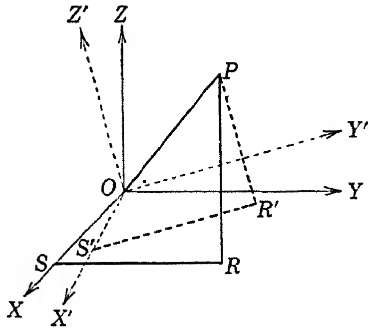


FIG. 73.

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0,$$
$$\lambda_1\lambda_3 + \mu_1\mu_3 + \nu_1\nu_3 = 0,$$
$$\lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3 = 0.$$

From the assumption that these numbers are sets of direction cosines, we have

$$\lambda_1^2 + \mu_1^2 + \nu_1^2 = 1, \qquad \lambda_2^2 + \mu_2^2 + \nu_2^2 = 1,$$
$$\lambda_3^2 + \mu_3^2 + \nu_3^2 = 1.$$

Now, from the definitions of these cosines, we have

$$\lambda_1 = \cos (X'OX),$$

$\lambda_2 = \cos (Y'OX)$, $\lambda_3 = \cos (Z'OX)$, so that the set $\lambda_1,\lambda_2,\lambda_3$ is the set of direction cosines of OX referred to the new axes. Similarly the sets μ_1,μ_2,μ_3 and ν_1,ν_2,ν_3 are the direction cosines of OY and OZ, respectively, referred to the new axes. From this it follows that

$$\lambda_1\mu_1 + \lambda_2\mu_2 + \lambda_3\mu_3 = 0, \qquad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1,$$
$$\lambda_1\nu_1 + \lambda_2\nu_2 + \lambda_3\nu_3 = 0, \qquad \mu_1^2 + \mu_2^2 + \mu_3^2 = 1,$$
$$\mu_1\nu_1 + \mu_2\nu_2 + \mu_3\nu_3 = 0, \qquad \nu_1^2 + \nu_2^2 + \nu_3^2 = 1.$$

Let, now, the coordinates of any point P be (x,y,z), referred to the original system of axes and (x',y',z'), referred to the new system. To find an expression for x, in terms of x', y', and z', note that it is the projection, upon OX of the directed line OP. Now, in Prob. 17, page 110, the student has been asked to prove a proposition which, interpreted for this figure, states that the projection, upon OX, of OP is the same as the sum of the projec-

tions upon OX of OS', $S'R'$, and $R'P$. Also, Exercise 12, page 109, shows that the projections of these directed lines upon OX are given by $\lambda_1 x'$, $\lambda_2 y'$, and $\lambda_3 z'$, respectively. We conclude that

$$\begin{aligned} x &= \lambda_1 x' + \lambda_2 y' + \lambda_3 z', \\ y &= \mu_1 x' + \mu_2 y' + \mu_3 z', \\ z &= \nu_1 x' + \nu_2 y' + \nu_3 z', \end{aligned} \tag{61}$$

the last two being readily established by arguments similar to the above.

Similarly we may compute, in two ways, the projection of OP upon each of OX', OY', and OZ' and obtain the relations

$$\begin{aligned} x' &= \lambda_1 x + \mu_1 y + \nu_1 z, \\ y' &= \lambda_2 x + \mu_2 y + \nu_2 z, \\ z' &= \lambda_3 x + \mu_3 y + \nu_3 z. \end{aligned} \tag{62}$$

Exercise 2. Verify that the system of Eqs. (62) is consistent with the system (61) by substituting the values given by (62) for x', y', and z', into (61), making use of the foregoing quadratic relations among the nine direction cosines $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2$, and ν_3.

Problems

1. Translate the axes to the point $(2, -5, 6)$ as new origin and find

(a) The new coordinates of the point $(2, -3, 4)$.

(b) The new equation corresponding to

$$4(x - 2)^2 + 9(y + 5)^2 = 36(z - 6).$$

(c) The new equations corresponding to $\dfrac{x - 2}{4} = \dfrac{y + 5}{3} = \dfrac{z - 6}{2}$.

(d) The new coordinates for the old origin.

2. Determine the translation which will remove the

(a) first-degree terms from $4x^2 - 6y^2 + 9z^2 + 12x - 12y - 18z = 36$;

Ans. $h = -3/2$, $k = -1$, $l = 1$.

(b) constant term and the first-degree terms in y' and z' from the equation in (a).

3. Show that rotation of axes will change the equation $x^2 + y^2 + z^2 = a^2$ into $x'^2 + y'^2 + z'^2 = a^2$.

4. Describe the rotation represented by the equations

$$\begin{aligned} x &= x' \cos \theta - y' \sin \theta, \\ y &= x' \sin \theta + y' \cos \theta, \end{aligned}$$

5. Apply the rotation $\left\{\begin{aligned} x &= \tfrac{2}{7}x' + \tfrac{3}{7}y' + \tfrac{6}{7}z' \\ y &= \tfrac{6}{7}x' + \tfrac{2}{7}y' - \tfrac{3}{7}z' \\ z &= -\tfrac{3}{7}x' + \tfrac{6}{7}y' - \tfrac{2}{7}z' \end{aligned}\right\}$ to the equation

(a) $2x + 6y - 3z = 14$; (b) $3x + 2y + 6z = -14$; (c) $6x - 3y - 2z = 0$; (d) $36x^2 - 12y^2 - 24z^2 + 55xy + 5yz + 11zx = 49$.

Ans. (d) $x'y' + y'z' + z'x' = 1$.

6. Find the direction cosines of the line about which the axes have been rotated in Prob. 5. HINT: For points on this line $x = x'$, $y = y'$, $z = z'$.

Ans. $3/\sqrt{19}, 3/\sqrt{19}, 1/\sqrt{19}$.

7. Is the transformation defined by $\left\{\begin{array}{l} x = x' \\ y = y' \\ z = -z' \end{array}\right\}$ a rotation?

37. Equations and Graphs. By the *graph* of an equation involving some or all of the variables x, y, and z we mean the figure consisting of all points whose coordinates satisfy the equation. Suppose a given equation involves y alone. It is, in general, satisfied by particular values of y, say $y = y_1, y_2, \ldots,$ and is satisfied by those values regardless of the values which we may assign to the missing variables, x and z. Its graph is evidently a system of planes, all parallel to the xz-plane. For example, the equation $y^2 - y - 6 = 0$ is satisfied by the coordinates of all points $(x,-2,z)$ and $(x,3,z)$, regardless of the values of x and z. Its graph consists of the pair of planes shown in Fig. 74. Similarly, the graphs of equations involving x alone and z alone are, in general, sets of planes parallel to the yz-plane and to the xy-plane, respectively.

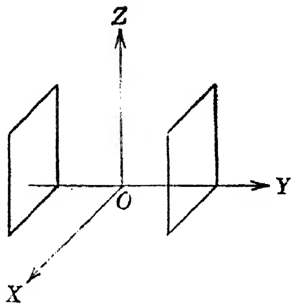


FIG. 74.

If an equation involves both x and y but is free of z, it is, in general, satisfied by all points $(x,y,0)$ on some curve in the xy-plane, and all points obtained from those by changing the value of z. It consists, then, of a surface generated by a straight line, always parallel to OZ, as it traces this curve in the xy-plane. Such a surface is known as a *cylindrical surface*, or, more simply, a *cylinder*, the straight lines on which are known as *elements*. Similarly we may readily observe that, in general, equations lacking x represent cylinders parallel to OX and equations lacking y represent cylinders parallel to OY. For example, the equation $\dfrac{(y-4)^2}{9} + \dfrac{(z-3)^2}{4} = 1$ represents the elliptical cylinder shown in Fig. 75.

The graph of an equation is, in general, a surface, regardless of how many of the coordinates (x,y,z) may actually be involved in the equation. Two surfaces, still speaking in general, will have in common, points on some curve of intersection. The coordinates of such points will satisfy the equations of both the surfaces so that, manifestly, the graph of a pair of simultaneous equations is a curve. If a system of three simultaneous equations has a graph, it can consist only of the points common to the curve determined by a pair of the equations and the surface determined by the remaining equation. This may be readily visualized by thinking of the special case in which each of three equations represents a plane, no two of which are parallel. Any two of these planes intersect in a line and that line pierces the third plane in one point.

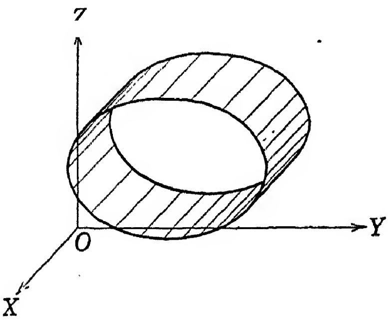


FIG. 75.

If a given equation lacks one or two of the coordinates (x,y,z), its graph is readily found from the foregoing discussions for such cases. To determine the graph of an equation involving all three variables, it is generally desirable to set one of the variables equal to a constant and plot the resulting curve. Thus, if we set $z = z_1$, we obtain an equation involving x and y, and the problem is reduced to one in plane analytic geometry, in the plane whose points have the coordinates (x,y,z_1). The resulting curve lies upon the surface sought and is, indeed, the section of that surface made by the plane $z = z_1$. By taking a number of such sections, including, possibly, sections x = constant and y = constant, the surface may be determined and drawn. A section of a given surface by a plane is called its *trace* in that plane, and the points common to a given curve and a given plane are called the *piercing points* of the curve in that plane.

Consider, as an illustration, $25x^2 + 9y^2 = 75z$. It is clear that z cannot be negative and that if $z = 0$, both x and y must be zero. Hence, the surface has no points below the xy-plane and only the origin in that plane. Letting z equal a constant, say

$c\ (>0)$ we may write the equation as

$$\frac{x^2}{75c/25} + \frac{y^2}{75c/9} = 1$$

which we recognize as an ellipse with semimajor axis $5\sqrt{3c}/3$, parallel to OY and semiminor axis $\sqrt{3c}$, parallel to OX. The center is given by the values $x = 0$, $y = 0$, in the plane $z = c$ and, hence, has the coordinates $(0,0,c)$. It is sufficient, now, to examine the yz-trace, given by the equations $\left\{\begin{matrix} 3y^2 = 25z \\ x = 0 \end{matrix}\right\}$, and the xz-trace, given by $\left\{\begin{matrix} x^2 = 3z \\ y = 0 \end{matrix}\right\}$. Both of these traces are parabolas with vertex at the origin and axis along OZ. The former is, evidently, the locus of the vertices of the ellipses $z =$ constant, while the latter is the locus of the end points of the minor axes of those ellipses. The surface, shown in Fig. 76, is called an *elliptic paraboloid*.

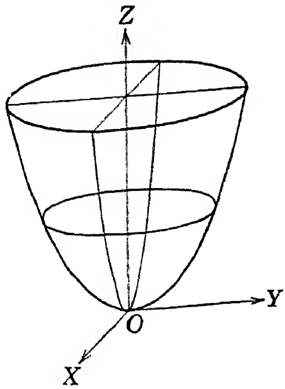


FIG. 76.

38. Parametric Representation. Let x, y, and z be given in terms of a parameter, say t, as, for example

$$\begin{aligned} x &= t^2 + 1, \\ y &= t, \\ z &= t^3 + 1. \end{aligned}$$

We note, under these circumstances, that the point whose coordinates are (x,y,z) is entirely determined by the value assigned to t, and raise the question, what is the locus of this point as t is allowed to take on all real values. If we eliminate t between the first two of these equations, we obtain, in this case, the equation $y^2 = x - 1$, the equation of a *parabolic cylinder* with elements parallel to OZ. If we eliminate t between the second and third equations, we obtain the equation $y^3 = z - 1$, which represents a *cylinder* with elements parallel to OX. Portions of these two cylinders are shown in Fig. 77 where may be seen their curve of intersection, which is known as a *twisted cubic*. This cubic is, evidently, the desired locus, and we conclude, on the basis of this example, that if x, y, *and* z are

given in terms of a parameter by three equations, the locus of the point (x,y,z) is some curve.

If x, y, and z are given in terms of two independent parameters, we know from algebra that both parameters cannot be eliminated without using all three equations, and that the result of this elimination can, in general, give us only one equation in x, y, and z.

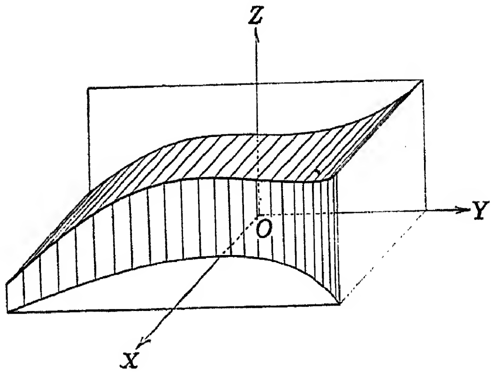


FIG. 77.

We conclude that *three equations expressing x, y, and z in terms of two independent parameters represent a surface.* If, under these circumstances, we assign a constant value to one parameter, say to u, we obtain thereby a curve upon that surface, known as a curve of constant u. If we allow u to vary but hold the other parameter, say v, constant, we obtain a curve of constant v. Should we assign constant values to both parameters, say $u = u_1$ and $v = v_1$, we obtain a point on the surface which is said to have the *curvilinear coordinates* (u_1,v_1).

Problems

1. Draw the surface represented by each of the equations:

(a) $x = 0$.

(b) $y^2 = 4$.

(c) $y(z - 1) = 0$.

(d) $y + z = 0$.

(e) $x - 3z = 6$.

(f) $x^2 + y^2 = 25$.

(g) $4y^2 + 9z^2 = 36$.

(h) $(x - 1)^2 + (z - 4)^2 = 16$.

(i) $xz = 12$.

(j) $z = y^3$.

(k) $x^2 + y^2 + z^2 = 16$.

(l) $(x - 4)^2 + (y + 4)^2 + (z - 4)^2 = 16$.

(m) $xyz = 216$.

(n) $x^{2/3} + y^{2/3} + z^{2/3} = 1$.

(o) $\frac{x^2}{16} + \frac{y^2}{9} + \frac{z^2}{4} = 1$.

(p) $\frac{x^2}{25} + \frac{y^2}{25} - \frac{z^2}{36} = 1$.

(q) $-\frac{x^2}{25} + \frac{y^2}{25} - \frac{z^2}{16} = 1$.

(r) $x^2 + z^2 = y$.

(s) $4y^2 + 9z^2 = 36x$.

(t) $z = \frac{8}{x^2 + y^2 + 4}$.

(u) $4z^2 = y^2(9 - x^2)$.

(v) $x^2 + z^2 = y^2$.

2. Draw the curves represented by the following pairs of equations:

(a) $\begin{cases} x = 3, \\ y = 5. \end{cases}$ (d) $\begin{cases} x^2 + y^2 = 9, \\ x + y = 3. \end{cases}$ (g) $\begin{cases} x^2 + y^2 + z^2 = 25, \\ x + y + z = 5. \end{cases}$

(b) $\begin{cases} z = 6, \\ x = y. \end{cases}$ (e) $\begin{cases} x^2 + y^2 + (z - 1)^2 = 16, \\ x^2 + y^2 + (z + 1)^2 = 16. \end{cases}$ (h) $\begin{cases} xy = 1, \\ yz = 1. \end{cases}$

(c) $\begin{cases} y^2 = 10z, \\ x = y. \end{cases}$ (f) $\begin{cases} x^2 + y^2 = 2z, \\ y = z. \end{cases}$ (i) $\begin{cases} x^2 + z^2 = 4, \\ 3y + 4z = 15. \end{cases}$

3. Draw the curves represented parametrically by the following sets of equations. HINT: Eliminate the parameter between pairs of equations, as in the text.

(a) $\begin{cases} x = 5 - 3t, \\ y = t - 6, \\ z = 5t + 3. \end{cases}$ (d) $\begin{cases} x = \dfrac{2}{1 - t}, \\ y = \dfrac{3}{1 - 2t}, \\ z = \dfrac{4}{1 - 3t}. \end{cases}$ (f) $\begin{cases} x = 1 - \cos 2\theta, \\ y = 1 + \cos 2\theta, \\ z = 2 \sin \theta. \end{cases}$

(b) $\begin{cases} x = 5 \cos \theta, \\ y = 5 \sin \theta, \\ \pi z = \theta. \end{cases}$ (g) $\begin{cases} x = 3t - t^3, \\ y = 3t^2, \\ z = 3t + t^3. \end{cases}$

(c) $\begin{cases} \pi x = \theta \cos \theta, \\ \pi y = \theta \sin \theta, \\ \pi z = \theta. \end{cases}$ (e) $\begin{cases} x = \dfrac{4 + 2t^2}{1 + t^2}, \\ y = \dfrac{3}{t}, \\ z = \dfrac{4 + 2t}{1 + t^2}. \end{cases}$

4. Draw or describe the surfaces represented parametrically by the following sets of equations. Locate the curves of constant u and the curves of constant v.

(a) $\begin{cases} x = u + v, \\ y = u - v, \\ z = 4. \end{cases}$ (d) $\begin{cases} x = u \cos v, \\ y = u \sin v, \\ z = u^2. \end{cases}$ (g) $\begin{cases} x = \dfrac{u + v}{1 + uv}, \\ y = \dfrac{1 - uv}{1 + uv}, \\ z = \dfrac{u - v}{1 + uv}. \end{cases}$

(b) $\begin{cases} x = u + 2v, \\ y = 2u + 3v, \\ z = 3u + 4v. \end{cases}$ (e) $\begin{cases} x = u \cos v, \\ y = u \sin v, \\ \pi z = v. \end{cases}$

(c) $\begin{cases} x = u + v, \\ y = u - v, \\ z = uv. \end{cases}$ (f) $\begin{cases} x = u \cos v, \\ y = u \sin v, \\ z = \sin (2v). \end{cases}$ (h) $\begin{cases} x = 4 \cos v, \\ y = (3 - u) \sin v, \\ z = u. \end{cases}$

5. Find the equation of the sphere with radius 6 and center at the point $(3, -4, 2)$. *Ans.* $x^2 + y^2 + z^2 - 6x + 8y - 4z = 7$.

6. Find the equation of the right circular cylinder whose radius is 5 and whose axis is the line $\begin{cases} y = 4, \\ z = -3. \end{cases}$

7. Find the equation of the parabolic cylinder whose elements are parallel to OY and whose xz-trace is a parabola with focus at $(2,0,4)$ and vertex at $(0,0,4)$.

8. A point moves so that it is always equidistant from the points P_1 $(3,-5,6)$ and P_2 $(-1,-1,2)$. Find the equation of the locus and name the surface. What relation exists between the coefficients of x, y, and z in the equation and the direction numbers of the line P_1P_2?

9. A point moves so that it is always twice as far from the origin as from the point $(2,1,-2)$. Find the equation of the locus. Draw and name the surface. *Ans.* $3x^2 + 3y^2 + 3z^2 - 16x - 8y + 16z + 36 = 0.$

10. A point moves so that it is always equidistant from the point $(0,0,6)$ and the xy-plane. Find the equation of the surface on which it lies (paraboloid of revolution).

11. Find the equation of the cone whose axis of revolution is the x-axis, whose vertex is the origin and whose semivertical angle is 45°. HINT: Find the equation first in polar space coordinates. *Ans.* $x^2 - y^2 - z^2 = 0.$

12. Find parametric equations for the line with direction numbers $4, -2, 1$ and passing through the point $(1,2,3)$. HINT: If (x,y,z) is a point on this line the differences $(x - 1,\ y - 2,\ z - 3)$ are a set of direction numbers of the line.

13. What is represented in cylindrical coordinates by the equation $r = 4$?

14. Find the equation of the locus of a point which moves so that

(*a*) the square of its distance from the x-axis equals its distance from the xy-plane;

(*b*) it is equidistant from the yz-plane and the x-axis; *Ans.* $x^2 - y^2 - z^2 = 0.$

(*c*) it is twice as far from the x-axis as from the z-axis;

(*d*) it is equidistant from the point $(0,6,0)$ and the z-axis;

(*e*) the difference of the squares of its distances from OX and OY is 5;

(*f*) its distance from OZ is equal to its distance from the xy-plane;

(*g*) the sum of its distances from the three planes of reference is equal to half its distance from the origin.

15. Find the equation of the plane which contains the x-axis and passes through the point $(0,2,4)$. *Ans.* $z = 2y.$

16. Find the equation of the sphere whose center has the coordinates (h,k,l) and whose radius is a.

39. Surfaces of Revolution. We are concerned, in particular here, with a surface which is generated by revolving, about a line in its plane, some plane curve whose equation is given or can be found. Let the plane be the xy-plane. If the line about which the curve is to be revolved is not one of the coordinate axes, we may readily make it so by suitable translation and rotation of the axes. It, therefore, entails no loss of generality to deal with rotation only about OX or OY. The method is made clear by means of an example. Let the curve to be revolved be the parabola $y^2 = 4x$. Then any point Q on this curve will have coordinates which satisfy the given equation (see Fig. 78). As it revolves about OX this point Q will generate a circle whose plane is normal to OX. If (x,y,z) is a point on this circle we see from the figure, that $y^2 + z^2 = r^2$ and $CP = CQ = r$. Hence the coordinates of Q in the xy-plane are (x,r), and $r^2 = 4x$.

Since $r^2 = y^2 + z^2$, we have the equation

$$y^2 + z^2 = 4x,$$

which is satisfied by the coordinates (x,y,z) of every point P on

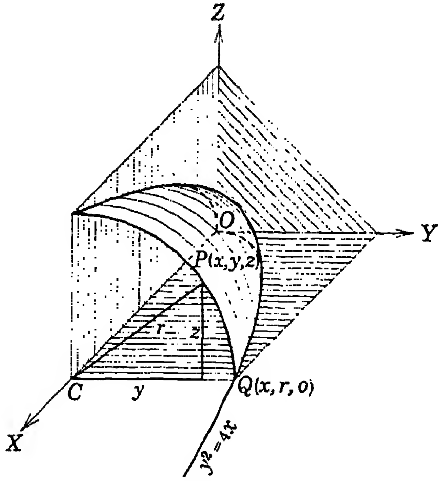
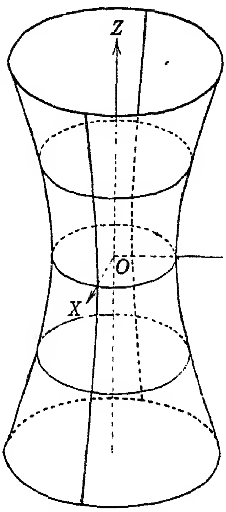


FIG. 78.

the surface. In general we may form the rule, *to find the equation of the surface obtained by revolving a given curve in the xy-plane about OX, change y to $\sqrt{y^2 + z^2}$ in the equation of the curve.*



Unparted Hyperboloid

FIG. 79.

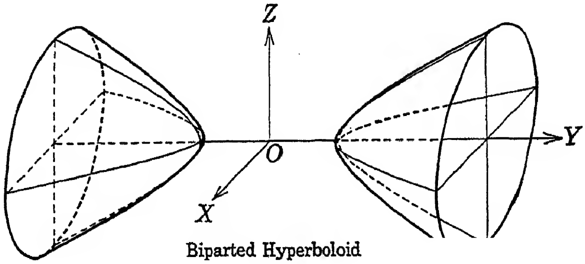
Similar rules may readily be given in the case of rotation about OY, as well as for rotation about the coordinate axes of curves in planes other than the xy-plane, but the student is encouraged to draw a figure for each case and obtain the desired equation by an argument similar to the one just used.

We state here certain exercises and exhibit certain figures which will be useful in the section on quadric surfaces appearing next.

Exercise 1. The *oblate spheroid* is the surface generated by revolving an ellipse about its *minor* axis. If the ellipse has semimajor axis a and semiminor axis b, find the equation of the oblate spheroid it generates for the two cases, first, when the vertices are on OX and, second, when the vertices are on OY. Draw the figures. *Ans.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1$, $\frac{x^2}{b^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} = 1$.

Exercise 2. The *prolate spheroid* is the surface generated by revolving an ellipse about its *major* axis. Find its equation for the ellipses of Exercise 1, and draw the figures. *Ans.* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1$, $\frac{x^2}{b^2} + \frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$.

Exercise 3. If a hyperbola is revolved about its *imaginary* axis, the surface generated is called an *unparted hyperboloid of revolution.* Find its



Biparted Hyperboloid

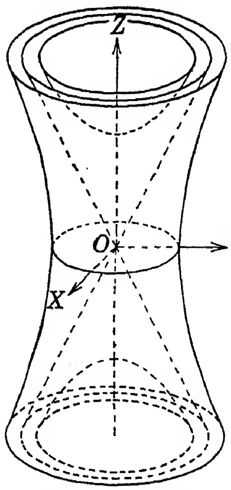
FIG. 80.

equation if the hyperbola is in the yz-plane, if its real axis is of length $2a$ along the y-axis and its imaginary axis is of length $2b$.

Ans. $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 1$.

Exercise 4. If a hyperbola is revolved about its *real* axis, the surface generated is called a *biparted hyperboloid of revolution.* Find its equation if the hyperbola is in the yz-plane, with its real axis of length $2a$ along the y-axis and its imaginary axis of length $2b$.

Ans. $\frac{y^2}{a^2} - \frac{x^2}{b^2} - \frac{z^2}{b^2} = 1$.



Hyperboloids and their Asymptotic Cone

FIG. 81.

Exercise 5. If, as a hyperboloid is generated by the revolution of a hyperbola about one of its axes of symmetry, the asymptotes are allowed to revolve, the cone generated is said to be *asymptotic* to the hyperboloid. Show that the asymptotic cone to either of the hyperboloids

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} \quad \pm 1$$

has the equation $\frac{x^2}{a^2} + \frac{y^2}{a^2} - \frac{z^2}{b^2} = 0$.

Exercise 6. If a parabola is revolved about its transverse axis, the surface generated is called a *paraboloid of revolution.* Find the equation of the paraboloid generated by revolving the parabola $y^2 = 2px$ about the x-axis. *Ans.* $z^2 + y^2 = 2px$.

Problems

1. Find the equation of the surface generated by revolving the given curve about the given line in each case. Draw the figure and name the surface if possible.

(a) $x - 2y = 0$: the x-axis.
(b) $y + 3x = 0$: the y-axis.
(c) $z + y = 1$: the y-axis.
(d) $2x - y = 3$: the x-axis.
(e) $y^2 = 2x - 8$: OX.
(f) $x^2 + y^2 + ax = 0$: OY.
(g) $(x - a)^2 + y^2 = b^2$, $a > b$: OY.
(h) $\frac{x^2}{16} + \frac{y^2}{9}$ 1: the x-axis.
(i) $\frac{x^2}{16} - \frac{y^2}{9}$ 1: the y-axis.
(j) $\frac{x^2}{16} - \frac{y^2}{9} = 1$: the x-axis.
(k) $\frac{(x - 3)^2}{9} + \frac{y^2}{4} = 1$: OX.
(l) $x^{2/3} + y^{2/3} = 1$: OY.
(m) $x^{1/2} + y^{1/2} = 1$: OY.
(n) $\frac{x^2}{144} - \frac{(y + 2)^2}{81} = 1$: OY.
(o) $y^2 = x^3$: the x-axis.
(p) $y^2 = z$: the y-axis.
(q) $y = e^x$: the x-axis.
(r) $y = 1 + \sin x$: OX.

Ans. (a) $x^2 - 4y^2 - 4z^2 = 0$. (b) $y^2 - 9x^2 - 9z^2 = 0$.
(c) $x^2 + z^2 - y^2 + 2y = 1$.

2. Find the equation of the surface generated by rotating the line $x - y = 0$ about the line $3x - 4y + 5 = 0$. KEY: By the transformation $\begin{Bmatrix} x' = \dfrac{4x + 3y}{5} \\ y' = \dfrac{3x - 4y + 5}{-5} \end{Bmatrix}$, combining the rotation $\begin{Bmatrix} x'' = \dfrac{4x + 3y}{5} \\ y'' = \dfrac{-3x + 4y}{5} \end{Bmatrix}$ with the subsequent translation $\begin{Bmatrix} x' = x'' \\ y' = y'' - 1 \end{Bmatrix}$, the line $x - y = 0$ becomes

$$x' - 7y' - 7 = 0$$

and the line $3x - 4y + 5 = 0$ becomes $y' = 0$. Hence the equation of the surface becomes

$$x' - 7\sqrt{y'^2 + z'^2} - 7 = 0,$$

which reduces to

$$(x' - 7)^2 = 49y'^2 + 49z'^2.$$

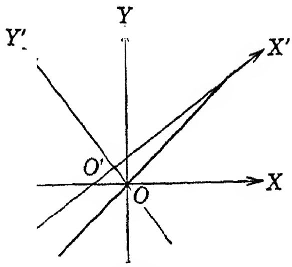


FIG. 82.

We may now return to the original x- and y-axes in the xy-plane by applying the transformation

$$x' = \frac{4x + 3y}{5},$$
$$y' = \frac{3x - 4y + 5}{-5},$$
$$z' = z$$

to the last equation, which gives the final result

$$17x^2 + 31y^2 + 49z^2 - 48xy + 70x - 70y = 0.$$

3. Find the equation of the surface generated by revolving the given curve about the given line in each case. Draw a figure.

(a) $x^2 + y^2 = 9$: the line $x = 5$.
(b) $y^2 = 4x$: its directrix.

Ans. $y^4 + 8y^2 - 16x^2 - 32x - 16z^2 = 0$.

(c) $3x - y + 5 = 0$: the line $y = -2$.
(d) $xy = 12$: the line $x = y$.
(e) $x^{2/3} + y^{2/3} = a^{2/3}$: the line $x = y$.
(f) $x^{1/2} + y^{1/2} = a^{1/2}$: the line $x = y$.

4. Prove that the locus of a point, which moves in space in such a way that the sum of its distances from $(0,0,3)$ and $(0,0,-3)$ is always equal to 10, is a prolate spheroid, by finding the equation of the locus.
Ans. $25x^2 + 25y^2 + 16z^2 = 400$.

5. Find the equation of the locus of a point which moves in space in such a way that the difference of its distances from $(0,5,0)$ and $(0,-5,0)$ is always equal to 6. Show from the equation that the surface is a biparted hyperboloid. *Ans.* $16y^2 - 9x^2 - 9z^2 = 144$.

6. A point moves so that its distance from the point $(6,0,0)$ divided by its distance from the yz-plane is always a positive constant e. Find the equation of the locus of the point and draw the surface for each of the three cases $e < 1$, $e = 1$, and $e > 1$. Name the surfaces.

7. A point moves so that its distance from the xz-plane is equal to its distance from the y-axis. Find the equation of the locus and name the surface.

8. Find the equation of a point which moves so that the product of its distances from $(4,0,0)$ and $(-4,0,0)$ is equal to a constant, k^2. How can the resulting locus be generated as a surface of revolution? See Ovals of Cassini, page 97.

40. Quadric Surfaces. A *quadric surface* is defined as one whose equation, in rectangular coordinates, is of the form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Kz + L = 0,$$

where A, B, C, D, E, F are not all zero, *i.e.*, an equation of the *second degree*. Certain of such surfaces have already been met with in the preceding sections, including cylinders whose cross sections are conics, and special surfaces of revolution, *viz.*, spheres, spheroids, cones, hyperboloids, and paraboloids.

In a brief treatment such as this, space cannot be found to classify completely all quadric surfaces and show how to identify a given equation of the above form as that of a quadric of one of those classes. In fact, we shall restrict such discussion as we do give to cases in which the coefficients D, E, and F are all zero, cases, then, in which the equation of the quadric is of the form

$$Ax^2 + By^2 + Cz^2 + Gx + Hy + Kz + L = 0. \tag{63}$$

Consider first those cases in which some two of the three coefficients A, B, and C are of like sign. This, of course, includes

all cases in which no one of the three is zero. Suppose that A and B, for example have like signs. We can write Eq. (63) in the form

$$A(x - m)^2 + B(y - n)^2 + Cz^2 + Kz + P = 0,$$

where the constants m, n, and P result from completing the squares for the terms in x and those in y. Let, now, the axes be translated to the new origin $(m,n,0)$. The equations for this transformation are

$$\begin{cases} x = x' + m \\ y = y' + n \\ z = z' \end{cases}$$

and the equation becomes

$$Ax'^2 + By'^2 + Cz'^2 + Kz' + P = 0. \tag{64}$$

To determine the surface represented by this equation, consider, alongside it, the equation

$$Ax'^2 + Ay'^2 + Cz'^2 + Kz' + P = 0. \tag{65}$$

It is readily noticeable that this equation represents a surface of revolution, and in particular, the surface generated by revolving about OZ' the curve, whose equation, in the $y'z'$-plane, is

$$Ay'^2 + Cz'^2 + Kz' + P = 0.$$

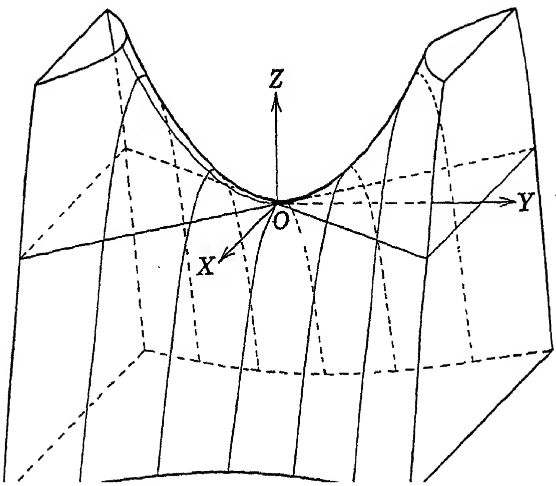
Now this equation is of the second degree and therefore represents a conic or one of its limiting forms such as one or two straight lines, if it has a locus. Furthermore, this curve has the z'-axis as an axis of symmetry, so that its revolution about that axis can only generate a sphere, spheroid, hyperboloid (unparted or biparted), paraboloid or cone, as discussed in Sec. 39, or a limiting form of one of those, including a point (on OZ'), a line (OZ') or one or two planes (perpendicular to OZ').

In order to conclude, now, what is represented by Eq. (64), in these cases, the student is asked, first, to work the following exercise:

Exercise 1. Prove that if the ordinate of a curve D bears a constant ratio to the ordinate of a circle with center at the origin, the ordinates being taken to correspond to the same abscissa, the curve D is an ellipse having

the coordinate axes as axes of symmetry and eccentricity dependent only upon the given ratio of ordinates.

Returning now to Eq. (64), we note that it represents a surface like the surface of revolution represented by (65), except that, corresponding to an assigned set of values of x' and z', the two equations are satisfied by values of y' which are in the ratio $(A/B)^{1/2}$. Hence, by the above exercise, the surface (64) differs from the surface of revolution (65) in that its sections



Hyperbolic Paraboloid
FIG. 83.

z' = constant, are all ellipses with transverse axes parallel and with the same eccentricity. We call such surfaces by the terminology already introduced, *viz.*, *hyperboloids, unparted and biparted, paraboloids and cones*, except that they are modified as *elliptic* when not surfaces of revolution. When the equation (65) represents a spheroid and (64) is not a surface of revolution, it is called an *ellipsoid*.

The cases of Eq. (63) not included in the above discussion reduce by suitable translations and choice of letters, to one or the other of the cases

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = cz \tag{66}$$

and

$$z^2 = ax + by. \tag{67}$$

If $c \neq 0$, the surface represented by (66) is called a *hyperbolic paraboloid.* It is shown in Fig. 83, but the details of its discussion are left to the student in the exercises below. Also we leave to the student the discussion of (66) in the event that $c = 0$ as well as the discussion of Eq. (67), which represents a *parabolic cylinder.*

Exercise 2. Show that sections of the surface $\frac{y^2}{a^2} - \frac{x^2}{b^2} = cz$, where $c \neq 0$, parallel to the xz-plane, are parabolas, and find the locus of their vertices; that sections parallel to the yz-plane are parabolas, and find the locus of their vertices; that sections parallel to the xy-plane are hyperbolas, and find the locus of their vertices.

Exercise 3. If a and b are not both zero, show that sections of the surface represented by the equation $z^2 = ax + by$ by a plane parallel to either the yz-plane or the xz-plane are parabolas. Find the locus of the vertices of these parabolas. Draw a figure showing the surface.

Problems

1. In each case below, find the traces in the coordinate planes, of the surface whose equation is given. Name the surface and show it in a figure.

(a) $4x^2 + y^2 + 9z^2 = 36.$

(b) $4x^2 + y^2 - 9z^2 = 36.$

(c) $4x^2 - y^2 - 9z^2 = 36.$

(d) $4x^2 - y^2 - 9z^2 = 0.$

(e) $\frac{(x-1)^2}{9} + \frac{(y-2)^2}{16} + \frac{(z+3)^2}{4} = 1.$

(f) $\frac{(x+1)^2}{9} + \frac{(y-3)^2}{16} - \frac{(z-5)^2}{4} = 1.$

(g) $\frac{x^2}{9} - \frac{y^2}{16} - \frac{(z+2)^2}{4} = 1.$

(h) $\frac{(x-5)^2}{9} + \frac{(y+4)^2}{16} - \frac{z^2}{25} = 0.$

(i) $\frac{x^2}{25} + \frac{y^2}{36} = 4z.$

(j) $\frac{x^2}{36} - \frac{y^2}{25} = 9z.$

(k) $x^2 = 2y + 4z.$

(l) $\frac{(x-1)^2}{4} + \frac{(y+3)^2}{9} = z + 6.$

(m) $\frac{(z+2)^2}{4} - \frac{(y+8)^2}{9} = 8 - 4x.$

(n) $(y+5)^2 = 4x - 3z + 2.$

(o) $\frac{y^2}{36} + \frac{z^2}{25} = 4.$

(p) $\frac{y^2}{9} - \frac{x^2}{16} = 25.$

(q) $y^2 = 6x.$

(r) $4x^2 + y^2 + 9z^2 = 0.$

2. Is the surface, generated by rotating a parabola about a line in its plane parallel to its directrix, a quadric surface?

3. Given the equation $xy = kz$, where k is constant, (a) draw the surface by studying plane sections of it by planes perpendicular to OZ and planes containing OZ; (b) rotate the axes in the xy-plane through the angle 45° and name the surface by identifying the result with one of the equations of the text.

41. (a) Planes. The planes thus far encountered in the text and problems have been parallel to at least one of the coordinate axes. We seek, now, the equation of a plane in any position whatever. Suppose that a plane passes through a fixed point (x_0,y_0,z_0) and its direction is defined as perpendicular to a line with the direction numbers A, B, C. If any point on the plane has the coordinates (x,y,z), the line which joins it to the point (x_0,y_0,z_0) has the direction numbers $x - x_0$, $y - y_0$, $z - z_0$, and, since this line is perpendicular to a line with direction numbers A, B, C, we have

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \tag{68}$$

as the equation sought. Since the direction numbers, A, B, C of a line cannot all three be zero, this equation is of the *first degree* in x, y, and z.

Consider, now, any given equation of the first degree, as

$$Ax + By + Cz + D = 0,$$

where, since the first degree is given, the coefficients A, B, and C are not all three zero. If any one of these coefficients is zero, at least one of the variables, x, y, and z, is missing from the equation, which, in view of the discussion on cylinders in Sec. 37, represents a plane parallel to at least one of the coordinate axes. If no one of A, B, and C is zero, the equation can certainly be written in the form

$$A(x - 0) + B(y - 0) + C\left(z + \frac{D}{C}\right) = 0.$$

This equation is the one which we should obtain if we set out to find the equation of the plane passing through the point $(0,0,-D/C)$ perpendicular to a line whose direction numbers are A, B, C. We conclude, therefore, that every equation of the *first degree* in x, y, and z represents a plane.

Of especial interest and use is a form of the equation of a plane known as the *normal form.* Let λ,μ,ν be the set of direction cosines of the line through the origin perpendicular to the plane, selected as follows:

First, if the plane does not pass through the origin, λ,μ,ν are the direction cosines of the directed segment *from* the origin *to* the plane.

Second, if the plane passes through the origin, then ν is taken positive unless it is zero.

Third, if the plane passes through the origin and $\nu = 0$, then μ is taken as positive, unless it also is zero.

Fourth, if the plane passes through the origin and $\mu = \nu = 0$, then $\lambda = 1$.

We now define the quantity p, known as the *normal intercept* of the plane, as a number, positive or zero, such that the point $(p\lambda,p\mu,p\nu)$ is on the plane in question, which quantity always exists and is unique. Manifestly, this point is p units from the origin and is the projection of the origin upon the plane.

In terms of these quantities p, λ, μ, ν, *i.e.*, of the normal intercept of the plane and its direction cosines, the equation of the plane takes the form

$$\lambda(x - p\lambda) + \mu(y - p\mu) + \nu(z - p\nu) = 0,$$

which reduces immediately to the form

$$\lambda x + \mu y + \nu z - p = 0,$$

known as the *normal form* of the equation of a plane.

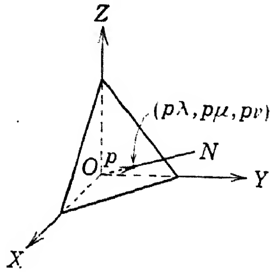


FIG. 84.

The above definitions of λ, μ, ν, and p, and the student's former experience with the normal form of the equation of a line in a plane, suggest at once that, in order to reduce the equation

$$Ax + By + Cz + D = 0$$

to normal form, we must divide by $\pm\sqrt{A^2 + B^2 + C^2}$, with the sign of the radical chosen:

opposite to the sign of D if $D \neq 0$;
the same as the sign of C if $D = 0$ and $C \neq 0$;
the same as the sign of B if $D = C = 0$ and $B \neq 0$;
the same as the sign of A if $D = C = B = 0$.

To illustrate, we exhibit here a few equations of planes before and after reduction to normal form

Given Equation	Normal Form
$2x - y - 2z + 12 = 0$	$-\frac{2}{3}x + \frac{1}{3}y + \frac{2}{3}z - 4 = 0$
$4x + y - 8z = 0$	$-\frac{4}{9}x - \frac{1}{9}y + \frac{8}{9}z = 0$
$3x + 4y = 0$	$\frac{3}{5}x + \frac{4}{5}y = 0$
$2x = 0$	$x = 0$
$2x + 5 = 0$	$-x - \frac{5}{2} = 0$

(b) Systems of Planes. If the planes π_1 and π_2, represented be the equations $A_1x + B_1y + C_1z + D_1 = 0$ and

$$A_2x + B_2y + C_2z + D_2 = 0,$$

respectively, are not parallel, the equation

$$A_1x + B_1y + C_1z + D_1 + k(A_2x + B_2y + C_2z + D_2) = 0 \quad (69)$$

is of the first degree, regardless of the value of the constant k. It is obviously satisfied by the coordinates (x,y,z) of any point which is on both the planes π_1 and π_2 and hence is the equation of a plane which passes through the line of intersection of the two. Moreover, by suitable choice of k we may obtain any one of the family of planes through that line, except the plane π_2. The details of proof of these statements as well as discussions of the exceptional cases are left to the student in the following exercises.

Exercise 1. Prove that if the two planes given above are not parallel, Eq. (69) is of the first degree. HINT: If the planes are not parallel the coefficients $A_1B_1C_1$ are not proportional to the coefficients A_2,B_2,C_2.

Exercise 2. If the two planes whose equations are given above are parallel but distinct, prove that Eq. (69) is of the first degree for all but one value of k.

Exercise 3. Under the conditions of Exercise 2 prove that any plane represented by Eq. (69) is parallel to the given planes.

Exercise 4. If the two planes whose equations are given above are coincident, prove that all planes represented by (69) coincide with them.

Exercise 5. If the two planes whose equations are given above are not coincident, prove that one of the planes represented by (69) passes through any given point which is not on π_2.

42. Distance from a Plane to a Point. Let a given plane π have an equation which reduces to the normal form

$$\lambda x + \mu y + \nu z - p = 0$$

and let (x_1,y_1,z_1) be the coordinates of a given point P. If we rotate the axes about the origin so that the new x-axis, OX' has the direction cosines λ, μ, ν, we have the relation (see page 114)

$$x' = \lambda x + \mu y + \nu z.$$

Hence, referred to the new axes, the plane π has the equation

$$x' - p = 0,$$

and, if the new coordinates of the point P are (x_1',y_1',z_1')

$$x_1' = \lambda x_1 + \mu y_1 + \nu z_1.$$

Evidently, it is possible for x_1' to be greater than, equal to, or less than p so that the quantity

$$x_1' - p$$

may be positive, zero, or negative. From the very nature of a system of rectangular coordinates we observe that the numerical value of this quantity measures the distance of the point P_1 (x_1',y_1',z_1') from the plane π, $x' - p = 0$, whose every point has the first coordinate $x_1' = p$. We note, also, that:

If $x_1' - p > 0$ the point P_1 is in the direction OX' from the plane π;

If $x_1' - p < 0$ the point P_1 is in the direction opposite to OX' from the plane π.

We may, therefore, adopt the quantity $x_1' - p$ as a measure of the *directed distance from* the plane π, $x' - p = 0$, *to* the point $P(x_1',y_1',z_1')$, directed in the same way as the axis OX'.

Interpreting, now, in terms of the original coordinate system, we conclude that *if $\lambda x + \mu y + \nu z - p = 0$ is the normal form of the equation of a plane, the directed distance from that plane to a point (x_1,y_1,z_1), is given by the formula*

$$d = \lambda x_1 + \mu y_1 + \nu z_1 - p.$$

Problems

1. Find the equation of the plane

(*a*) through the point $(5,-4,2)$ with $(5,6,2)$ as direction numbers of its normal. *Ans.* $5x + 6y + 2z = 5$.

(*b*) through $(2,1,-5)$ perpendicular to the line joining $(-4,2,2)$ and $(3,-1,7)$.

(*c*) with intercepts, 2 on OX, 5 on OY, and -6 on OZ. *Ans.* $15x + 6y - 5z = 30$.

(*d*) through the points $(2,3,0)$, $(0,6,2)$ and $(5,0,1)$. *Ans.* $9x + 8y - 3z = 42$.

(*e*) through the points $(4,3,-9)$, $(2,1,\frac{1}{2})$, and $(7,-9,-6)$.

(*f*) through $(5,3,1)$ and having equal intercepts on the coordinate axes. *Ans.* $x + y + z = 9$.

(*g*) through $(5,3,1)$ whose normal has its three direction angles equal.

(*h*) through $(0,3,2)$ and $(2,-1,1)$ and parallel to OX. *Ans.* $y - 4z + 5 = 0$.

(i) through $(3,6,-1)$ and parallel to the plane $3x - 6y + 5z = 0$.

(j) 2 units from the origin and perpendicular to a line whose direction numbers are $(8,4,1)$. *Ans.* $8x + 4y + z \pm 18 = 0$.

(k) 3 units from the origin and parallel to the plane

$$x - 2y + 2z = 10.$$

(l) tangent at the point $(6,-3,-2)$ to a sphere with center at the origin.

(m) through the points $(1,1,1)$ and $(2,-1,-3)$ and perpendicular to the plane $2x - 3y - 5z = 0$.* *Ans.* $2x + 3y - z = 4$.

(n) through the point $(0,2,3)$ and perpendicular to the planes $x - y - z + 1 = 0$, and $8x + 4y - 2z = 7$.

Ans. $x - y + 2z - 4 = 0$.

(o) bisecting, at right angles, the line segment joining $(4,3,-6)$ and $(0,5,2)$.

(p) perpendicular, at $(-3,-1,0)$, to a line segment from $(5,3,-1)$.

(q) parallel to the plane $2x - 6y + 3z + 10 = 0$ and such that the point $(1,2,3)$ is midway between the two planes.

(r) parallel to the plane $x + 2y + 2z = 0$ and making with the coordinate planes a tetrahedron of unit volume.

2. Reduce each of the given equations to normal form. Show the plane by a figure and find the distances from it to the given points.

(a) $2x - y + 2z = 6$; $(2,-4,2)$, $(0,3,-1)$. *Ans.* 2, $-1\frac{1}{3}$.

(b) $4x - 3z + 10 = 0$; $(1,-2,6)$, $(2,3,5)$.

(c) $3x - 6y + 2z + 21 = 0$; $(4,2,-6)$, $(-3,3,1)$.

(d) $3x + 4y - 12z = 0$; $(4,3,4)$, $(5,6,3)$. *Ans.* $24/13$, $-3/13$.

(e) $4x - 4y + 2z + 9 = 0$; $(1,5,2)$, $(2,3,1)$.

(f) $5x - 12z + 26 = 0$; $(8,2,1)$, $(8,2,6)$.

(g) $12x - 5y = 0$; $(4,4,4)$, $(1,3,5)$.

3. Find the distance between the two parallel planes

(a) $3x - 4y + 12z = 5$, $4y = 3x + 12z + 8$. *Ans.* 1.

(b) $2x - y + 2z = 6$, $2x - y + 2z = 2$.

(c) $8x - 4y + z - 7 = 0$, $8x - 4y + z + 11 = 0$.

(d) $4x - 3y = 6$, $8x - 6y - 27 = 0$.

4. Find the equation of the locus of a point which moves so that

(a) its distance from the plane $3x - 6y + 2z - 28 = 0$ is always equal to its distance from the origin.

(b) its distance from the plane $x - 2y + 2z - 12 = 0$ is equal to its distance from the line through the point $(1,0,0)$ with direction cosines $(0,0,1)$.

(c) the sum of the squares of its distances from the three planes $x + y + z = 0$, $x - z = 0$, $x - 2y + z = 0$ is always equal to 9;

Ans. $x^2 + y^2 + z^2 = 9$.

(d) the sum of the squares of its distances from the two planes $x + y + z = 0$ and $x - 2y + z = 0$ is equal to the square of its distance from the plane $x - z = 0$.

---

* Note that the angle between two planes is equal to the angle between their normals.

5. If a, b, and c are each different from zero, find the intercepts on the coordinate axes of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

6. Express the normal intercept p of a plane not passing through the origin, in terms of its intercepts, a, b, and c.

7. The xy-trace and xz-trace of a plane have the equations $\left\{\begin{matrix} z = 0 \\ x - 2y = 4 \end{matrix}\right\}$ and $\left\{\begin{matrix} y = 0 \\ 2x + z = 8 \end{matrix}\right\}$, respectively. Find the equation of the plane.

Ans. $2x - 4y + z = 8$.

8. Find the angle between the planes in each of the following cases. HINT: See footnote to Prob. 1(m).

(a) $2x + y - z = 9$, $x - 2y + 2z = 6$. *Ans.* $\cos^{-1} \frac{2}{3\sqrt{6}}$.

(b) $4x - 7y + 4z = 6$, $3x + 4y = 0$. *Ans.* $\cos^{-1} 16/45$.

(c) $2x + 3y + 6z = 14$, $3x + 4y + 12z = 13$.

9. Determine k so that the plane $x - ky + 2kz = 18k$ shall

(a) be 8 units from the origin. *Ans.* $k = \pm 4$.

(b) be parallel to the plane $2x + y - 2z = 5$. *Ans.* $k = -1/2$.

(c) be perpendicular to the plane $3x - 3y - 2z = 1$.

(d) pass through the point $(4,-8,6)$.

(e) pass through the points $(0,-4,7)$ and $(0,-8,5)$.

10. For what value of k will the equations $kx - y + 2z = 6$ and $9x + ky - 6z + 18 = 0$ represent perpendicular planes?

11. Find the equations of the planes which bisect the dihedral angle between the two given planes, in each case below. HINT: Points on such a plane are equally distant from the two given planes.

(a) $x - 2y + 2z - 1 = 0$, $8x - 4y - z - 12 = 0$.

Ans. $5x + 2y - 7z - 9 = 0$, $11x - 10y + 5z - 15 = 0$.

(b) $3x - 4y - 10 = 0$, $5x - 14y + 2z = 0$.

(c) $3x + 4y + 12z = 0$, $12x - 5z = 0$.

12. Write the equation of the system of planes which pass through the line of intersection of the planes $2x + 4y + z - 3 = 0$ and

$$x + 2y + z - 1 = 0.$$

Find the plane in the system which

(a) passes through the point $(1,1,2)$. *Ans.* $3x + 6y - z = 7$.

(b) is parallel to the z-axis. *Ans.* $x + 2y - 2 = 0$.

(c) is perpendicular to the plane $4x + 4y + 5z = 10$.

(d) is one unit from the origin (two answers).

13. Write the equation of the system of planes passing through the line of intersection of the planes $6x - 7y - z - 24 = 0$ and

$$8x - 26y + 7z - 42 = 0.$$

Find the plane of the system which

(a) passes through the origin; *Ans.* $2x + 11y - 7z = 0$.

(b) is parallel to the y-axis;

(c) is perpendicular to the plane $3x + 2y + 4z = 6$;
(d) parallel to the plane $4x - 18y + 6z + 7 = 0$;
(e) tangent to a sphere with center at the origin and radius equal to 2. *Ans.* $2x + y - 2z - 6 = 0$, and $4x - 8y + z - 18 = 0$.

14. Show that the planes $2x - 3y + z - 4 = 0$, $3x + y + 2z - 5 = 0$, $x + 15y + 2z + 1 = 0$ have a line in common.

15. Show that the planes $x - y - 2z + 3 = 0$, $2x + y + z - 3 = 0$, $x - 2y + z - 8 = 0$, and $3x + 3y + 2z - 3 = 0$ have a point in common.

16. What is represented by the equation

$$(4x - y + 3z - 1)(2x + 3y - z - 6) = 0?$$

17. The edge, through two vertices, of a tetrahedron with vertices P_1, P_2, P_3, and P_4, is bisected by a plane perpendicular to the edge through the other two vertices. Prove, analytically, that the six such planes possible have a point in common.

18. Prove that the four points (x_1,y_1,z_1), (x_2,y_2,z_2), (x_3,y_3,z_3), and (x_4,y_4,z_4) lie in the same plane if, and only if, the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

has the value zero.

43. Straight Lines. Let a given straight line have direction numbers l,m,n and pass through the point (x_0,y_0,z_0). If the coordinates of any other point on the line are represented by (x,y,z), it follows, from the definition of direction numbers, that the set $x - x_0$, $y - y_0$, $z - z_0$ is proportional to the set l,m,n. We portray this property by the equations

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n}, \tag{70}$$

if no one of the numbers l,m,n is zero. These equations are satisfied by every point (x,y,z) on the line and by no other sets (x,y,z), and hence may be taken as equations for the line. They are known as the *symmetric form* of the equations of the line. The form is impossible if any one of the numbers l,m,n is zero, although we may use the form, even then, if we follow the practice of interpreting a formal equality in which one member is $a/0$ as equivalent to the equality, $a = 0$.

If we set each of the three members of the set of equations in (70) equal to t and clear of fractions in each case, we obtain

$$\begin{cases} x = x_0 + lt \\ y = y_0 + mt \\ z = z_0 + nt, \end{cases}$$

a parametric representation for the line, since, manifestly, t is not constant.

It is obvious that a line may also be represented as a pair of simultaneous linear equations, the equations of any two distinct planes passing through the line. While this method of representation is widely used, it presents some difficulty due to the multiplicity of such representations for a single line. Through any one line pass infinitely many distinct planes, any two of which determine the line. No property of the line, such as its direction numbers or the coordinates of one of its points, is recognizable at the sight of the two equations of those planes. Hence, given two such pairs of first-degree equations, it is not easy, in general, to detect whether or not they represent the same line.

Consider, for example, the straight lines represented by

$$\begin{cases} x - y + z - 1 = 0 \\ 2x + y + z - 3 = 0 \end{cases} \quad \text{and} \quad \begin{cases} 4x - y + 3z - 5 = 0 \\ 5x - 2y + 4z - 6 = 0. \end{cases}$$

Since, in each case, the two given planes are nonparallel, each pair represents some line. To find where the first line is located, we eliminate z between the two equations. This gives the equation $x + 2y - 2 = 0$, which represents a plane through the given line and parallel to OZ. It is, indeed the plane by which the given line is projected upon the xy-plane. Similarly, the line is projected upon the xz-plane by the plane whose equation, $3x + 2z - 4 = 0$, is found by eliminating y between the given equations. These planes are shown in Fig. 85. Their intersection is the given line.

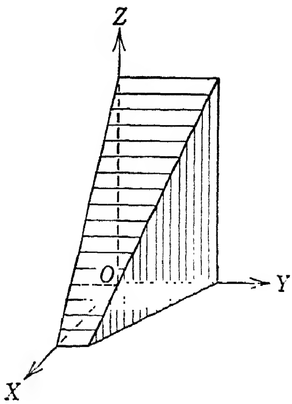


FIG. 85.

If we turn, now, to the second pair of equations given above and eliminate first z and then y, we obtain, again the equations $x + 2y - 2 = 0$ and $3x + 2z - 4 = 0$, thus proving that the two given pairs of equations represent the same line.

To find direction numbers for a line which is represented by two simultaneous equations, we may find first two points whose coordinates satisfy both equations and find the desired direction numbers from these coordinates. An alternative procedure is to reduce the equations to the symmetric form as in the following example. Let us begin by finding the equations of planes through the given line and parallel to two of the coordinate axes. Using the case of the last paragraph, this gives

$$x + 2y - 2 = 0, \qquad 3x + 2z - 4 = 0.$$

Solving the first for x in terms of y and the second for x in terms of z, we have

$$x = -2y + 2, \qquad x = \frac{-2z + 4}{3},$$

or

$$\frac{x}{1} = \frac{-2y + 2}{1} = \frac{-2z + 4}{3},$$

which can further be reduced to the form

$$\frac{x}{1} = \frac{y - 1}{-\frac{1}{2}} = \frac{z - 2}{-\frac{3}{2}},$$

or, finally,

$$\frac{x}{2} = \frac{y - 1}{-1} = \frac{z - 2}{-3}.$$

This last, being in the symmetric form, shows that a set of direction numbers for the line is $(2, -1, -3)$ and that the line passes through the point $(0,1,2)$.

Problems

1. Show each of the following lines by a figure and find the equations of the planes by which it is projected upon the coordinate planes. Find the direction numbers for each line.

(a) $\begin{cases} 3x - 2y + 5z - 30 = 0, \\ 2x + 3y - 10z - 6 = 0. \end{cases}$ *Ans.* $(5,40,13)$.

(b) $\begin{cases} 2x + 2y - 3z - 5 = 0, \\ x - 3y + z - 6 = 0. \end{cases}$

(c) $\begin{cases} x + 2y + 2z + 2 = 0, \\ x - 4y - z - 1 = 0. \end{cases}$

(d) $\begin{cases} x - 2y - 3z + 6 = 0, \\ x + y + z - 1 = 0. \end{cases}$

2. Show each of the following lines in a figure and find the coordinates of the points in which it pierces the coordinate planes.

(a) $\dfrac{x-1}{2} = \dfrac{y+3}{-2} = \dfrac{z-2}{-1}$.

(b) $\dfrac{x-6}{9} = \dfrac{y-8}{8} = \dfrac{z+4}{16}$.

(c) $\dfrac{x-2}{3} = \dfrac{y+1}{3} = \dfrac{z}{-2}$.

(d) $\dfrac{x-3}{4} = \dfrac{y-7}{2} = \dfrac{z-3}{-3}$.

(e) $\begin{cases} x = 5 - 4t, \\ y = 2 + 3t, \\ z = -1 - t. \end{cases}$

Ans. $\begin{cases} (9,-1,0) \\ (23/3, 0, -1/3) \\ (0, 23/4, -9/4) \end{cases}$

(f) $\begin{cases} x = 1 + 6t, \\ y = 3 - 3t, \\ z = -4 + 2t. \end{cases}$

3. Find, in the symmetric form, equations for the line passing

(a) through the point (3,2,5) with direction numbers $(1,-1,4)$;

(b) through the point $(2,-3,1)$ with direction numbers $(4,2,-5)$;

(c) through the points (2,1,6) and (3,5,7);

Ans. $\dfrac{x-2}{1} = \dfrac{y-1}{4} = \dfrac{z-6}{1}$.

(d) through the points (1,2,1) and $(5,-1,1)$;

(e) through the point $(4,-3,2)$ with direction cosines, $\cos\alpha = \frac{1}{2}$, $\cos\beta = -\frac{1}{4}$;

(f) through the point $(4,-5,20)$ and perpendicular to the plane $x + 3y - 6z - 8 = 0$;

(g) through the point $(-7,5,-6)$ and parallel to the line

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{6};$$

(h) through the origin and perpendicular to lines with direction numbers (4,2,1) and $(-3,-2,1)$; *Ans.* $x/4 = y/-7 = z/-2$.

(i) through the origin and meeting the line $\dfrac{x-10}{4} = \dfrac{y}{3} = \dfrac{z}{2}$ at right angles. *Ans.* $x/13 = y/-12 = z/-8$.

4. Show that the two lines $\begin{Bmatrix} 2x - y + z - 5 = 0 \\ 4x + y - 4z + 8 = 0 \end{Bmatrix}$ and $\begin{Bmatrix} x + 7y - z + 2 = 0, \\ x + 3y + 3z - 10 = 0 \end{Bmatrix}$ meet.

5. Prove the lines of Prob. 4 to be perpendicular.

6. Prove that the line $\begin{Bmatrix} 6x + 2y + z + 6 = 0 \\ 3x - 2y + 2z + 4 = 0 \end{Bmatrix}$ is parallel to the plane $3x - 2y + 2z + 1 = 0$. HINT: Prove that the normal of the plane is perpendicular to the line.

7. Find the equation of the plane determined by the intersecting lines $\dfrac{x-2}{4} = \dfrac{y+3}{-1} = \dfrac{z+2}{3}$ and $\begin{Bmatrix} 3x + 2y + z + 2 = 0 \\ x - y + 2z - 1 = 0 \end{Bmatrix}$.

Ans. $4x + 7y - 3z + 7 = 0$.

8. Find the equation of the plane determined by the line

$$\frac{x-1}{5} = \frac{y+2}{6} = \frac{z-3}{7}$$

and the point (6,2,4). *Ans.* $11x - 15y + 5z - 56 = 0$.

9. Prove that the plane $x + 4y - 3z = 0$ contains the line

$$\begin{cases} 3x + 2y + z - 2 = 0 \\ x - y + 2z - 1 = 0. \end{cases}$$

10. Find the coordinates of the point in which the line of Prob. 9 meets the plane $x - 4y + 3z + 6 = 0$. *Ans.* $(-3, 3.6, 3.8)$.

11. Find the equation of the plane which passes through the point $(3,2,5)$ and is parallel to each of the lines

$$\frac{x+2}{-2} = \frac{y-1}{2} = \frac{z+3}{1} \quad \text{and} \quad \frac{x+1}{-2} = \frac{y+3}{6} = \frac{z-11}{3}.$$

Ans. $y - 2z + 8 = 0$.

12. Find the equation of the plane passing through the points $(6,-1,2)$ and $(-1,-2,3)$ and parallel to the line $\frac{x-2}{4} = \frac{y+2}{3} = \frac{z+1}{1}$.

Ans. $4x - 11y + 17z - 69 = 0$.

13. Find the perpendicular distance from the given point to the given line, in each case.

(*a*) $(10,2,0)$; $\frac{x-9}{2} = \frac{y-6}{6} = \frac{z-9}{3}$ *Solution.* The plane perpendicular to the given line, l, and passing through the given point,

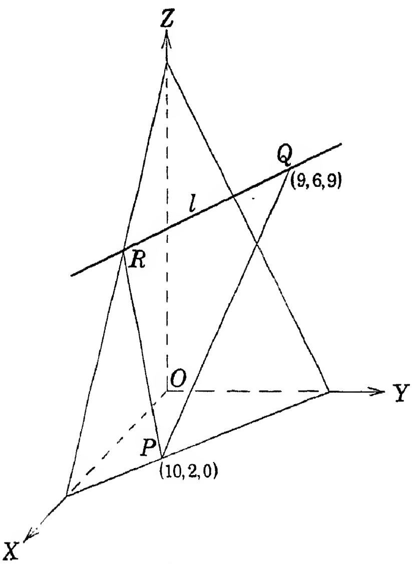


FIG. 86.

$$P\ (10,2,0),$$

has the equation

$$2(x - 10) + 6(y - 2) + 3z = 0,$$

$$2x + 6y + 3z - 32 = 0,$$

and, evidently, meets l in a point R such that PR is perpendicular to l. Therefore, PR is the distance sought. Now the normal form of the equation of the above mentioned plane is

$$\tfrac{2}{7}x + \tfrac{6}{7}y + \tfrac{3}{7}z - \tfrac{32}{7} = 0,$$

and the distance from the plane to Q $(9,6,9)$, a point on l, is

$$\begin{aligned} RQ &= \tfrac{2}{7}(9) + \tfrac{6}{7}(6) + \tfrac{3}{7}(9) - \tfrac{32}{7} \\ &= \tfrac{49}{7} = 7. \end{aligned}$$

The distance PQ is readily found to be $\sqrt{1 + 16 + 81} = \sqrt{98}$. Hence, we obtain, finally,

$$PR = \sqrt{98 - 49} = \sqrt{49} = 7.$$

(*b*) $(0,0,0)$; $\frac{x-4}{\ } = \frac{y+3}{2} = -z$ *Ans.* $\frac{1}{3}\sqrt{219}$.

(*c*) $(3,-1,2)$; $\begin{cases} 2x + y = 10 \\ 3x - y + z = 4. \end{cases}$

14. In each case, below, find the *shortest* distance between the two nonintersecting and nonparallel lines given:

(*a*) one line through (1,2,0), with direction numbers $(2,3,-1)$; the other through (3,0,2), with direction numbers $(1,-2,3)$.

Solution: Pass a plane through one line, say the first, parallel to the other line. The normal to that plane must be perpendicular to both lines. If we call the direction numbers of that normal A,B,C, we have

$$2A + 3B - C = 0, \qquad A - 2B + 3C = 0.$$

From these two equations, we obtain $A:B:C = -1:1:1$. Hence, the equation of that plane is $-1(x - 1) + 1(y - 2) + 1(z - 0) = 0$, or

$$-x + y + z = 1.$$

The distance from (3,0,2) to that plane is the distance sought. It is readily found to be of length $2/\sqrt{3}$. (The distance from any other point on the second line to the plane found would, of course, do as well.)

(*b*) the line $\dfrac{x}{3} = y - 3 = \dfrac{z + 1}{2}$, and the line

$$x + 1 = -y - 2 = \frac{z - 3}{3}. \qquad \textit{Ans. } \frac{14}{3\sqrt{10}}.$$

(*c*) the line $\begin{cases} 3x - y + 6 = 0 \\ 2y + 3z = 0 \end{cases}$, and the line

$$\begin{cases} 3y - 2x + 7 = 0 \\ x + 3z - 2 = 0 \end{cases}.$$

CALCULUS

CHAPTER VII

VARIABLES, FUNCTIONS, LIMITS, AND CONTINUITY

44. Constants and Variables. A *constant* is a number such as 0, 6, -2, $\sqrt{5}$, and π, or a symbol which is used, throughout one discussion, to represent just one number. A *variable* is a symbol which is used, in a discussion, to represent any one of a set of numbers, called its *range*. Thus, if we say that x is a number which satisfies the equation

$$x^3 - 6x^2 + 11x - 6 = 0,$$

then x is a variable whose range is the set of numbers (1,2,3). It is convenient, at times, to represent the range of a variable, x, by the symbol $R(x)$. If we say that

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot \cdot \cdot n,$$

we are using n as a variable whose range, $R(n)$, is the set of all positive integers. If we state that

$$(a + b)^2 = a^2 + 2ab + b^2.$$

we may call a and b variables, where each of their ranges, $R(a)$ and $R(b)$, is the set of all real numbers.

If a is a constant less than the constant b, the set of all real numbers x such that $a \leqq x$ and $b \geqq x$* is called the *interval*† from a to b. If c is a constant and ϵ is a constant greater than zero, the interval from $c - \epsilon$ to $c + \epsilon$ is spoken of as the ϵ-neighborhood of c. These terms, interval, neighborhood, open, closed, and many other terms used in analysis, are suggestive of geometry from which they are borrowed. Since any real number r may be identified with a point on a line, we may say that the interval from a to b consists of all the points on the line between and

---

* These inequalities may be written more concisely $a \leqq x \leqq b$.

† As here defined, the interval is known more technically as the *closed* interval from a to b. By the *open* interval from a to b is meant the set of all real numbers x for which $a < x < b$.

including a and b. See Fig. 87 for a representation of the interval from a to b and the ϵ-neighborhood of c. By the *deleted* ϵ-neighborhood of c is meant the set of all numbers in the ϵ-neigh-

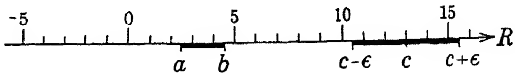


FIG. 87.

borhood of c, except c itself. A representation of the deleted ϵ-neighborhood of c is shown in Fig. 88.

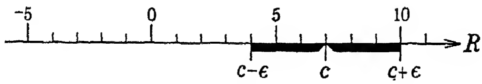


FIG. 88.

The student should note that if x is in the ϵ-neighborhood of c then $|x - c| \leqq \epsilon$ and conversely. Similarly, the set of inequalities $0 < |x - c| \leqq \epsilon$ is equivalent to the statement that x is in the deleted ϵ-neighborhood of c. We shall often make such statements in the form of inequalities for the sake of brevity in speech and writing.

Problems

1. State the range $R(u)$ for which each of the following is true:

(*a*) $u^2 - 5u - 6 = 0$.

(*b*) u is a multiple of 7 under 40.

(*c*) $(u + 1)! = (u + 1)(u!)$.

(*d*) $\theta = \sin^{-1} u$. *Ans.* $-1 \leqq u \leqq 1$.

(*e*) $\csc \theta = u$.

(*f*) $2^u = 2 \cdot 2 \cdot 2 \cdot \cdot \cdot 2$ (u factors in all).

(*g*) $(u - a)(u + a) = u^2 - a^2$.

2. Draw a real axis showing each of the following:

(*a*) The interval from 2 to 5.

(*b*) The ϵ-neighborhood of 9 if $\epsilon = 3$.

(*c*) The deleted ϵ-neighborhood of -3 if $\epsilon = 2$.

3. Express by inequalities the sets of numbers described in problem 2.

Ans. (*b*) $|x - 9| \leqq 3$.

4. Describe the interval from -1 to 4 as an ϵ-neighborhood of some number.

5. Draw an axis showing the range of x for each of the following cases, and describe the ranges as intervals or neighborhoods.

(*a*) $|x - 5| \leqq 3$.

(*b*) $0 < |x - 12| \leqq 2$.

(*c*) $-7 \leqq x \leqq 0$.

45. Functions. If a variable y is related to a variable x in such a way that to each number in the range of x corresponds a unique number in the range of y, then the variable y is said to be a *function* of the variable x. For example, let $R(x)$ be the set of positive integers in the interval from 1 to 10 and $R(y)$ be the numbers (3, 4, 5). Let the correspondence be established by the table

x	1	2	3	4	5	6	7	8	9	10
y	4	5	5	3	4	3	3	4	5	4

Then y is a function of x.

If y is a function of x and the range of x is $R(x)$, the function y is said to be *defined on the range* $R(x)$. The variable x of the definition is said to be the *independent* variable, while the variable y is called the *dependent* variable. If the correspondence is unique no matter which variable is chosen first, as it is in the table

u	3	5	8	12	17	23	30	38	47	57	68
v	10	9	8	7	6	5	4	3	-1	-6	-7

either variable may be taken as the independent one. However, we do not have the same function in the two cases, but one function and its inverse. For example, let the independent variable have the range (1,2,3,4,5) and the corresponding values of the dependent variable be obtained by squaring. This gives the table

Independent variable.	1	2	3	4	5
Dependent variable.	1	4	9	16	25

while the inverse function is given by the table

Independent variable.	1	4	9	16	25
Dependent variable.	1	2	3	4	5

Obviously, not every function has an inverse, since none exists for the function defined by the table

Independent variable.............	Every real number, r
Dependent variable................	Greatest integer $\leqq r$

It is customary to use such symbols as $f(x)$, $g(y)$, $h(z)$, and $F(t)$ to represent the dependent variable and, at the same time, to call attention to the independent variable. That is, if x is an independent variable, the symbol $f(x)$ may be used to represent the dependent variable. Under this notation, furthermore, if we write $f(3)$ we mean that number in the range of the dependent variable $f(x)$ which corresponds to the number 3 in the range of x. For example, in the table,

x	1	2	3	4	5	6	7	8	9	10	$g(y)$
$f(x)$	2	4	6	8	10	12	14	16	18	20	y

$f(2)$ stands for 4, $f(5) = 10$, $f(6) = 12$, and $f(10) = 20$. The function defined by this table evidently has an inverse which we may call $g(y)$ defined on the range of its independent variable y. We have, then, $f(4) = 8$, and $g(8) = 4$. Hence $f[g(8)] = 8$ and $g[f(4)] = 4$. Similarly $g[f(7)] = 7$ and, in general,

$$f[g(a)] = a,$$

$g[f(b)] = b$, a property evidently possessed by any function f and its inverse g.

If the range of the independent variable is finite, the correspondence can be readily given in a table, as in the above cases. But if the range possesses infinitely many elements, some other method must be resorted to to establish the correspondence. Several methods are discussed below.

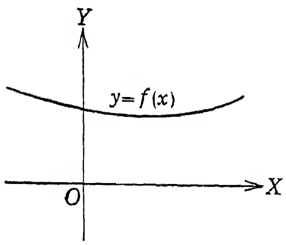


FIG. 89.

Graphical Method. Let a point have the number a, in the range of x, as abscissa, and the functional value, $f(a)$, as ordinate. One such point may thus be established for each element in the range of x. The totality of such points is called

the *graph* of the function. It may, or may not, form a connected curve.

Method of Construction. If the numbers of the range of x and those of the range of $f(x)$ are represented by the points on two lines, it is sometimes possible to so arrange the two lines that the correspondence between the numbers of the two sets is given by a construction with ruler and compass. For example, see Fig. 90.

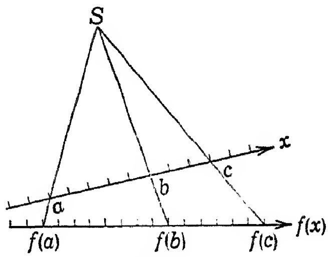


FIG. 90.

Mechanical Devices. These, like geometrical methods, are of unlimited variety but, of course, not applicable to every function. We shall mention only one well-known one, the clock. If we consider the reading of the hands as a function of the number of ticks after a given one, the range of the independent variable is the set of all positive integers. The dependent variable, being the reading of the hands, goes through a finite sequence of values repeatedly and affords us an illustration of a so-called *periodic* function, *viz.*, one for which there is some constant p such that $f(a + p) = f(a)$, no matter what the choice of a.

Function Scales. An exceedingly useful device for showing functional values consists of a straight line graduated so that

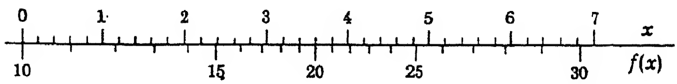


FIG. 91.

opposite the graduation a lies the graduation $f(a)$, as shown in Fig. 91. A commonly known instance of such a scale is the ordinary slide rule.

Formulas. From the standpoint of the calculus the most useful method of obtaining the functional value $f(a)$ corresponding to a is by means of formulas, as $f(x) = 2x$, $f(x) = \log_{10} x$, $f(x) = \sin x$. Of special importance are the functions of the following classification:

ALGEBRAIC FUNCTIONS:*

Powers, i.e., functions of the form x^n where n is a constant.

* Not all algebraic functions are included in this list.

Polynomials, i.e., functions of the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

where n is a positive integer and the a's are constants.

Rational functions, i.e., functions of the form

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials such that $Q(a) \neq 0$ if a is in the range of x.

TRANSCENDENTAL FUNCTIONS:*

Trigonometric functions.
Inverse trigonometric functions.
Logarithmic functions, i.e., functions of the form

$$\log_a x$$

where a is a positive constant not equal to 1.

Exponential functions, i.e., functions of the form

$$a^x$$

where a is a constant.

Problems

1. If the range of x is the interval from -2 to 5 and $f(x) = 2x - 3$ exhibit the function (*a*) graphically; (*b*) on a function scale; (*c*) by a construction.

2. Find a formula for the inverse of the function of Prob. 1. That is, if $f(x)$ is represented as y, express x as a function of y.

3. Draw the graph of the function defined by the table

x	$x < -1$	$-1 \leqq x \leqq 2$	$x > 2$
$f(x)$	2	$1 - x$	$x - 3$

4. On the same set of axes draw the graph of the two functions $\sqrt{4 - x^2}$ and $-\sqrt{4 - x^2}$. What single equation involving both x and $f(x)$ is true no matter which of the given functions is meant by $f(x)$?

5. Given $f(x) = x^2 - \dfrac{1}{x}$ where $x \neq 0$, find $f(1)$, $f(-1)$, $f(3)$, $f(\frac{1}{2})$, $f(s)$, $f(t^2)$ and $f(1/x)$.

* Not all transcendental functions are included in this list.

6. Given $F(x) = 1 - x$, find $F(0)$, $F(-4)$, $F(\frac{1}{3})$, $F(a)$, $F(x^2)$, and $F[F(x)]$. What is the form of the inverse function?

7. If the range of x is the set of all integers and $f(x)$ equals 1 when x is odd and 2 when x is even, draw the graph of the function from $x = -10$ to $x = 19$. Verify that the function is given by the formula

$$f(x) = \tfrac{3}{2} + (-1)^x \tfrac{1}{2}.$$

8. If a value for x and the corresponding value for $f(x)$ are found by assigning a value to t in the equations

$$x = 3t - 5$$
$$f(x) = t^2 + 2t,$$

draw the graph of the function $f(x)$. Find an equation between x and $f(x)$ which is free of t.

9. Devise a mechanical scheme to obtain the values of $\sin x$.

10. Over what ranges of x can the following define a function which is real when x is real?

(*a*) $x^3 - 6x^2 + x$.

(*b*) $\dfrac{4}{x-5}$.

(*c*) $\log_{10} x$.

(*d*) $\sin^{-1}(4x)$.

(*e*) $\sqrt{1 - x^2}$.

(*f*) $\sqrt{x^2 - 4}$.

(*g*) $\sqrt{1 + |3x - 5|}$.

(*h*) $\dfrac{1}{1 + |x|}$.

11. What formulas are possible for $g(x)$ if $x[g(x)]^2 = 25$? Over what range $R(x)$ could $g(x)$ be defined? Show graphically.

12. Draw the graph of the function $f(x)$ = greatest integer less than or equal to x.

13. Regarding the time, in hours, after 12 o'clock noon as independent variable and the reading of the hour hand of a clock as dependent variable, draw the graph of the function from $t = 0$ to $t = 36$.

46. Limits. Suppose that a variable x takes on consecutively the values

$$\tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \tfrac{4}{5},\ \tfrac{5}{6},\ \ldots$$

On a real axis, as shown in Fig. 92, x assumes the consecutive positions P_1, P_2, P_3, P_4, P_5, . . . , each of which is to the right

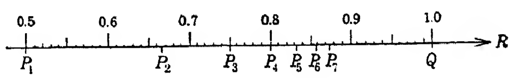


FIG. 92.

of its predecessor but to the left of the point Q whose coordinate is 1. The distance from the point (x) to the point Q is given by the expression

$$|1 - x| = 1 - x$$

and runs consecutively through the values

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}, \ldots$$

and, manifestly, can be made as small as we please by following the sequence sufficiently far. That is, for any positive quantity ϵ (however small) we can find a place in the sequence

$$\tfrac{1}{2}, \tfrac{2}{3}, \tfrac{3}{4}, \tfrac{4}{5}, \tfrac{5}{6}, \ldots$$

beyond which every number in the sequence differs from 1 by an amount less than ϵ. Geometrically, we see that for every $\epsilon > 0$ there exists an m such that if $n > m$ then P_n lies in the ϵ-neighborhood of Q. This condition is an illustration of a variable, x, approaching its *limit*, 1, and is represented symbolically by any one of the notations:

$\text{limit } x = 1$	(read "the limit of x equals 1")
$\lim x = 1$	(read as the above)
$x \to 1$	(read "x approaches 1")

In general, we shall say that a variable u has the limit a (that is, $u \to a$) if u runs through a sequence of numbers such that, corresponding to any positive number ϵ, there is a definite place in the sequence beyond which every value of u satisfies $|u - a| < \epsilon$.

We shall, upon occasion, use the notation

$$u \to +\infty,$$

read "u approaches plus infinity" or "u increases beyond all bound" to mean that u runs through a sequence of numbers such that, any positive number b having been chosen, there exists a place in the sequence beyond which every value of u is greater than b.

Similarly, the notation

$$u \to -\infty,$$

read "u approaches minus infinity" or "u decreases beyond all bound" will be used, meaning that u runs through a sequence of numbers such that, any negative number b having been chosen, there exists a place in the sequence beyond which every value of u is less than b.

If the notations

$$u \to \pm\infty, \qquad u \to \infty$$

are used, they are to be interpreted to mean that

$$|u| \to +\infty,$$

no restriction being placed upon the sign of u.

We shall have use for the following three theorems on limits.

Theorem 1. *If, in running through a sequence of values, a variable, x, never decreases and never exceeds a finite constant M, then x approaches a limit a, where $a \leqq M$.*

Theorem 2. *If, in running through a sequence of values, a variable, x, never increases and is never less than some finite constant m, then x approaches a limit b, where $b \geqq m$.*

Theorem 3. *If, in running through a sequence of values,*

$$a_1,\ a_2,\ a_3,\ a_4,\ \ldots$$

a variable, x, approaches a limit, then, for every $\epsilon > 0$ there exists a finite integer n such that $|a_n - a_{n+p}| < \epsilon$ for every positive integer p.

The proofs of these theorems depend upon the foundations of our system of real numbers and are beyond the scope of this book.

Suppose now that we have a function of x defined in a deleted neighborhood of 2 and that $x \to 2$ by running through a sequence of values. Then the function runs through the sequence of corresponding values. We give two illustrations:

x	1.9	2.1	1.99	2.01	1.999	2.001	1.9999	. . .
$f(x)$	3	4	5	6	7	8	9	. . .

x	1.9	2.1	1.99	2.01	1.999	2.001	1.9999	. . .
$g(x)$	3.2	2.8	3.02	2.98	3.002	2.998	3.0002	. . .

It appears that, in the second illustration, $g(x)$ is approaching 3 as x approaches 2. If, furthermore, the sequence of values of $g(x)$ has 3 as a limit, for *every* sequence of value of x having 2

as a limit, we write

$$\lim_{x \to 2} g(x) = 3.$$

In general, we shall say that

$$\lim_{x \to a} f(x) = b$$

if $f(x)$ is defined in some deleted neighborhood of a and if, for every $\epsilon > 0$ there exists a $\delta > 0$ such that x in the deleted δ-neighborhood of a implies that $f(x)$ is in the ϵ-neighborhood of b, *i.e.*. that

$$|f(x) - b| \leqq \epsilon$$

for every x which satisfies

$$0 < |x - a| \leqq \delta.$$

ILLUSTRATIONS OF LIMITS

Illustration 1. If $y = 3x + 2$, then $\lim_{x \to 1} y = 5$. For, let an arbitrary ϵ-neighborhood be assigned to 5. To select x so that $|5 - y| \leqq \epsilon$ is to select x so that $|5 - (3x + 2)| \leqq \epsilon$, *i.e.*, so that $|3 - 3x| \leqq \epsilon$. Set $\delta = \epsilon/3$ and it follows that $f(x)$ is in the ϵ-neighborhood of 5 whenever x is in the deleted δ-neighborhood of 1. Note the neighborhoods in Fig. 93.

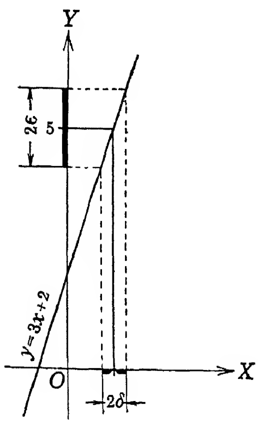


FIG. 93.

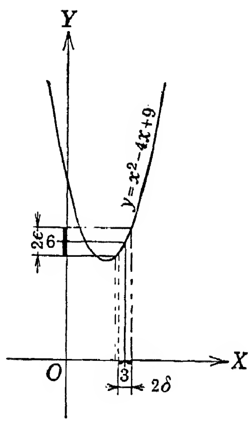


FIG. 94.

Illustration 2. If $y = x^2 - 4x + 9$, then $\lim_{x \to 3} y = 6$. For, if $x = 3 + \xi$ then $y = (3 + \xi)^2 - 4(3 + \xi) + 9$, or

$$y = 6 + 2\xi + \xi^2.$$

To select x so that $|y - 6| \leqq \epsilon$, *i.e.*, so that $|\xi^2 + 2\xi| \leqq \epsilon$, it is sufficient to select x so that

$$\xi^2 + 2|\xi| \leqq \epsilon,$$

i.e., so that

$$\xi^2 + 2|\xi| + 1 \leqq \epsilon + 1.$$

This will be accomplished if

$$|\xi| + 1 \leqq \sqrt{\epsilon + 1},$$

or if

$$|\xi| \leqq \sqrt{\epsilon + 1} - 1.$$

Hence, choose $\delta = \sqrt{\epsilon + 1} - 1$, and it follows that y is in the ϵ-neighborhood of 6 whenever x is in the deleted δ-neighborhood of 3.

Illustration 3. $\lim\limits_{t \to 0} (1 + t)^{1/t}$. If a sum of P dollars is deposited in a bank and draws interest at the rate r compounded annually for n years, the amount becomes

$$A = P(1 + r)^n.$$

If, in particular, the principal is \$1, the interest rate is 4 per cent, and the period 25 years, the amount is

$$(1 + 4/100)^{25} = (1 + 1/25)^{25} = 2.66584.$$

If interest is compounded twice yearly, the amount is

$$\left(1 + \frac{4}{2 \cdot 100}\right)^{2(25)} = \left(1 + \frac{1}{50}\right)^{50} = 2.69159.$$

Compounded quarterly, the amount becomes

$$\left(1 + \frac{4}{4 \cdot 100}\right)^{4(25)} = \left(1 + \frac{1}{100}\right)^{100} = 2.70481.$$

Continuing in this way, we find the amount of

$$\left(1 + \frac{4}{12 \cdot 100}\right)^{12(25)} = \left(1 + \frac{1}{300}\right)^{300} = 2.71377$$

for monthly conversion of interest,

$$\left(1 + \frac{4}{52 \cdot 100}\right)^{52(25)} = \left(1 + \frac{1}{1300}\right)^{1300} = 2.71688$$

for weekly conversion, and

$$\left(1 + \frac{4}{365 \cdot 100}\right)^{365(25)} = \left(1 + \frac{1}{9125}\right)^{9125} = 2.71812$$

for daily conversion. This process leads, naturally, to the question, what would be the amount if interest were converted instantaneously, or, in other words, what amount is approached as the conversion period is made

to approach zero in length. An answer is found in the expression

$$\lim_{t \to 0} (1 + t)^{1/t}$$

To show rigorously that this limit exists, is too difficult for a first course in the calculus. However, it is comparatively easy to show (see Prob. 21, below) that, if it does exist, its value is approximated by adding more and more terms of the infinite series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots.$$

The limit is conventionally represented by the letter e, and its value, correct to ten decimal places, is

$$e = 2.7182818285.$$

This limit is of great importance in the subject of calculus, and a problem directing the student through the proof of the existence of that limit, in case $1/t$ is a positive integer, is found at the close of this section.

Illustration 4. $\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right)$. This limit is another one of much importance for subsequent work. If the angle BOD in Fig. 95 is equal to 2θ and

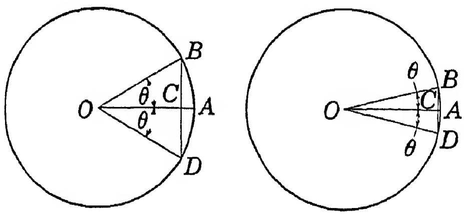


FIG. 95.

the radius equals unity, then $CB = \sin \theta$ and the arc AB equals θ. Hence

$$\frac{\sin \theta}{\theta} = \frac{2 \sin \theta}{2\theta} = \frac{\text{chord } DCB}{\text{arc } DAB}.$$

Now the length of the circumference of a circle is defined as the limit of the perimeter of an inscribed or circumscribed regular polygon as the number of sides increases indefinitely. This definition amounts to the hypothesis, for circular arcs, that the limit of the ratio of the chord to the corresponding arc approaches unity, as the length of the arc approaches zero. This hypothesis is fundamental to all work bearing upon lengths of circular arcs and is either expressly or tacitly assumed. We conclude, therefore, that

$$\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right) = 1.$$

We shall now give four important properties of limits. The first three of these are given as Theorems 4, 5, and 6 below, while the fourth is stated as Exercise 1.

Theorem 4. *The limit of a product of two functions of a common variable is equal to the product of their limits, or,*

$$\lim_{x \to a} [f(x) \cdot g(x)] = [\lim_{x \to a} f(x)] \cdot [\lim_{x \to a} g(x)].$$

Proof: Let $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$. Then for any $\epsilon_1 > 0$ there exists a $\delta_1 > 0$ such that if $0 < |x - a| \leqq \delta_1$ then $|f(x) - b| \leqq \epsilon_1$, and for any $\epsilon_2 > 0$ there exists a $\delta_2 > 0$ such that if $0 < |x - a| \leqq \delta_2$, then $|g(x) - c| \leqq \epsilon_2$. We wish to show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$0 < |x - a| \leqq \delta$$

then $|f(x) \cdot g(x) - bc| \leqq \epsilon$. Verify by algebra that

$$|f(x) \cdot g(x) - bc| = |\{g(x) - c\}\{f(x) - b\} + c\{f(x) - b\} + b\{g(x) - c\}|.$$

Making all the products on the right-hand side positive can only increase that member, so we have

$$|f(x) \cdot g(x) - bc| \leqq |g(x) - c| \cdot |f(x) - b| + |c| \cdot |f(x) - b| + |b| \cdot |g(x) - c|.$$

If, now, we keep $|x - a|$ positive but less than the smaller of δ_1 and δ_2, then both of the conditions $0 < |x - a| \leqq \delta_1$ and $0 < |x - a| \leqq \delta_2$ will be satisfied and hence both of the conclusions $|f(x) - b| \leqq \epsilon_1$ and $|g(x) - c| \leqq \epsilon_2$ will follow. Hence, from the above we have

$$|f(x) \cdot g(x) - bc| \leqq \epsilon_1\epsilon_2 + |c|\epsilon_1 + |b|\epsilon_2,$$

for any positive values whatever of the quantities ϵ_1 and ϵ_2. Now, for any $\epsilon > 0$ we may so choose ϵ_1 and ϵ_2 that each of the three terms $\epsilon_1\epsilon_2$, $|c|\epsilon_1$, and $|b|\epsilon_2$ is less than $\epsilon/3$. The δ which we seek may be taken as the smaller of the δ_1 and δ_2 determined by this choice of ϵ_1 and ϵ_2.

Theorem 5. *The limit of a sum of two functions of a common variable is equal to the sum of their limits, or*

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x),$$

Proof: Let $\lim_{x \to a} f(x) = b$ and $\lim_{x \to a} g(x) = c$. Let ϵ be any positive number and let $\epsilon_1 = \epsilon_2 = \frac{1}{2}\epsilon$. From the definition of a limit,

there exist positive numbers δ_1 and δ_2 such that whenever $0 < |x - a| \leqq \delta_1$ then $|f(x) - b| \leqq \epsilon_1$, and whenever

$$0 < |x - a| \leqq \delta_2$$

then $|g(x) - c| \leqq \epsilon_2$. Both of these sets of conditions will hold whenever $0 < |x - a| \leqq \delta$, where δ is the smaller of δ_1 and δ_2. We thus have established the existence of a δ such that whenever $0 < |x - a| \leqq \delta$, then $|f(x) - b| \leqq \epsilon_1$ and $|g(x) - c| \leqq \epsilon_2$ and, therefore,

$$\begin{aligned} |[f(x) + g(x)] - [b + c]| &= |[f(x) - b] + [g(x) - c]| \\ &\leqq |f(x) - b| + |g(x) - c| \\ &\leqq \epsilon_1 + \epsilon_2 \\ &\leqq \epsilon. \qquad Q.\ E.\ D. \end{aligned}$$

Theorem 6. *If* $\lim\limits_{x \to a} f(x) \neq 0$ *then* $\lim\limits_{x \to a} \dfrac{1}{f(x)} = \dfrac{1}{\lim\limits_{x \to a} f(x)}$.

Proof: Let $\lim\limits_{x \to a} f(x) = b \neq 0$, ϵ be any positive number, $\epsilon_1 = \frac{1}{2}|b|$, and $\epsilon_2 = \frac{1}{2}b^2\epsilon$. Corresponding to ϵ_1 and ϵ_2 there exist two positive numbers δ_1 and δ_2 such that whenever

$$0 < |x - a| \leqq \delta_1$$

then $|f(x) - b| \leqq \epsilon_1$ and whenever $0 < |x - a| \leqq \delta_2$ then $|f(x) - b| \leqq \epsilon_2$, and both conclusions follow whenever

$$0 < |x - a| \leqq \delta,$$

where δ is the smaller one of δ_1 and δ_2. From the form of ϵ_1 we have, under these conditions, $f(x)$ differing from b by not more than $\frac{1}{2}|b|$, so that $f(x)$ lies between $\frac{1}{2}b$ and $\frac{3}{2}b$, whence $|f(x)| \geqq \frac{1}{2}|b|$. Finally, when $0 < |x - a| \leqq \delta$, we have the relations

$$\left|\frac{1}{f(x)} - \frac{1}{b}\right| = \left|\frac{b - f(x)}{b \cdot f(x)}\right| = \frac{|b - f(x)|}{|b| \cdot |f(x)|} \leqq \frac{\epsilon_2}{|b| \cdot |\frac{1}{2}b|} = \frac{2\epsilon_2}{b^2} = \epsilon,$$

and the theorem is proved.

Exercise 1. As a corollary to Theorems 4 and 6, prove that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$$

if

$$\lim g(x) \neq 0.$$

Problems

1. What limit, if any, is approached by x as it runs through the sequence

(a) $4, 3, 2, 2\frac{1}{3}, 2 + \frac{1}{3^2}, 2 + \frac{1}{3^3}, 2 + \frac{1}{3^4}, \ldots$?

(b) 1.0, 0.9, 0.8, 0.7, . . . , 0.2, 0.1, 0.09, 0.08, 0.07, 0.06, . . . , 0.02, 0.01, 0.009, 0.008, . . . ?

(c) 4, 4.4, 4.44, 4.444, 4.4444, . . . ?

(d) $1, -1, 1, -1, 1, -1, 1, -1, \ldots$?

(e) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$?

(f) 1, 3, 5, 7, 9, 11, . . . ?

(g) 2.1, 2, 2.11, 2, 2.111, 2, 2.1111, 2, . . . ?

2. In each case in Prob. 1, above, what limit, if any, is approached by a variable which runs through the sequence remaining after the first, third, fifth, and all odd-numbered elements are dropped?

3. If $x \to 0$, prove that $cx \to 0$, if c is a finite constant.

4. If $x \to 0$, prove that $xy \to 0$, if y is a variable such that $|y| < c$ where c is some finite constant.

5. If $x \to 0$, prove that $c/x \to \infty$, if c is a constant not zero.

6. If $x \to 0$, prove that $y/x \to \infty$, if y is a variable such that $|y| > c$ where c is some positive constant.

7. Show by illustration that if $x \to 0$ and $y \to 0$, then the fraction x/y may approach zero, may approach a finite constant c, not zero, or may increase beyond all bound.

8. If $x \to \infty$, prove that $c/x \to 0$, if c is a finite constant.

9. If $x \to \infty$, and $|y| < c$, where c is a finite positive constant, prove that $y/x \to 0$.

10. Show by illustration that if $x \to \infty$ and $y \to \infty$, then x/y may approach zero, a finite constant different from zero, or increase beyond all bound.

11. Prove that $\lim\limits_{x \to 4} (2x - 1) = 7$. HINT: Follow the method of Illustration 1 of this section.

12. Prove that $\lim\limits_{x \to a} (5x - 4) = 5a - 4$, for every finite number a.

13. Prove that $\lim\limits_{x \to 1} (3x^2 - 4x + 4) = 3$, using the method of Illustration 2 of this section.

14. Prove that $\lim\limits_{x \to d} (ax^2 + bx + c) = ad^2 + bd + c$, where a, b, c, and d are any constants.

15. Prove that $\lim\limits_{x \to a} x^n = a^n$, where a is any constant and n is any positive integer.

HINT: The proof may be made using the binomial theorem, or by induction, using the result proved in Theorem 4 of this section.

16. Prove that $\lim\limits_{x \to a} P(x) = P(a)$, where $P(x)$ is any polynomial in x.

17. Show that $\lim\limits_{x \to 0} (-1)^x$ does not exist, by considering first that x runs through a sequence of decreasing fractions of the form $1/p$, where p is an

odd integer, and second, that x runs through a sequence of decreasing fractions of the form $2/q$, where q is an odd integer.

18. Prove that if $f(x) = g(x)$ in some deleted neighborhood of a, then $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x)$, if the limits exist.

19. Can $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ exist if $\lim\limits_{x\to a} g(x) = 0$?

20. Find the following limits, if they exist.

(a) $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$. *Ans.* 4.

(b) $\lim\limits_{x\to 3} \dfrac{x+1}{x-3}$.

(c) $\lim\limits_{x\to 0} \dfrac{x^3-5x^2+2x}{x}$. *Ans.* 2.

(d) $\lim\limits_{x\to 3} \dfrac{x^2-9}{x^2-5x+6}$. *Ans.* 6.

(e) $\lim\limits_{x\to 0} \dfrac{x^3+2x^2}{x^2-x}$. *Ans.* 0.

(f) $\lim\limits_{x\to -1} \dfrac{x^2+x+1}{x+1}$.

(g) $\lim\limits_{\theta\to 0} \dfrac{\sin 3\theta}{\sin \theta}$. *Ans.* 3.

(h) $\lim\limits_{\theta\to \frac{\pi}{2}} \dfrac{\cos\theta}{\frac{\pi}{2}-\theta}$. *Ans.* 1.

(i) $\lim\limits_{\theta\to 0} \dfrac{1-\cos\theta}{\theta}$. *Ans.* 0.

(j) $\lim\limits_{\theta\to 0} \dfrac{\tan\theta}{\sin\theta}$. *Ans.* 1.

(k) $\lim\limits_{x\to 0} \dfrac{10^x}{\cos\left(x+\frac{\pi}{2}\right)}$.

(l) $\lim\limits_{x\to\infty} (\sqrt{x+1}-\sqrt{x})$. *Ans.* 0.

(m) $\lim\limits_{x\to 0} \dfrac{\sin x}{2x}$.

(n) $\lim\limits_{\theta\to 0} \dfrac{\theta}{\tan\theta}$.

21. Prove that $\lim\limits_{t\to 0} (1+t)^{1/t}$ exists if $1/t$ is restricted to values which are positive integers. HINT: The limit is the same as $\lim\limits_{z\to\infty}\left(1+\dfrac{1}{z}\right)^z$, where z is held to positive integral values. Consider the two series

(A) $1+1+\dfrac{1}{2!}+\dfrac{1}{3!}+\dfrac{1}{4!}+\cdots$

(B) $1+1+\dfrac{1}{2}\left(1-\dfrac{1}{z}\right)+\dfrac{1}{3!}\left(1-\dfrac{1}{z}\right)\left(1-\dfrac{2}{z}\right)+\dfrac{1}{4!}\left(1-\dfrac{1}{z}\right)\left(1-\dfrac{2}{z}\right)\left(1-\dfrac{3}{z}\right)+\cdots$

each to $z+1$ terms, and show that the series (B) is the binomial expansion of $\left(1+\dfrac{1}{z}\right)^z$. Show next that for any positive value of z any term in the series (B) is less than, or equal to, the corresponding term in series (A). Show also that, as z increases, the terms of series (B) increase in magnitude and in number and approach the corresponding terms in series (A). This establishes the fact that $\left(1+\dfrac{1}{z}\right)^z$ increases as z increases but cannot exceed the sum of the first $z+1$ terms of the series (A). Now let the series (A) be represented by the notation

$$(A') \qquad a_0+a_1+a_2+a_3+a_4+\cdots.$$

Evidently $a_0 = 1$ and $a_{i+1} = \dfrac{a_i}{i+1}$ for every value of i. Hence

$$\begin{aligned}
a_0 &= 1 = 1.0\\
a_1 &= \frac{a_0}{1} = 1.0\\
a_2 &= \frac{a_1}{2} = 0.5\\
a_3 &= \frac{a_2}{3} < 0.1666666667\\
a_4 &= \frac{a_3}{4} < 0.0416666667\\
a_5 &= \frac{a_4}{5} < 0.0083333334\\
a_6 &= \frac{a_5}{6} < 0.0013888889\\
a_7 &= \frac{a_6}{7} < 0.0001984127\\
a_8 &= \frac{a_7}{8} < 0.0000248016\\
a_9 &= \frac{a_8}{9} < 0.0000027558.
\end{aligned}$$

Hence

$$a_0 + a_1 + \cdots + a_9 < 2.7182815258.$$

Now, since $a_9 < 0.000003$ and $a_{10} = a_9/10$,

$$\begin{aligned}
a_{10} &< 0.0000003\\
a_{11} &< 0.00000003\\
a_{12} &< 0.000000003\\
&\cdots\cdots
\end{aligned}$$

and the sum of any number of terms in the series (A) after a_9 is less than 0.000000333333 . . . , which is less than 0.0000003334. The sum of any number of terms of series (A) is, therefore, less than

$$2.7182815258 + 0.0000003334 = 2.7182818592.$$

Since the quantity $\left(1 + \frac{1}{z}\right)^z$ increases with z but is always less than 2.7182818592, it approaches a limit by virtue of Theorem 1 of this section.

22. Find $\lim\limits_{x \to \infty} \dfrac{3x^2 - 2}{4x^2 - 5}$. HINT: If x is finite and not zero,

$$\frac{3x^2 - 2}{4x^2 - 5} = \frac{3 - \dfrac{2}{x^2}}{4 - \dfrac{5}{x^2}}.$$

As x increases, the numerator approaches 3 while the denominator approaches 4. Hence the limit is $\frac{3}{4}$.

23. Find the following limits if they exist.

(*a*) $\lim\limits_{x \to \infty} \dfrac{2x - 1}{5x + 2}$. *Ans.* $\frac{2}{5}$. (*b*) $\lim\limits_{x \to \infty} \dfrac{2x^3 - 3x^2}{5x^2 - 6}$.

(c) $\lim_{x \to \infty} \frac{4x - 6}{x^2 + x + 1}$. *Ans.* 0. (d) $\lim_{x \to \infty} \frac{7x^3 - 2x^2 + 8x + 11}{9x^3 + 5x^2 + 10x + 15}$

47. Continuity. If $f(x) = x^2 + x + 1$, we see readily that

First: $f(3) = 13$,

Second: $\lim_{x \to 3} f(x) = 13$,

We speak of the function $x^2 + x + 1$ as being *continuous* at $x = 3$ since it possesses the three properties set forth in the following definition:

If a function, $f(x)$, has the three properties

First: *$f(a)$ is defined,*

Second: $\lim_{x \to a} f(x)$ *exists,*

Third: $\lim_{x \to a} f(x) = f(a)$,

the function is said to be continuous at $x = a$.

If one or more of these properties fails to hold at $x = a$, then we say that the function is *discontinuous* at $x = a$.

ILLUSTRATIONS

Illustration 1. The function $f(x) = 3x^2 - 5x$ is everywhere continuous, as we may show as follows. If a is any constant, we have the function defined, since

$$f(a) = 3a^2 - 5a.$$

To show that $\lim_{x \to a} f(x)$ exists, and, moreover, that it is equal to $3a^2 - 5a$, we must show that, for every $\epsilon > 0$ there exists a $\delta > 0$ such that whenever x is in the deleted δ-neighborhood of a, then $f(x)$ is in the ϵ-neighborhood of $3a^2 - 5a$. Let $x - a = \xi$ or $x = a + \xi$. Hence

$$\begin{aligned} f(x) &= 3(a + \xi)^2 - 5(a + \xi) \\ &= 3a^2 + 6a\xi + 3\xi^2 - 5a - 5\xi \\ &= (3a^2 - 5a) + \xi(6a - 5) + 3\xi^2. \end{aligned}$$

If, now, we select $\delta > 0$ so small that

$$\delta|6a - 5| \leqq \frac{\epsilon}{2}$$

and

$$3\delta^2 \leqq \frac{\epsilon}{2}$$

and keep

$$|\xi| \leqq \delta,$$

we shall have

$$|\xi(6a - 5) + 3\xi^2| \leqq |\xi(6a - 5)| + 3\xi^2 \leqq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence

$$|f(x) - (3a^2 - 5a)| \leqq \epsilon,$$

and the function is continuous at $x = a$. Since no restrictions have been placed upon a in the proof, the function is everywhere continuous.

Illustration 2. Let the point $[0,f(x)]$ be determined on the y-axis by projecting the point $(x,0)$, on the x-axis, from the fixed point $(4,-3)$. Then the points $[0, f(a) + \epsilon]$ and $[0, f(a) - \epsilon]$ determine an interval on the y-axis and a corresponding interval on the x-axis from $a - \delta_1$ to $a + \delta_2$. If δ is the smaller of δ_1 and δ_2 then, whenever $0 < |x - a| \leqq \delta$, we have

$$|f(x) - f(a)| \leqq \epsilon.$$

This establishes the condition

$$\lim_{x \to a} f(x) = f(a),$$

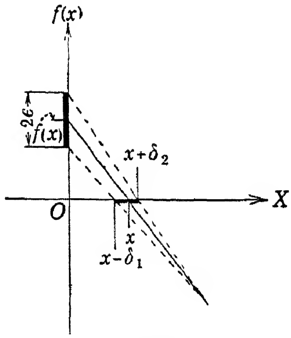


FIG. 96

or that the function $f(x)$ is continuous at such points as render valid these geometrical arguments. The one point of discontinuity is $x = 4$, since $f(x)$ is not defined at that point.

Exercise 1. Obtain a formula for $f(x)$ as used in Illustration 2. Note from the formula whether $f(x)$ is defined for $x = 4$.

In later sections of this book we shall have need of two properties of functions which are continuous on an interval.

Property 1. *If $f(x)$ is continuous on the interval $a \leqq x \leqq b$, and $f(a) \neq f(b)$, then for any constant c between $f(a)$ and $f(b)$, there exists an x_1 between a and b such that $f(x_1) = c$.*

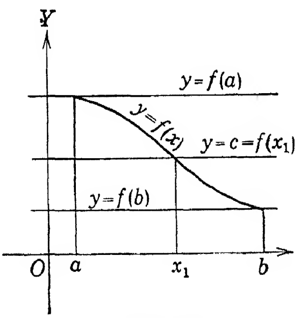


FIG. 97.

Geometrically, this means that the line $y = c$ has upon it a point $[x_1, f(x_1)]$ which lies on the curve $y = f(x)$, if the line $y = c$ lies between the lines $y = f(a)$ and $y = f(b)$ and, moreover, x_1 lies between a and b.

Property 2. *If $f(x)$ is continuous on the interval $a \leqq x \leqq b$, then there exist an x_1 and an x_2, $a \leqq x_1 \leqq b$, and $a \leqq x_2 \leqq b$, such that $m = f(x_1) \leqq f(x) \leqq f(x_2) = M$ for every x on the interval.*

The geometrical significance of this property is made clear by a glance at Fig. 98.

These properties seem obviously true, when viewed from the geometrical point of view, and the student is asked to accept them on faith. A rigorous demonstration that these properties hold is well beyond the scope of this book.

As an illustration of the content of Property 2, consider the function $4/(x - 1)$ for the range $1 < x \leqq 5$. Evidently it has no upper bound, *i.e.*, no number M stated for Property 2. (Is there a lower bound, *i.e.*, a number m stated for property 2?) This function fails to meet the hypotheses of Property 2 in that that the value 1 is not included in the range of x.

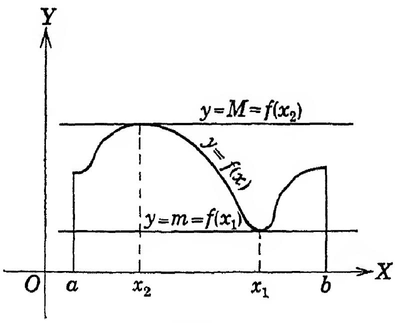


FIG. 98.

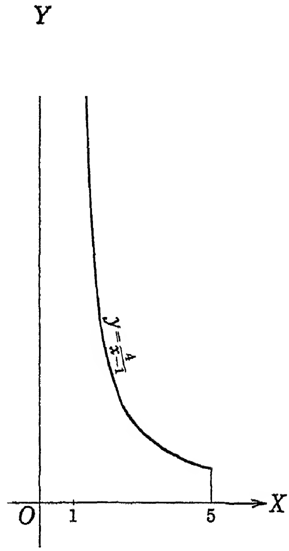


FIG. 9

Problems

1. Prove that the sum of two functions of x, both of which are continuous at $x = a$, is continuous at $x = a$. HINT: Make use of the result of Theorem 5 of the preceding section (page 155).

2. Prove that the product of two functions of x, both of which are continuous at $x = a$, is continuous at $x = a$. HINT: Make use of the result of Theorem 4 of the preceding section (page 155).

3. Prove that if $f(x)$ and $g(x)$ are both continuous at $x = a$ and $g(a) \neq 0$, then $f(x)/g(x)$ is continuous at $x = a$. HINT: Make use of Exercise 1 of the preceding section (page 156).

4. Prove that x^n is everywhere continuous if n is a positive integer.

5. Prove, as a corollary to Probs. 1 and 4, that any polynomial is everywhere continuous.

6. Prove, as a corollary to Probs. 3 and 5, that a rational function is continuous at every point at which its denominator is not zero.

7. Prove that any rational function is discontinuous at any point at which its denominator is zero.

8. In each of the following cases tell for what values of x the given function is discontinuous.

(a) $\dfrac{3x+1}{x^4-x}$.

(b) $\dfrac{x+2}{4x}$.

(c) $\dfrac{4x^3-2x^2}{x^4-16x^2}$.

(d) $\dfrac{12}{x^3-8}$.

(e) $\dfrac{2x-5}{1+x^2}$.

(f) $\dfrac{1}{x^2-4}-\dfrac{2}{x(x^2-4)}$.

(g) $\dfrac{(x-1)(x-2)^2}{(x-1)^2(x-2)}$.

(h) $\dfrac{x^2}{(x-2)(x-3)}-\dfrac{3}{x-3}-\dfrac{4}{x(x-2)}$.

9. Draw the graph of the function $\dfrac{1}{1+|x|}$. For what values of x is it discontinuous?

10. If $f(x) = \dfrac{x^2-3x}{3-x}$ when $x \neq 3$ and $f(x) = -3$ when $x = 3$, find for what values of x it is continuous.

11. If $f(x) = 3$ when $x \leqq 2$ and $f(x) = 2x + 1$ when $x > 2$, draw the graph of the function and state for what range the function is continuous.

12. If $f(x) = 0$ when $x < 0$, $f(x) = 1$ when $0 < x < 1$, and $f(x) = 2$ when $x > 1$, find the points of discontinuity.

13. Prove that $\sin x$ is continuous for every value of x. HINT: Write

$$\sin(a+\xi) - \sin a = 2\cos\left(a+\frac{\xi}{2}\right)\sin\frac{\xi}{2}.$$

Hence

$$\sin(a+\xi) = \sin a + 2\cos\left(a+\frac{\xi}{2}\right)\sin\frac{\xi}{2}.$$

Now examine

$$\lim_{\xi\to 0}\sin(a+\xi).$$

14. Prove that $\cos x$ is everywhere continuous.

15. Prove that $\tan x$ is continuous except at $x = n\pi/2$, where n is any odd integer.

16. If $f(x) = x\sin\dfrac{1}{x}$ when $x \neq 0$ and $f(x) = 0$ when $x = 0$, draw the graph of the function and discuss its continuity.

17. Find the discontinuities of the function

$$f(x) = \lim_{n\to\infty}\frac{nx}{1+nx}$$

and draw the graph.

18. Find the discontinuities of the function

$$f(x) = \frac{1}{10^{\frac{1}{x}}+1} + \frac{1}{10^{-\frac{1}{x}}+1}$$

and draw the graph.

19. Find the discontinuities of the function

$$f(x) = \lim_{n\to+\infty}\frac{x}{x^n+1}$$

and draw the graph.

20. Find the discontinuities of the function

$$f(x) = \lim_{n \to +\infty} (\tan^{-1} nx)$$

and draw the graph.

21. (*a*) Does the function $f(x) = x^2 - 1$ satisfy the hypothesis of Property 1 of this section (page 161) in the interval $0 \leqq x \leqq 1$? If so, find the value of x_1 corresponding to $c = -\frac{3}{4}$.

(*b*) For the function and interval stated in (*a*) find m, M, x_1, and x_2 as related to Property 2 of this section.

22. Does Property 2 of this section apply to $f(x) = \dfrac{3}{4 - x^2}$ in the interval $1 \leqq x \leqq 2$? If so, find m, M, x_1, and x_2.

CHAPTER VIII

DERIVATIVES

48. The Derivative of a Function. Let a function, $f(x)$, be continuous in a neighborhood of $x = x_0$, and let u be some other value of x in that neighborhood. The functional values at $x = x_0$ and $x = u$ are $f(x_0)$ and $f(u)$, and the difference

$$f(u) - f(x_0)$$

represents the change in the function, induced by causing x to change from x_0 to u. Also the fraction

$$\frac{f(u) - f(x_0)}{u - x_0} \tag{71}$$

represents the average rate of change in the functional value per unit change in x, as x changes from x_0 to u. If this fraction (71) approaches a limit as u approaches x_0, we may write it as

$$\lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0},$$

and call it the *instantaneous* rate of change of the functional value, $f(x)$, per unit change in x, at $x = x_0$. This limit, if it exists, is evidently dependent upon x_0, and considering x_0 as a variable over the range for which the limit exists, we may write

$$f'(x_0) = \lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0}.$$

We shall call this function of x_0 the *derived* function, or the *derivative of* $f(x)$, *with respect* to x, *at* $x = x_0$.

We have already made an interpretation of this derivative as the instantaneous rate of change of the dependent variable, $f(x)$, per unit change of the independent variable x, at a particular value, x_0, of the latter. If the independent variable is t, representing time, and if the functional value $s(t)$ represents the corresponding distance from some fixed point on a straight line

to a particle moving on that line, the difference

$$s(t) - s(t_0)$$

represents the distance the particle has traveled while the time has changed from t_0 to t, *i.e.*, during an elapsed time

$$t - t_0.$$

Hence, the difference quotient

$$\frac{s(t) - s(t_0)}{t - t_0}$$

represents the average velocity of the particle between the instants t_0 and t. As we let t approach t_0, this average velocity is computed over an ever shorter time after t_0. Hence we regard

$$s'(t_0) = \lim_{t \to t_0} \frac{s(t) - s(t_0)}{t - t_0},$$

the derivative of $s(t)$ with respect to t at $t = t_0$, as the instantaneous velocity of the particle at the time t_0.

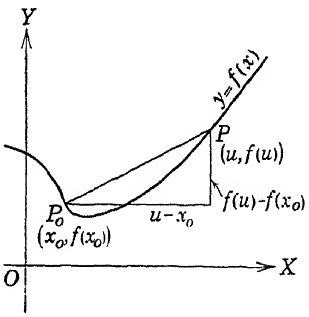


FIG. 100.

The geometrical significance of the derivative is readily grasped. The points P_0 $[x_0,f(x_0)]$ and P $[u,f(u)]$ are points on the graph of the function $f(x)$, and the difference quotient

$$\frac{f(u) - f(x_0)}{u - x_0}$$

is the slope of the chord P_0P. As $u \to x_0$, the point P approaches P_0 along the curve, while the line P_0P turns about the fixed point P_0 and approaches the line tangent to the curve at the point P_0. Its slope,

$$\frac{f(u) - f(x_0)}{u - x_0},$$

approaches the slope of that line. Hence $f'(x_0)$, the derivative of $f(x)$ with respect to x, at $x = x_0$, represents the slope of the line tangent to the curve at the point P_0.

Exercise 1. Prove that the existence of the limit defining $f'(x_0)$ implies that $f(x)$ is continuous at $x = x_0$.

Illustrations

Illustration 1. Given $f(x) = 1 + x^2$, find $f'(2)$. METHOD:

$$f(u) = 1 + u^2, \qquad f(2) = 5, \qquad f(u) - f(2) = u^2 - 4.$$

Hence

$$\frac{f(u) - f(2)}{u - 2} = \frac{u^2 - 4}{u - 2} = u + 2,$$

and

$$f'(2) = \lim_{u \to 2} \frac{f(u) - f(2)}{u - 2} = \lim_{u \to 2} (u + 2) = 4.$$

The derivative of the function $1 + x^2$, with respect to x, at $x = 2$, is thus found to be 4.

In view of the text above, this signifies that the function, $1 + x^2$, changes at the rate of 4 units, per unit change in x, when $x = 2$; also that the slope of the line tangent to the curve $y = 1 + x^2$, at the point (2,5), equals 4.

Illustration 2. Find $f'(x_0)$ for $f(x) = 1 + x^2$. METHOD:

$$f(u) = 1 + u^2, \qquad f(x_0) = 1 + x_0^2.$$

Hence

$$f(u) - f(x_0) = u^2 - x_0^2,$$

and

$$\frac{f(u) - f(x_0)}{u - x_0} = \frac{u^2 - x_0^2}{u - x_0} = u + x_0.$$

Hence

$$f'(x_0) = \lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0} = \lim_{u \to x_0} (u + x_0) = 2x_0.$$

Note that, since x_0 denotes any fixed value of x, the last result gives at once $f'(2) = 4$ (Illustration 1), $f'(5) = 10$, $f'(a) = 2a$, etc.

Problems

1. Find $f'(x_0)$ when $f(x)$ and x_0 are as designated below.
   (*a*) $f(x) = 1 + 2x$, $x_0 = 2$. *Ans.* 2.
   (*b*) $f(x) = 1 + 2x^2$, $x_0 = -1$. *Ans.* -4.
   (*c*) $f(x) = x^3 + 2$, $x_0 = 1$. *Ans.* 3.
   (*d*) $f(x) = \dfrac{2}{1 - x}$, $x_0 = -2$. *Ans.* $\frac{2}{9}$.
   (*e*) $f(x) = \dfrac{x + 1}{2x^2 - 3}$, $x_0 = 3$. *Ans.* $-\frac{11}{75}$.
   (*f*) $f(x) = \sin x$, $x_0 = 0$. *Ans.* 1.
   (*g*) $f(x) = \tan x$, $x_0 = 0$. *Ans.* 1.
2. Find $f'(x_0)$ when $f(x)$ is as designated below.
   (*a*) $f(x) = 2 - 3x^2$. *Ans.* $-6x_0$.
   (*b*) $f(x) = 3 + 2x^3$. *Ans.* $6x_0^2$

(c) $f(x) = \dfrac{x}{3 + 4x}$. *Ans.* $\dfrac{3}{(3 + 4x_0)^2}$.

(d) $f(x) = \dfrac{x^2}{2 - 3x}$. *Ans.* $\dfrac{4x_0 - 3x_0^2}{(2 - 3x_0)^2}$.

(e) $f(x) = \sqrt{1 + x}$. *Ans.* $\dfrac{1}{2\sqrt{1 + x_0}}$.

3. Find the slope of the tangent for each curve designated below, at the point indicated. From that find the equation of the tangent.

(a) $y = 2x^2 + 3$ at the point (2,11). *Ans.* $y - 8x + 5 = 0$.

(b) $y = \dfrac{5}{2 + x}$ at the point whose abscissa is 1.

Ans. $5x + 9y = 20$.

(c) $y = 1 - x^3$ at the point whose abscissa is -2.

Ans. $12x + y + 15 = 0$.

(d) $y = \dfrac{3}{2 + x^2}$ at the point whose ordinate is ½.

Ans. $2x + 6y = 7$, $2x - 6y + 7 = 0$.

(e) $y = \sqrt{1 - 2x}$ at the point whose ordinate is 3.

Ans. $x + 3y = 5$.

49. Derivatives of Algebraic Functions. In this section we shall develop a number of theorems that will enable the student to write down the derivative of an algebraic function at sight, without the necessity of developing the derivative from its definition.

Theorem 1. *The derivative of a constant is zero.*

Proof: By hypothesis

$$f(x) = c.$$

Hence

$$f(u) = c, \qquad f(x_0) = c$$

and

$$f'(x_0) = \lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0} = \lim_{u \to x_0} \frac{c - c}{u - x_0} = 0.$$

Example. If $f(x) = 5$, $f'(x_0) = 0$, for every x_0.

Theorem 2. *The derivative of the sum of two functions of x is the sum of the derivatives of the two functions.*

Proof: By hypothesis,

$$f(x) = g(x) + h(x).$$

Hence

$$f(u) = g(u) + h(u), \qquad f(x_0) = g(x_0) + h(x_0),$$

and

$$\begin{aligned}
f'(x_0) &= \lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0} \\
&= \lim_{u \to x_0} \frac{g(u) + h(u) - g(x_0) - h(x_0)}{u - x_0} \\
&= \lim_{u \to x_0} \left[\frac{g(u) - g(x_0)}{u - x_0} + \frac{h(u) - h(x_0)}{u - x_0} \right] \\
&= \lim_{u \to x_0} \frac{g(u) - g(x_0)}{u - x_0} + \lim_{u \to x_0} \frac{h(u) - h(x_0)}{u - x_0} \\
&= g'(x_0) + h'(x_0).
\end{aligned}$$

Corollary. *The derivative of a sum of any finite number of functions of x equals the sum of their derivatives.*

Theorem 3. *If $f(x) = ax^n$, where a is any constant and n is any rational constant, then $f'(x_0) = anx_0^{n-1}$.*

Proof: First let n be a positive integer. Then

$$\begin{aligned}
f'(x_0) &= \lim_{u \to x_0} \frac{au^n - ax_0^n}{u - x_0} \\
&= a \lim_{u \to x_0} \frac{u^n - x_0^n}{u - x_0} \\
&= a \lim_{u \to x_0} (u^{n-1} + u^{n-2}x_0 + u^{n-3}x_0^2 + \cdots + ux_0^{n-2} + x_0^{n-1}) \\
&= anx_0^{n-1}.
\end{aligned}$$

This result holds if n is any rational number, but the proof for that, more general, case is more easily given later. The student may use the formula for the present without proof.

Example. If $f(x) = 3\sqrt{x} = 3x^{1/2}$, $f'(x_0) = \frac{3}{2}x_0^{-1/2}$, if $x_0 \neq 0$.

Theorem 4. *If $f(x) = [g(x)][h(x)]$, then*

$$f'(x_0) = [g(x_0)][h'(x_0)] + [h(x_0)][g'(x_0)].$$

Proof: From the definition of the derivative,

$$\begin{aligned}
f'(x_0) &= \lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0} \\
&= \lim_{u \to x_0} \frac{g(u)h(u) - g(x_0)h(x_0)}{u - x_0} \\
&= \lim_{u \to x_0} \frac{g(u)h(u) - g(u)h(x_0) + g(u)h(x_0) - g(x_0)h(x_0)}{u - x_0} \\
&= \lim_{u \to x_0} g(u)\frac{h(u) - h(x_0)}{u - x_0} + \lim_{u \to x_0} h(x_0)\frac{g(u) - g(x_0)}{u - x_0} \\
&= g(x_0) \cdot h'(x_0) + h(x_0) \cdot g'(x_0).
\end{aligned}$$

Theorem 5. *If* $f(x) = \dfrac{g(x)}{h(x)}$, $f'(x_0) = \dfrac{h(x_0)g'(x_0) - g(x_0)h'(x_0)}{[h(x_0)]^2}$, *if* $h(x_0) \neq 0$.

Proof: From the definition of the derivative,

$$\begin{aligned}
f'(x_0) &= \lim_{u\to x_0} \frac{\dfrac{g(u)}{h(u)} - \dfrac{g(x_0)}{h(x_0)}}{u - x_0} \\
&= \lim_{u\to x_0} \frac{h(x_0)g(u) - g(x_0)h(u)}{h(u)h(x_0)(u - x_0)} \\
&= \lim_{u\to x_0} \frac{h(x_0)g(u) - h(x_0)g(x_0) - g(x_0)h(u) + g(x_0)h(x_0)}{h(x_0)h(u)(u - x_0)} \\
&= \lim_{u\to x_0} \frac{h(x_0)[g(u) - g(x_0)] - g(x_0)[h(u) - h(x_0)]}{h(x_0)h(u)(u - x_0)} \\
&= \lim_{u\to x_0} \frac{h(x_0)\dfrac{g(u) - g(x_0)}{u - x_0} - g(x_0)\dfrac{h(u) - h(x_0)}{u - x_0}}{h(x_0)h(u)} \\
&= \frac{h(x_0)g'(x_0) - g(x_0)h'(x_0)}{[h(x_0)]^2}.
\end{aligned}$$

Theorem 6. *If* $f(x) = g(v)$ *where* $v = h(x)$ *and if* $g'(v_0)$ and $h'(x_0)$ exist, *where* $v_0 = h(x_0)$, *then* $f'(x_0)$ *exists and is given by*

$$f'(x_0) = g'(v_0)h'(x_0).$$

Proof: Let $s = h(u)$, and write

$$\begin{aligned}
f'(x_0) &= \lim_{u\to x_0} \frac{f(u) - f(x_0)}{u - x_0} \\
&= \lim_{u\to x_0} \frac{g(s) - g(v_0)}{u - x_0} \\
&= \lim_{\substack{u\to x_0 \\ s\to v_0}} \left[\frac{g(s) - g(v_0)}{s - v_0} \cdot \frac{s - v_0}{u - x_0}\right]^* \\
&= \left[\lim_{s\to v_0} \frac{g(s) - g(v_0)}{s - v_0}\right] \cdot \left[\lim_{u\to x_0} \frac{h(u) - h(x_0)}{u - x_0}\right] \\
&= g'(v_0) \cdot h'(x_0).
\end{aligned}$$

* Note that $h'(x_0)$ exists by hypothesis, implying that $v = h(x)$ is continuous at $x = x_0$ and, hence, that $s \to v_0$ as $u \to x_0$.

The multiplication of numerator and denominator by $s - v_0$, as performed here, is legitimate if, in some deleted neighborhood of x_0, there is no u such that $s = h(u)$ and $v_0 = h(x_0)$ have the same values. If, on the other

Corollary. *If* $f(x) = a[h(x)]^n$, *then* $f'(x_0) = an[h(x_0)]^{n-1}h'(x_0)$.

Exercise 1. Prove the above corollary.

Theorem 7. *If* $y = f(x)$ *is a function of* x *having an inverse* $x = g(y)$, *then* $g'(y_0) = \dfrac{1}{f'(x_0)}$ *where* $y_0 = f(x_0)$.

Proof: Let $w = f(u)$, then

$$f'(x_0) = \lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0} = \lim_{u \to x_0} \frac{w - y_0}{u - x_0}$$

and

$$\begin{aligned} g'(y_0) &= \lim_{w \to y_0} \frac{g(w) - g(y_0)}{w - y_0} = \lim_{w \to y_0} \frac{u - x_0}{w - y_0} \\ &= \lim_{w \to y_0} \frac{1}{\dfrac{w - y_0}{u - x_0}} \\ &= \frac{1}{\displaystyle\lim_{w \to y_0} \frac{w - y_0}{u - x_0}} \\ &= \frac{1}{\displaystyle\lim_{u \to x_0} \frac{w - y_0}{u - x_0}} \\ &= \frac{1}{f'(x_0)}. \end{aligned}$$

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hand, there is a u in every deleted neighborhood of x_0 for which $h(u) = h(x_0)$, then, by taking only such values, we have

$$f'(x_0) = \lim_{u \to x_0} \frac{f(u) - f(x_0)}{u - x_0} = \lim_{u \to x_0} \frac{g(v_0) - g(v_0)}{u - x_0} = 0.$$

Then, also, taking only those values of u, we have

$$h'(x_0) = \lim_{u \to x_0} \frac{h(u) - h(x_0)}{u - x_0} = \lim_{u \to x_0} \frac{h(x_0) - h(x_0)}{u - x_0} = 0.$$

This special approach to x_0 by u is legitimate since $h'(x_0)$ is defined as the limit of $\dfrac{h(u) - h(x_0)}{u - x_0}$, regardless of the mode of approach of u to x_0. By hypothesis $g'(v_0)$ is defined, and hence finite, and in the case at hand both $f'(x_0)$ and $h'(x_0)$ are zero, so that the relation

$$f'(x_0) = g'(v_0)h'(x_0)$$

is still satisfied.

NOTE: What justification is there in using $u \to x_0$ and $w \to y_0$ interchangeably?

At this stage, let us observe that the symbol x_0 that we have employed in writing $f'(x_0)$ denoted *any* value of x for which the limit of $\dfrac{f(u) - f(x_0)}{u - x_0}$, as $u \to x_0$, exists. Should we call that value just x, the derivative, computed for that value, would evidently be designated as $f'(x)$. In fact, we may define $f'(x)$ as $\lim\limits_{u \to x} \dfrac{f(u) - f(x)}{u - x}$, and all the results that have been so far developed will carry over, with the change from x_0 to x. With that in mind, we recapitulate the seven theorems developed in this section as

Theorem 1. *If* $f(x) = c, \quad f'(x) = 0.$

Theorem 2. *If* $f(x) = g(x) + h(x), \quad f'(x) = g'(x) + h'(x).$

Theorem 3. *If* $f(x) = ax^n, \quad f'(x) = nax^{n-1}.$

Theorem 4. *If* $f(x) = g(x) \cdot h(x),$

$$f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x).$$

Theorem 5. *If* $f(x) = \dfrac{g(x)}{h(x)}, f'(x) = \dfrac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2}.$

Theorem 6. *If* $f(x) = g(v)$, *where* $v = h(x), f'(x) = g'(v) \cdot h'(x).$

Corollary. *If* $f(x) = a[h(x)]^n; \quad f'(x) = an[h(x)]^{n-1}h'(x).$

Theorem 7. *If* $y = f(x)$, *where* $x = g(y), \quad g'(y) = \dfrac{1}{f'(x)}.$

Exercise 2. If $f(x) = g(x) \cdot h(x) \cdot s(x)$, prove that

$$f'(x) = g(x) \cdot h(x) \cdot s'(x) + g(x) \cdot s(x) \cdot h'(x) + h(x) \cdot s(x) \cdot g'(x).$$

Exercise 3. If $f(x) = a \cdot g(x)$, where a is constant, prove that

$$f'(x) = a \cdot g'(x).$$

Problems

1. Find $f'(x)$ in each of the following cases, employing the indicated theorem:

(*a*) $f(x) = 7.$ (Theorem 1.)
(*b*) $f(x) = 5x^3.$ (Theorem 3.)
(*c*) $f(x) = 2x^4 - 3x^2.$ (Theorems 2 and 3.)

(*d*) $f(x) = 8x^3 - 5x^2 + 9x + 2$. (Theorems 1, 2, and 3.)
(*e*) $f(x) = (x^4 - x^3)(2x^2 - x)$. (Theorem 4.)
(*f*) $f(x) = (x - x^5)(x^4 + x^2)$. (Theorem 4.)
(*g*) $f(x) = \dfrac{x^2 - 8x + 12}{2x - 1}$. (Theorem 5.)
(*h*) $f(x) = \dfrac{1 - x}{1 + x}$. *Ans.* $\dfrac{-2}{(1 + x)^2}$.
(*i*) $f(x) = 10(3x^2 - 2x)^5$. (Corollary to Theorem 6.)
(*j*) $f(x) = 7(2x^3 - 8x^2)^4$. *Ans.* $28(2x^3 - 8x^2)^3(6x^2 - 16x)$.
(*k*) $f(x) = 3(4x^5 - 8x^3)^7 + 9(2x - 1)^5$.
(*l*) $f(x) = \dfrac{x^3}{(2 - x)^3}$. *Ans.* $\dfrac{6x^2}{(2 - x)^4}$.
(*m*) $f(x) = (x - x^2)^4(3 + x^3)^5$.
Ans. $(x - x^2)^3(3 + x^3)^4(19x^3 - 23x^4 - 24x + 12)$.
(*n*) $f(x) = ax^4(2 - 3x)^5$. *Ans.* $ax^3(2 - 3x)^4(8 - 27x)$.
(*o*) $f(x) = \sqrt{x}$. (Theorem 3.)
(*p*) $f(x) = 4\sqrt[5]{2x^3}$. *Ans.* $\dfrac{24x^2}{5(2x^3)^{\frac{4}{5}}}$.
(*q*) $f(x) = 2\sqrt{x^3 + x^2 + x}$. *Ans.* $\dfrac{3x^2 + 2x + 1}{\sqrt{x^3 + x^2 + x}}$.
(*r*) $f(x) = \dfrac{1}{x}$. (Theorem 3.)
(*s*) $f(x) = \dfrac{15}{(2x^4 - x^2 + 5)^3}$. (Theorem 6, corollary.)

2. In each of the following cases find the equation of the line tangent to the given curve at the given point:

(*a*) $y = 2x^3 - 3x$, (2,10). KEY: $f(x) = 2x^3 - 3x$, so $f'(x) = 6x^2 - 3$. Hence, $f(2) = 10$ and $f'(2) = 21$. The equation of the tangent is therefore $y - 10 = 21(x - 2)$.
(*b*) $y = 4x^2 - 3x + 2$ at the point where $x = 1$. *Ans.* $5x - y = 2$.
(*c*) $y = x^7 - x^6$, (0,0). *Ans.* $y = 0$.
(*d*) $y = 5x^4 - 4x^5 + 6$, (0,6). *Ans.* $y = 6$.
(*e*) $y = x^3 + x^2 + x + 4$, (0,4). *Ans.* $y = x + 4$.
(*f*) $y = (x^2 - 3x + 1)^5$, (0,1). *Ans.* $15x + y = 1$.
(*g*) $y = \dfrac{2 - x}{3 + x^2}$, (2,0). *Ans.* $x + 7y = 2$.

3. Find the equation of the line *normal* to the given curve at the given point, *i.e.*, perpendicular, at the point of contact, to the line tangent to the curve at the given point.

(*a*) $y = 3x^2 + 4x - 5$ at $x = 1$. *Ans.* $x + 10y = 21$.
(*b*) $y = 3x^2 + 2x + 4$ at (0,4). *Ans.* $x + 2y = 8$.
(*c*) $y = -3x^6 + 15x$ at (0,0). *Ans.* $x + 15y = 0$.
(*d*) $y = x^4 + x^3 - 8x + 3$ at (1,−3). *Ans.* $x - y = 4$.
(*e*) $y = (3x^3 + x - 2)^3$ at (0,−8). *Ans.* $x + 12y + 96 = 0$.
(*f*) $y = (x^3 - x + 1)(x^2 + x - 2)$ at (0,−2).
Ans. $x + 3y + 6 = 0$.

4. Find $f'(x)$ in the following cases:

(a) $f(x) = \sqrt{4x^2 - 3x}$.

(b) $f(x) = \dfrac{6}{x^3}$.

(c) $f(x) = \sqrt{x^2 + x}$. *Ans.* $\dfrac{2x + 1}{2\sqrt{x^2 + x}}$.

(d) $f(x) = \dfrac{1}{x}$.

(e) $f(x) = \sqrt[3]{4x^2 - x}$.

(f) $f(x) = \dfrac{-2}{\sqrt{x}}$.

(g) $f(x) = \dfrac{\sqrt[3]{x}}{\sqrt{x + 1}}$. *Ans.* $\dfrac{2 - x}{6x^{2/3}(x + 1)^{3/2}}$.

(h) $f(x) = \sqrt[4]{x^3}\sqrt[5]{x + 5}$. *Ans.* $\dfrac{19x + 75}{20x^{1/4}(x + 5)^{4/5}}$.

(i) $f(x) = \sqrt{2ax}$.

(j) $f(x) = ax^2\sqrt{a - x}$. *Ans.* $\dfrac{ax(4a - 5x)}{2\sqrt{a - x}}$.

(k) $f(x) = \sqrt{\dfrac{x - a}{x + a}}$. *Ans.* $\dfrac{a}{(x + a)^2}\sqrt{\dfrac{x + a}{x - a}}$.

(l) $f(x) = \dfrac{\sqrt{x - 2}}{\sqrt{4 - x}}$. *Ans.* $\dfrac{1}{(4 - x)^{3/2}\sqrt{x - 2}}$.

5. Find the coordinates of the points on the following curves at which the given value of m is the slope of the line tangent to the curve.

(a) $y = 2x^3 + 3x^2 - 10x + 6$, $m = 2$.

KEY: We have

$$f(x) = 2x^3 + 3x^2 - 10x + 6,$$

hence

$$f'(x) = 6x^2 + 6x - 10.$$

Setting this equal to the given value of m, we have

$$\begin{aligned} 6x^2 + 6x - 10 &= 2, \\ x^2 + x - 2 &= 0, \\ x = 1, \qquad x &= -2. \end{aligned}$$

At $x = 1$, we have $y = 1$. At $x = -2$, we have $y = 22$. Hence, $(1,1)$ and $(-2,22)$ are the required points.

(b) $y = x^2 - 6x + 11$, $m = 0$. *Ans.* $(3,2)$.

(c) $y = 3x^2 - 5x + 4$, $m = -2/3$.

(d) $y = x^4 - \frac{8}{3}x^3 - 16x^2 - 12x + 12$, $m = -12$.

Ans. $(0,12)$, $(4, -620/3)$, $(-2, 28/3)$.

6. Find the angle between the two curves at their points of intersection, *i.e.*, the angle between their tangents drawn at their points of intersection.

(a) $y = x^2 - 5$, $y = 2x^2 - 14$. *Ans.* $\tan^{-1} 6/73$.

(b) $y = x^3 + 1$, $y = 2x^2 + 1$. *Ans.* $0°$, $\tan^{-1} 4/97$.

(c) $y = 3x^2 - 4$, $y = 9x - 10$. *Ans.* $\tan^{-1} 3/55$, $\tan^{-1} 3/109$

7. A ball rolling down an inclined plane is found to travel a distance of s ft. in t sec., where $s = 6t^2$. What is its velocity at the time t? At the end of 2 sec.? When will its velocity be 60 ft. per second? What is the rate of change of its velocity? HINT. Recall that the velocity is given by $s'(t)$. *Ans.* $12t$, 24, 5, 12.

8. A ball is rolled up an inclined plane and found to follow the law $s = 24t - 2t^2$, where s represents distance traveled in feet and t represents time in seconds. What is its velocity in terms of the time t? What is its velocity at the end of 4 sec.? At the end of 7 sec.? Explain. At what time and at what distance does it stop?

9. Find the equation of the line tangent to the curve $y = 3x^2 - 4x$ and perpendicular to the line $x + 2y = 6$. *Ans.* $2x - y = 3$.

10. Find the equation of the line tangent to the curve $y = x^3 - 4x$ and parallel to the line $x + y = 5$. *Ans.* $x + y = \pm 2$.

11. Find the equation of the line normal to the curve $y = x^4 - 3x$ and parallel to the line $2x + 2y = 7$. *Ans.* $x + y + 1 = 0$.

12. Find the equation of the line tangent to the curve $y = 2x^2 - 1$ and passing through $(4, 13)$. HINT. If (u,v) is the point of tangency, the slope of the tangent is $4u$. That slope is also equal to $\frac{13 - v}{4 - u}$. A second equation arises from the fact that (u,v) is on the curve.

Ans. $4x - y = 3$, $28x - y = 99$.

13. Find the equation of the line tangent to the curve $y = 2x^2 + x - 8$ and passing through $(4,10)$. *Ans.* $5x - y = 10$, $29x - y = 106$.

14. Find the area of the triangle formed by the coordinate axes and the line tangent to the curve $y = x^3 - x$ at $(2,6)$. *Ans.* $^{128}/_{11}$.

15. If $y = f(x)$ is defined by $x = 3 - 4y^2 + y^3$, find $f'(x)$. HINT. Use Theorem 7.

16. Find the equation of the line tangent to the curve $x = 4 - y + y^4$ at the point $(4,1)$. *Ans.* $x - 3y = 1$.

17. Find the equation of the line tangent to the curve $x = (y^3 + y - 1)^3$ at the point $(1,1)$. *Ans.* $x - 12y + 11 = 0$.

18. Prove Theorem 3 for the case when n is a negative integer. HINT. Put $n = -m$, obtaining $f(x) = a/x^m$, and apply Theorem 5.

50. The Derivative as a Quotient. Suppose a body is moving on a straight line according to the law

$$s = 5t^2 + 12t,$$

where s represents distance, say in feet, from a fixed position, t sec. after starting. If we wish to know the distance traveled over in the one-half second after $t = 37$, we can substitute $t = 37.5$ and $t = 37$ in the equation and subtract the results. However, using the formal relation

$$\textit{distance} = \textit{velocity} \times \textit{time},$$

and taking as velocity the derivative $10t + 12$ with $t = 37$, and $\frac{1}{2}$ as the time, we obtain

$$ds = (10t + 12)(\tfrac{1}{2}) = (370 + 12)(\tfrac{1}{2}) = {}^{382}\!/_{2} = 191 \text{ ft.}$$

(The meaning of the symbol, ds, will be explained presently.)

By substituting $t = 37$ and $t = 37.5$ directly in the given equation and subtracting, we obtain

$$\begin{aligned}
5(37.5)^2 + 12(37.5) - 5(37)^2 - 12(37) \\
&= 5[(37.5)^2 - (37)^2] + 12(37.5 - 37) \\
&= 5(37.5 - 37)(37.5 + 37) + 12(37.5 - 37) \\
&= \frac{5}{2}(74.5) + \frac{12}{2} = \frac{372.5 + 12}{2} = \frac{384.5}{2} = 192.25
\end{aligned}$$

The discrepancy between the two results, 191 and 192.25, is due to the fact that the first method assumes a constant velocity over the half second, which is not exactly true. However, it affords a much faster computation and a good approximation to the result.

In the same way, since $f'(x)$ represents the instantaneous rate of change of $f(x)$ per unit change in x, the approximate change in $f(x)$, due to a change dx in x, is given by

$$f'(x)\,dx.$$

If we represent $f(x)$ by y and write

$$y = f(x),$$

we can then write

$$dy = f'(x)\,dx \tag{72}$$

where dx is an arbitrary change in x, called the *differential* of x, and dy is the *approximate* change in y, called the *differential* of y, obtained by assuming that the rate of change of $f(x)$ remains constant over the whole interval from x to $x + dx$.

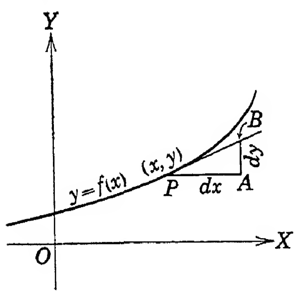

FIG. 101.

In the previous illustration, the relation used amounted to

$$ds = s'(t)\,dt,$$

where dt is the change in the value of t and ds is the corresponding (approximate) change in the value of s.

From the geometrical point of view, if $y = f(x)$ is the equation of a curve, then $f'(x)$ is the slope of the line tangent to the curve at the point whose abscissa is x, and we may think of dx as an arbitrary change or increment in x. In Fig. 101, dx is represented by the line PA and dy by the line AB (both lines directed).

In view of (72), we may, and will freely, from now on, write $f'(x)$ as dy/dx, where $y = f(x)$. For example, if $y = f(x) = x^3$, we shall represent the derivative of $f(x)$ with respect to x indifferently as $f'(x) = 3x^2$ or $dy/dx = 3x^2$.

In this quotient notation for the derivative we now restate Theorems 1 to 7 of the preceding section:

Theorem 1. *If* $y = c$, $dy/dx = 0$.

Theorem 2. *If* $y = u + v$, $\dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$.

Theorem 3. *If* $y = ax^n$, $dy/dx = anx^{n-1}$.

Theorem 4. *If* $y = uv$, $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

Theorem 5. *If* $y = \dfrac{u}{v}$, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$.

Theorem 6. *If* $y = f(u)$, *where* u *is a function of* x,

$$\frac{dy}{dx} = \frac{d[f(u)]}{du} \cdot \frac{du}{dx}.$$

Corollary. *If* $y = au^n$, $\dfrac{dy}{dx} = anu^{n-1}\dfrac{du}{dx}$.

Theorem 7. *If* $y = f(x)$, $\dfrac{dy}{dx} = \dfrac{1}{\dfrac{dx}{dy}}$.

Exercise 1. Prove: If $y = f(x)$ and $y_0 = f(x_0)$, then as x increases from x_0, y increases from y_0, if $f'(x_0)$ is positive.

Exercise 2. Prove: If $y = f(x)$, $y_0 = f(x_0)$ and $f'(x_0)$ are negative, then as x increases from x_0, y decreases from y_0 and as x decreases from x_0, y increases from y_0.

Exercise 3. Prove:

(a) If $y = c$, $dy = 0$.

(b) If $y = au^n$, where u is a function of x, or of any independent variable whatever, then $dy = nau^{n-1}\,du$, in other words, $d(au^n) = nau^{n-1}\,du$.

(c) $d(u + v) = du + dv$,
(d) $d(uv) = u\,dv + v\,du$,
(e) $d\left(\dfrac{u}{v}\right) = \dfrac{v\,du - u\,dv}{v^2}$,

where u and v are functions of any independent variable whatever.

Problems

1. Find the approximate change in the square of a number x caused by increasing x by the amount $1/100$.

KEY: Let $y = x^2$. Then $dy = 2x\,dx$. Putting $dx = 1/100$, we have $dy = 2x/100 = x/50$.

2. Find the approximate change in the cube of a number n, caused by increasing the number by $1/1000$. *Ans.* $3n^2/1000$.

3. Find the approximate change in the volume v of a sphere of radius r, caused by increasing the radius 1 per cent. HINT: $dr = r/100$.
Ans. $\pi r^3/25$.

4. A circular plot of garden has a radius of 100 ft. Find the approximate increase in area if the radius is extended 1 ft. *Ans.* 200π sq. ft.

5. Find the approximate change in $x^3 - 5x^2$ when x changes from 10 to 9.97. HINT: Find $d(x^3 - 5x^2)$ if $x = 10$ and $dx = -0.03$. *Ans.* -6.

6. Find the approximate value of

(a) $\sqrt[3]{8.02}$.

(b) $\dfrac{1}{(5.01)^2 + 5.01}$.

7. Approximately how much can x be increased above 10 without changing the value of $5x^2 - 2x$ more than 0.1? *Ans.* $1/980$.

8. In measuring a cube its edge is found to be 15 in. but is in doubt by 0.01 in. By what approximate amount is its volume in doubt? The area of one of its faces? *Ans.* 6.75 cu. in.; 0.3 sq. in.

9. If the length of a pendulum, in feet, is given by the formula $4\pi^2 l = gt^2$, where $g = 32.2$ and t is the time, in seconds, for one complete cycle, find the length of a pendulum which swings through one cycle per second. If a clock with a pendulum 1 ft. long gains 5 min. per day, by what approximate amount should the pendulum be lengthened?

10. Find dy in terms of x and dx, and also dy/dx, for each of the following:

(a) $y = 5x^3 - 7x^2$. *Ans.* $dy = (15x^2 - 14x)dx$.

(b) $y = (8x^4 - 2x)^6$.

(c) $y = (7x^2 - 4x + 1)^3 - (2x - 1)^6$.

(d) $y = (5x^7 - 8x^4)(10x^2 + x)$.

(e) $y = (3x - 1)^4(4x^2 - 2)^2$.

(f) $y = \dfrac{2x - 3x^2}{(4x + 5)^2}$.

51. Derivatives of Higher Order. Up to the present we have spoken of the derivative of $f(x)$ at a fixed point $[x, f(x)]$. However, since this derivative depends upon x, it is a function of x if the latter is regarded as a variable. As such, $f'(x)$ may itself be differentiated,* its derivative being defined as

$$\lim_{u \to x} \frac{f'(u) - f'(x)}{u - x},$$

if this limit exists. This derivative of $f'(x)$ is called the *second* derivative of $f(x)$ with respect to x, and is designated by the symbol $f''(x)$—read "f second of x." Continuing in this way, we shall call the derivative of $f''(x)$ the third derivative of $f(x)$ and represent it by the symbol $f'''(x)$—read "f third of x"—and,

* To differentiate a function means to obtain its derivative.

in general, call the derivative of the nth derivative the $(n + 1)$st derivative, representing them by $f^{(n)}(x)$—read "f upper n of x"—and $f^{(n+1)}(x)$. The quotient notation for the higher derivatives as well as the D_x notation (not previously introduced but widely used) is made clear by the table

$f'(x)$	$f''(x)$	$f'''(x)$	$f^{iv}(x)$	. . .
$\dfrac{df(x)}{dx}$	$\dfrac{d^2f(x)}{dx^2}$	$\dfrac{d^3f(x)}{dx^3}$	$\dfrac{d^4f(x)}{dx^4}$	. . .
$D_xf(x)$	$D_x{}^2f(x)$	$D_x{}^3f(x)$	$D_x{}^4f(x)$	. . .

If $y = f(x)$, these same quantities may also be expressed as

y'	y''	y'''	y^{iv}	. . .
$\dfrac{dy}{dx}$	$\dfrac{d^2y}{dx^2}$	$\dfrac{d^3y}{dx^3}$	$\dfrac{d^4y}{dx^4}$	. . .
D_xy	$D_x{}^2y$	$D_x{}^3y$	$D_x{}^4y$	. . .

It may be observed at once if $s = f(t)$, where t represents time and s distance on a straight line, that, since s' represents the instantaneous velocity, s'' represents the rate of change of the velocity per unit time, or the instantaneous *acceleration.*

The geometrical significance of the second and higher derivatives will be brought out later.

Problems

1. Find all the derivatives of $f(x) = x^4 - 7x^3 + 12x^2 - 4x + 9$ which are different from zero.

2. Find y', y'', and y''' in the following cases:

(*a*) $y = x^4 + 8x^3 - 2x^2.$

(*b*) $y = \dfrac{x^3}{x - 1}.$

(*c*) $y = \sqrt{4 + t^2}.$

(*d*) $y = \dfrac{x}{\sqrt{x + 3}}.$

(*e*) $y = \dfrac{x^2}{1 + x}.$

(*f*) $y = \dfrac{1 + x}{1 - x}.$

(*g*) $y = x\sqrt{2x - 3}.$

(*h*) $y = 5(x - 7)^{10}.$

3. Find y in terms of x if y'' has the form specified and, when $x = 0$, y and y' have the values specified.

(*a*) $y'' = 5x^3 - 6x + 2,\ y' = 3\ \ y = 2.$

KEY: $y' = \frac{5}{4}x^4 - 3x^2 + 2x + c$, where c is a constant.

$y = \frac{1}{4}x^5 - x^3 + x^2 + cx + k$, where k is a constant.

$x = 0$ gives $y = k = 2$ and $y' = c = 3$.

Hence, the required result is $y = \frac{1}{4}x^5 - x^3 + x^2 + 3x + 2$.

(b) $y'' = 0$, $y' = -5$, $y = 4$. *Ans.* $y = -5x + 4$.

(c) $y'' = 9x + 4$, $y' = 7$, $y = 0$. *Ans.* $y = \dfrac{3x^3}{2} + 2x^2 + 7x$.

(d) $y'' = 12x^2 - 6x + 2$, $y' = -2$, $y = 5$.

Ans. $y = x^4 - x^3 + x^2 - 2x + 5$.

4. Find an expression for $\dfrac{d^n(1/x)}{dx^n}$.

5. Find the nth derivative of $\dfrac{3}{2x+5}$ with respect to x.

6. Find the nth derivative of $(1 + x)^{3/2}$ with respect to x.

7. If $y = uv$, where u and v are functions of x, prove that

(a) $y'' = uv'' + 2u'v' + u''v$.

(b) $y''' = uv''' + 3u'v'' + 3u''v' + u'''v$.

(c) $y^{iv} = uv^{iv} + 4u'v''' + 6u''v'' + 4u'''v' + u^{iv}v$.

8. Show that $D_x{}^2y$ and $D_y{}^2x$ are not reciprocals in general, by finding them for the case $xy = 1$. HINT: $y = 1/x$ and $x = 1/y$.

9. Show that if $y = (x + \sqrt{a^2 + x^2})^n$,

$$(x^2 + a^2)y'' + xy' - n^2y = 0.$$

10. Find the equation of the tangent and of the normal to the curve $y = x^4 - 3x^2 + x - 6$ at the point where $y'' = 6$.

11. Find the value of y'' at that point of the curve $y = x^3 - 3x^2 + 3x$ where the tangent is parallel to the line $2y - 6x = 5$. *Ans.* ± 6.

12. A particle moves on a straight line in such a way that

$$s = 16t^2 + 10 \quad (s \text{ in feet, } t \text{ in seconds}).$$

Find the velocity and acceleration at the instant when the particle is 74 ft. from the starting point.

13. A particle moves on a straight line in such a way that

$$s = t^3 + 2t^2 - t + 1 \quad (t \text{ in seconds and } s \text{ in feet}).$$

(a) Find at what instant of time—at what value of t—its acceleration is 16 ft./sec/sec. *Ans.* $t = 2$.

(b) What is its velocity at that instant? *Ans.* 19 ft./sec.

(c) What is its distance from the origin—what is the value of s—at that instant?

(d) How far is the particle from the starting point—the position when $t = 0$—at that instant? *Ans.* 14 ft.

14. At what rate, per unit change of the abscissa, is the slope of the curve $y = x^3 + 1$ changing at the point on the curve where $x = 1$? *Ans.* 6.

52. The Derivative of $\log_a v$. To obtain the derivative of $\log_a v$ (where v is a function of x) with respect to x, we take first the

Case $v = x$. To find $f'(x)$, where $f(x) = \log_a x$, consider

$$\lim_{u \to x} \frac{f(u) - f(x)}{u - x} = \lim_{u \to x} \frac{\log_a u - \log_a x}{u - x},$$

which limit, by definition, is the derivative sought. Since the numerator and denominator of the above fraction simultaneously tend to zero as a limit as $u \to x$, we proceed, using the properties of the logarithm, to throw the fraction into another form, as follows:

$$\begin{aligned}
D_x \log_a x &= \lim_{u \to x} \frac{\log_a u - \log_a x}{u - x} \\
&= \lim_{u \to x} \frac{\log_a \left(\frac{u}{x}\right)}{u - x} \\
&= \lim_{u \to x} \frac{\log_a \left(1 + \frac{u - x}{x}\right)}{u - x} \\
&= \lim_{u \to x} \left[\frac{1}{x} \cdot \frac{\log_a \left(1 + \frac{u - x}{x}\right)}{\frac{u - x}{x}}\right] \\
&= \lim_{u \to x} \left[\left(\frac{1}{x}\right) \cdot \log_a \left(1 + \frac{u - x}{x}\right)^{\frac{x}{u - x}}\right] \\
&= \lim_{u \to x} \left(\frac{1}{x}\right) \cdot \lim_{u \to x} \log_a \left(1 + \frac{u - x}{x}\right)^{\frac{x}{u - x}} \\
&= \frac{1}{x} \cdot \log_a \left[\lim_{u - x \to 0} \left(1 + \frac{u - x}{x}\right)^{\frac{x}{u - x}}\right].
\end{aligned}$$

Now, the last limit is a case of $\lim\limits_{t \to 0} (1 + t)^{1/t}$, treated on page 154, and therefore equals the number e. We have thus arrived at the result:

$$\frac{d}{dx}(\log_a x) = \left(\frac{1}{x}\right) \cdot \log_a e. \tag{73}$$

Before considering directly the case in which v is any function of x, let us note that it may now be solved by use of Theorem 6 of Sec. 50 (page 177), *viz.*:

$$\frac{d}{dx}[f(v)] = \frac{d}{dv}[f(v)] \cdot \frac{dv}{dx}.$$

We thus obtain

$$\frac{d}{dx}(\log_a v) = \frac{d}{dv}(\log_a v) \cdot \frac{dv}{dx},$$

or, in view of (73),

$$\frac{d}{dx}(\log_a v) = \frac{1}{v} \cdot \log_a e \cdot \frac{dv}{dx}. \tag{74}$$

Examples: $\frac{d}{dx}[\log_{10} (5x^3 - 2x)] = \frac{1}{5x^3 - 2x} \cdot \log_{10} e \cdot (15x^2 - 2);$

$$\frac{d}{dx}[\log_5 x] = \frac{1}{x} \cdot \log_5 e.$$

General Case. It seems advisable, also, to obtain directly the value of $\frac{d}{dx}(\log_a v)$, without the intervention of formula (73). To that end, it will be convenient to introduce the symbol Δ which is very frequently met with in the calculus.

Definition. If a variable has its value changed from x to u, we define Δx as $u - x$, i.e., $u = x + \Delta x$.

The symbol Δx, thus, designates the change, or increment, in the value of the variable whose initial value is x. Likewise, Δy designates the change, or increment, in the value of the variable whose initial value is y (and, hence, the new value $y + \Delta y$).

If, then, when dealing with a function of x, say $f(x)$, we set $y = f(x)$, a change, Δx, in the value of x causes a change, Δy, in the value of y, and we obtain

$$y + \Delta y = f(x + \Delta x).$$

In this notation, the derivative of $f(x)$ with respect to x may be written as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{f(x + \Delta x) - f(x)}{\Delta x}\right]. \tag{75}$$

N.B. In the case of an independent variable, x, *i.e.*, a variable whose values are assigned arbitrarily, the increment Δx, as

defined above, has precisely the same value as dx, the differential of x. Not so with a dependent variable, say y, defined as $y = f(x)$. For in its case, $\Delta y = f(x + \Delta x) - f(x)$, while $dy = f'(x) \cdot dx$ (Sec. 50), and the two are, in general, not the same.

Illustration. For $y = x^2 + x$, compute Δy and dy for $x = 1$ and

$$\Delta x = dx = \tfrac{1}{2}$$

(hence $x + \Delta x = \tfrac{3}{2}$).

Computation:

$$\begin{aligned}\Delta y &= [(x + \Delta x)^2 + (x + \Delta x)] - (x^2 + x)\\ &= [(\tfrac{3}{2})^2 + \tfrac{3}{2}] - (1^2 + 1) = \tfrac{7}{4}.\\ dy &= \frac{d}{dx}(x^2 + x) \cdot dx = (2x + 1) \cdot dx = 3 \cdot \frac{1}{2} = \frac{3}{2}.\end{aligned}$$

The exact change in y, as x changes from 1 to $\tfrac{3}{2}$, is, of course, the value of Δy, *i.e.*, $\tfrac{7}{4}$. The value, $\tfrac{3}{2}$, of dy is an approximation to that change. It is, indeed, the change which y would have taken on if the rate of change of y with respect to x had remained equal to 3, its value when $x = 1$.

Exercise 1. Show, in the case $y = x^2 + x$, that $\Delta y - dy = \overline{\Delta x}^2$.

Exercise 2. Show, in the case $y = x^3 - 2x$, that $\Delta y - dy = 3x\,\overline{\Delta x}^2 + \overline{\Delta x}^3$.

Exercise 3. Verify for both Exercise 1 and Exercise 2 that

$$\lim_{\Delta x \to 0} \frac{\Delta y - dy}{\Delta x} = 0.$$

Exercise 4. Prove that for any function, y, of x, $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y - dy}{\Delta x} = 0$.

Let us now employ this notation to differentiate

$$y = \log_a v,$$

where v, and hence y, are functions of x. In starting with a certain value of x and the corresponding values of v and y, we let x take on the increment Δx. Designating the corresponding increments in v and y by Δv and Δy, respectively, we obtain

$$y + \Delta y = \log_a (v + \Delta v),$$

or

$$\Delta y = \log_a (v + \Delta v) - \log_a v,$$

and

$$\frac{\Delta y}{\Delta x} = \frac{\log_a (v + \Delta v) - \log_a v}{\Delta x}.$$

Hence

$$\frac{d}{dx}(\log_a v) = \frac{dy}{dx} = \lim_{\Delta x\to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x\to 0}\left[\frac{\log_a (v+\Delta v) - \log_a v}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\frac{\log_a\left(\frac{v+\Delta v}{v}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\frac{\log_a\left(1+\frac{\Delta v}{v}\right)}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left[\left(\frac{1}{v}\right)\cdot\left(\frac{v}{\Delta v}\right)\cdot\log_a\left(1+\frac{\Delta v}{v}\right)\cdot\left(\frac{\Delta v}{\Delta x}\right)\right]$$

$$= \lim_{\Delta x\to 0}\left[\frac{1}{v}\cdot\log_a\left(1+\frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}\cdot\frac{\Delta v}{\Delta x}\right]$$

$$= \lim_{\Delta x\to 0}\left(\frac{1}{v}\right)\cdot\lim_{\Delta x\to 0}\left[\log_a\left(1+\frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}\right]\cdot\lim_{\Delta x\to 0}\left(\frac{\Delta v}{\Delta x}\right)$$

$$= \left(\frac{1}{v}\right)\cdot\log_a\left[\lim_{\substack{\Delta x\to 0\\ \therefore \Delta v\to 0\\ \therefore \frac{\Delta v}{v}\to 0}}\left(1+\frac{\Delta v}{v}\right)^{\frac{v}{\Delta v}}\right]\cdot\frac{dv}{dx},$$

or, finally,

$$\frac{d}{dx}(\log_a v) = \frac{1}{v}\cdot\log_a e\cdot\frac{dv}{dx}. \tag{76}$$

From this, in turn, setting $v = x$, we obtain

$$\frac{d}{dx}(\log_a x) = \frac{1}{x}\cdot\log_a e\cdot\frac{dx}{dx} = \frac{1}{x}\cdot\log_a e,$$

i.e., formula (73).

Exercise 5. By setting $a = e$ in formula (76), obtain

$$\frac{d}{dx}\log_e v = \frac{1}{v}\cdot\frac{dv}{dx} \tag{77}$$

and

$$\frac{d}{dx}\log_e x = \frac{1}{x}. \tag{78}$$

We shall understand by $\log v$, with no base indicated, $\log_e v$. (Logarithms to this base are found in any good table.) The last two formulas can, thus, be displayed as

$$\frac{d}{dx}\log v = \frac{1}{v}\cdot\frac{dv}{dx}$$

and

$$\frac{d}{dx} \log x = \frac{1}{x}.$$

53. Logarithmic Differentiation. It is frequently convenient to take the logarithm of a function before differentiating. For example, let

$$y = u \cdot v \cdot w,$$

where u, v, and w are functions of x. Then

$$\log y = \log u + \log v + \log w$$

and differentiation, with respect to x, gives

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{u} \cdot \frac{du}{dx} + \frac{1}{v} \cdot \frac{dv}{dx} + \frac{1}{w} \cdot \frac{dw}{dx}.$$

Solving for dy/dx and replacing y by $u \cdot v \cdot w$, we have

$$\frac{dy}{dx} = vw\frac{du}{dx} + uw\frac{dv}{dx} + uv\frac{dw}{dx}.$$

Similarly, the rules for the derivatives of products and quotients, as given on page 177 can be most easily derived by logarithmic differentiation. Thus, the formula

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

can be derived very simply by logarithmic differentiation and, moreover, the proof holds for any rational value of n.

Let

$$y = x^n.$$

Then

$$\log y = n \log x$$

and

$$\frac{1}{y} \cdot \frac{dy}{dx} = n \cdot \frac{1}{x},$$

hence

$$\frac{dy}{dx} = \frac{ny}{x} = \frac{nx^n}{x} = nx^{n-1}$$

Employing this method to differentiate a^v, where a is positive, take the logarithm of both sides, to the base e, in the equality $y = a^v$, obtaining

$$\log y = v \log a.$$

Differentiating with respect to x, if v is a function of x, we obtain

$$\frac{1}{y} \cdot \frac{dy}{dx} = \log a \cdot \frac{dv}{dx},$$

hence

$$\frac{dy}{dx} = y \cdot \log a \cdot \frac{dv}{dx}$$
$$= a^v \cdot \log a \cdot \frac{dv}{dx},$$

that is

$$\frac{d}{dx}(a^v) = a^v \cdot \log a \cdot \frac{dv}{dx}. \tag{79}$$

In particular, if $a = e$, we have

$$\frac{d}{dx}(e^v) = e^v \cdot \frac{dv}{dx} \tag{80}$$

and, if $v = x$,

$$\frac{d}{dx}(e^x) = e^x. \tag{81}$$

Problems

1. Find the derivative of each of the following with respect to x:

(a) $\log x^2$. *Ans.* $2/x$.

(b) $\log (5x^2 - 3x)$. *Ans.* $\dfrac{10x - 3}{5x^2 - 3x}$.

(c) $\log_7 x$.

(d) $\log_5 (9x^2)$.

(e) $\log \dfrac{x - 1}{x + 1}$. *Ans.* $\dfrac{2}{x^2 - 1}$.

(f) $\log u^5$.

(g) $\log (\log x)$.

(h) $\log \sqrt[3]{1 - 5x}$. *Ans.* $\dfrac{-5}{3(1 - 5x)}$.

(i) e^{5x}.

(j) e^{-x}.

(k) e^{4x^3}.

(l) $\log (e^x + e^{-x})$.

(m) 10^x.

(n) $(5)2^{x^4}$.

(o) $\log \dfrac{x^2 - a^2}{x^2 + a^2}$. *Ans.* $\dfrac{4a^2x}{x^4 - a^4}$.

(p) $\log \sqrt{\dfrac{2 - 3x}{1 + x}}$. *Ans.* $-\dfrac{5}{2(1 + x)(2 - 3x)}$.

(q) $e^{\log 5x}$. *Ans.* 5.

(r) $10^{\log_{10}\sqrt{2-x}}$. *Ans.* $-\dfrac{1}{2\sqrt{2 - x}}$.

(s) $\log (e^{\sqrt{x}})$. *Ans.* $\dfrac{1}{2\sqrt{x}}$.

(t) $\log_{a^2} (a^{x^2-1})$. *Ans.* x.

2. Find dy/dx by logarithmic differentiation for each of the following cases:

(a) $y = 2^{x^2+2x}$.

(b) $y = x^5$.

(c) $y = x^x$. *Ans.* $x^x(1 + \log x)$.

(d) $y = (2x - 1)^{(x^2-2)}$.

(e) $y = x^{1/3}(1 - x)^{4/3}(2 - x)^{5/3}$.

(f) $y = \sqrt[4]{\dfrac{x^3 - 2}{x^2(x + 1)^3}}$.

(g) $y = (x + 3)^5(x - 1)^6(4x^2 - 3x)^7$.

(h) $y = uvwt$.

(i) $y = x^{(e^x)}$.

Ans. $e^x \cdot x^{e^x}\left(\dfrac{1}{x} + \log x\right)$.

(j) $y = (\log x)^{x^2}$.

3. Find a formula for $\dfrac{d^n(e^{ax})}{dx^n}$.

4. Find a formula for $\dfrac{d^n(\log x)}{dx^n}$.

5. Find the points on the curve $y = \log x^2$ where

(a) the line tangent to the curve has slope $\frac{1}{2}$;

(b) the normal has the slope -1.

6. Find the equation of the tangent and of the normal to the curve $y = e^{2x-3}$ at the point where $x = 1$.

7. Show that the curves $y = \log (x - 3)$ and $y = x^2 - 6x + 8$ have the point (4,0) in common. Find the angle between them at that point.

8. Find Δy and dy for the functions indicated below at the indicated values of x and $dx = \Delta x$.

(a) $y = x^2 - 2x$, $x = 1$, $\Delta x = 0.2$.

(b) $y = \log x^3$, $x = e$, $\Delta x = -e/10$.

(c) $y = e^{4x+2}$, $x = \frac{1}{4}$, $\Delta x = \frac{1}{16}$.

54. The Derivatives of Trigonometric Functions. Let x be measured in radians. Then, if we set $y = \sin x$,

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0}\left(\frac{\Delta y}{\Delta x}\right) = \lim_{\Delta x \to 0}\left(\frac{\sin (x + \Delta x) - \sin x}{\Delta x}\right) \\
&= \lim_{\Delta x \to 0}\left(\frac{2 \cos \left(x + \dfrac{\Delta x}{2}\right) \sin \dfrac{\Delta x}{2}}{\Delta x}\right) \\
&= \lim_{\Delta x \to 0}\left[\cos \left(x + \frac{\Delta x}{2}\right) \cdot \frac{\sin \dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}}\right] \\
&= \left[\lim_{\Delta x \to 0} \cos \left(x + \frac{\Delta x}{2}\right)\right] \cdot \left[\lim_{\frac{\Delta x}{2} \to 0} \frac{\sin \dfrac{\Delta x}{2}}{\dfrac{\Delta x}{2}}\right] \\
&= \cos x,
\end{aligned}$$

since $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$, as shown on page 154. Hence

$$\frac{d}{dx}(\sin x) = \cos x. \tag{82}$$

Using the formula

$$\frac{d}{dx}[f(v)] = f'(v) \cdot \frac{dv}{dx}$$

and the result just derived, we find that

$$\frac{d}{dx}(\sin v) = (\cos v) \cdot \frac{dv}{dx}. \tag{83}$$

Exercise 1. Obtain formula (83) directly, without the intervention of formula (82).

To find the derivative of $\cos x$, note that

$$\begin{aligned}\frac{d}{dx}(\cos x) &= \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} + x\right)\right] = \cos\left(\frac{\pi}{2} + x\right)\frac{d}{dx}\left(\frac{\pi}{2} + x\right)\\ &= \cos\left(\frac{\pi}{2} + x\right) = -\sin x,\end{aligned}$$

that is,

$$\frac{d}{dx}(\cos x) = -\sin x, \tag{84}$$

and, hence,

$$\frac{d}{dx}(\cos v) = -\sin v\frac{dv}{dx}. \tag{85}$$

Exercise 2. Obtain formula (84) by setting $y = \cos x$ and finding $\lim_{\Delta x \to 0} (\Delta y/\Delta x)$.

The derivatives of the remaining trigonometric functions follow readily from their expressions in terms of $\sin v$ and $\cos v$. Thus,

$$\begin{aligned}\frac{d}{dx}(\tan v) &= \frac{d}{dx}\frac{\sin v}{\cos v} = \frac{\cos^2 v + \sin^2 v}{\cos^2 v} \cdot \frac{dv}{dx}\\ &= \frac{1}{\cos^2 v} \cdot \frac{dv}{dx} = \sec^2 v \cdot \frac{dv}{dx},\end{aligned} \tag{86}$$

$$\begin{aligned}\frac{d}{dx}(\operatorname{ctn} v) &= \frac{d}{dx}\frac{\cos v}{\sin v} = \frac{-\sin^2 v - \cos^2 v}{\sin^2 v} \cdot \frac{dv}{dx}\\ &= \frac{-1}{\sin^2 v} \cdot \frac{dv}{dx} = -\csc^2 v \cdot \frac{dv}{dx},\end{aligned} \tag{87}$$

$$\frac{d}{dx}(\sec v) = \frac{d}{dx}(\cos v)^{-1} = -(\cos v)^{-2} \cdot \frac{d}{dx}(\cos v)$$
$$= \frac{\sin v}{\cos^2 v} \cdot \frac{dv}{dx} = \sec v \tan v \cdot \frac{dv}{dx}, \quad (88)$$

and

$$\frac{d}{dx}(\csc v) = \frac{d}{dx}(\sin v)^{-1} = -(\sin v)^{-2} \cdot \frac{d}{dx}(\sin v)$$
$$= \frac{-\cos v}{\sin^2 v} \cdot \frac{dv}{dx} = -\csc v \operatorname{ctn} v \cdot \frac{dv}{dx}. \quad (89)$$

The derivatives of the inverse trigonometric functions are found as follows: If

$$y = \sin^{-1} v,^*$$

then

$$\sin y = v$$

and, upon differentiating, we have

$$\cos y \frac{dy}{dx} = \frac{dv}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{\cos y} \cdot \frac{dv}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}} \cdot \frac{dv}{dx} = \frac{1}{\sqrt{1 - v^2}} \cdot \frac{dv}{dx},$$

i.e.,

$$\frac{d}{dx}(\sin^{-1} v) = \frac{1}{\sqrt{1 - v^2}} \cdot \frac{dv}{dx}. \quad (90)$$

Note that the substitution of the positive quantity $\sqrt{1 - \sin^2 y}$, for $\cos y$ is legitimate in view of the restriction upon $\sin^{-1} v$.

The formulas for the derivatives of other inverse trigonometric formulas are similarly derived. The work is left to the student in the following exercises.

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* Here $\sin^{-1} v$ is understood to be a single-valued function of v, *viz.*, that angle between $-\pi/2$ and $+\pi/2$, inclusive, whose sine is equal to v, and which is known as the *principal value* of $\sin^{-1} v$. For such values of y, $\cos y$ is positive and equals $\sqrt{1 - \sin^2 y}$. Similarly, $\cos^{-1} v$, $\sec^{-1} v$, and $\csc^{-1} v$ are understood to designate the principal values of those angles, as indicated in Exercises 3, 6, and 7. For such values it follows that $\sin(\cos^{-1} v) = \sqrt{1 - v^2}$, $\tan(\sec^{-1} v) = \sqrt{v^2 - 1}$ and $\operatorname{ctn}(\csc^{-1} v) = \sqrt{v^2 - 1}$.

Exercise 3. Prove that

$$\frac{d}{dx}(\cos^{-1} v) = \frac{-1}{\sqrt{1 - v^2}} \cdot \frac{dv}{dx} \tag{91}$$

if $0 \leqq \cos^{-1} v \leqq \pi$.

Exercise 4. Prove that

$$\frac{d}{dx}(\tan^{-1} v) = \frac{1}{1 + v^2} \cdot \frac{dv}{dx}. \tag{92}$$

Exercise 5. Prove that

$$\frac{d}{dx}(\text{ctn}^{-1} v) = \frac{-1}{1 + v^2} \cdot \frac{dv}{dx}. \tag{93}$$

Exercise 6. Prove that

$$\frac{d}{dx}(\sec^{-1} v) = \frac{1}{v\sqrt{v^2 - 1}} \cdot \frac{dv}{dx} \tag{94}$$

if $0 \leqq \sec^{-1} v < \pi/2$ when $v \geqq 1$, and $-\pi \leqq \sec^{-1} v < -\pi/2$ when $v \leqq -1$.

Exercise 7. Prove that

$$\frac{d}{dx}(\csc^{-1} v) = \frac{-1}{v\sqrt{v^2 - 1}} \cdot \frac{dv}{dx} \tag{95}$$

if $0 < \csc^{-1} v \leqq \pi/2$ when $v \geqq 1$, and $-\pi < \csc^{-1} v \leqq -\pi/2$ when $v \leqq -1$.

Problems

1. Find the derivative with respect to x, of each of the following functions:

(*a*) $\sin (2x)$. *Ans.* $2 \cos 2x$.

(*b*) $\cos (5x/2)$. *Ans.* $-\frac{5}{2} \sin (5x/2)$.

(*c*) $\tan (1 - x)$. *Ans.* $-\sec^2 (1 - x)$.

(*d*) $\text{ctn} (x/3)$.

(*e*) $\sec (2x - 4)$.

(*f*) $\csc (5x/2)$.

(*g*) $\sin (x^2)$.

(*h*) $\cos^2 (5x)$. *Ans.* $-5 \sin 10x$.

(*i*) $\sin^3 (x^{2/3})$.

(*j*) $\tan^3 (4x)$. *Ans.* $12 \tan^2 4x \sec^2 4x$.

(*k*) $\log (\tan x)$. *Ans.* $2 \csc 2x$.

(*l*) $\log (\sec x + \tan x)$.

(*m*) $\sec \left(\frac{1 + x}{1 - x}\right)$.

(*n*) $x - \tan x$.

(*o*) $e^{\sin 3x}$. *Ans.* $3e^{\sin 3x} \cos 3x$.

(*p*) $\sqrt{1 - \cos x}$.

(*q*) $\sqrt[3]{\tan \frac{x}{2}}$. *Ans.* $\dfrac{\sec^2 \frac{x}{2}}{6\left(\tan \frac{x}{2}\right)^{2/3}}$.

(*r*) $10^{\sqrt{\cos 2x}}$.

(*s*) $\sin^{-1} (x/a)$. *Ans.* $1/\sqrt{a^2 - x^2}$.

(*t*) $\cos^{-1} \left(\frac{1}{x}\right)$. *Ans.* $\dfrac{1}{x\sqrt{x^2 - 1}}$.

(*u*) $\tan^{-1} \left(\frac{x}{a}\right)$. *Ans.* $\dfrac{a}{a^2 + x^2}$.

(*v*) $\sec^{-1} \left(\frac{x}{a}\right)$. *Ans.* $\dfrac{a}{x\sqrt{x^2 - a^2}}$.

(*w*) $\tan (\sin^{-1} 2x)$. *Ans.* $2/(1 - 4x^2)^{3/2}$.

(*x*) $\cos^{-1} \left(\text{ctn} \frac{x}{2}\right)$.

(*y*) $(\sin x)^x$. *Ans.* $(\sin x)^x(x \,\text{ctn}\, x + \log \sin x)$.

(*z*) $\text{ctn}^{-1} \left(\frac{1 - x^2}{2x}\right)$. *Ans.* $\dfrac{2}{x^2 + 1}$.

2. Show that if $y = 2e^{3x} \sin 2x$, then $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 0$.

3. If $y = x^3(2 \cos \log x^2 - \sin \log x^2)$, find the value of

$$13y - 5x\frac{dy}{dx} + x^2\frac{d^2y}{dx^2}.$$

4. Find $\frac{d^2y}{dx^2}$ for $y = \tan^{-1}\left(\frac{e^x + e^{-x}}{e^x - e^{-x}}\right)$.

5. Show that when x is measured in degrees $\frac{d}{dx}(\sin x) = \frac{\pi}{180} \cos x$.

6. Find $\frac{d}{dx}(\tan x)$ if x is measured in degrees.

7. Find $\frac{d}{dx}(\sin^{-1} v + \cos^{-1} v)$ and show that the result is consistent with the relation known to hold between $\sin^{-1} v$ and $\cos^{-1} v$.

8. Find $\frac{d}{dx}(\tan^{-1} v + \operatorname{ctn}^{-1} v)$ and show that the result is consistent with the relation known to hold between $\tan^{-1} v$ and $\operatorname{ctn}^{-1} v$.

9. From the formula $\sin 2x = 2 \sin x \cos x$, obtain, by differentiation, a formula for $\cos 2x$.

10. Verify by differentiation that $\tan^{-1}\left(\frac{2x}{1 - x^2}\right) = 2 \tan^{-1} x$.

11. Verify by differentiation that $\cos^{-1}\left(\frac{\sqrt{1 - x^2} - x\sqrt{3}}{2}\right) - \sin^{-1} x$ is a constant.

12. Show that the curves $y = e^{ax^2} \sin bx$ and $y = e^{ax^2}$ are tangent to each other at every point that they have in common.

13. Find the angle between the curves $y = \tan 2x$ and $y = \sin 2x$ at the origin. *Ans.* 0°.

14. A particle moves on a straight line so that its distance s from the origin and the time t that it has been moving, are related by $s = 5 - 2 \cos 3t$.

(*a*) Find its velocity and acceleration at the end of $\pi/2$ units of time.

(*b*) Show that its acceleration at any instant is proportional to its distance from a certain point on the line and directed toward that point. What is the point?

(*c*) What is the numerical value of the greatest velocity and the greatest acceleration that it ever attains?

NOTE: The property proved in (*b*) defines the motion as *simple harmonic motion*.

15. Show that the motion on a straight line defined by

$$s = 2 \sin 3t - 3 \cos 3t$$

is simple harmonic motion. (See note to Problem 14.)

16. Find Δy and dy for each function indicated below and the indicated values of x and $\Delta x = dx$.

(*a*) $y = \sin 3x, \quad x = \pi/6, \quad \Delta x = \pi/18.$

(*b*) $y = \tan 4x, \quad x = \pi/4, \quad \Delta x = -\pi/16.$

55. Hyperbolic Functions and Their Derivatives. The function

$$\frac{e^x - e^{-x}}{2}$$

often occurs in mathematics, and is called the *hyperbolic sine* of x. It is written as

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

The *hyperbolic cosine, hyperbolic tangent, etc.*, are defined as

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}},$$

and

$$\operatorname{csch} x = \frac{2}{e^x - e^{-x}}.$$

These functions possess relations very closely analogous to those for the trigonometric functions, and the formulas for their derivatives follow those for the derivatives of trigonometric functions quite closely. The derivation of these formulas from the definitions is left to the student to verify in the form of exercises.

Exercise 1. Verify that

$$\frac{d}{dx}(\sinh v) = \cosh v \cdot \frac{dv}{dx}. \tag{96}$$

Exercise 2. Verify that

$$\frac{d}{dx}(\cosh v) = \sinh v \cdot \frac{dv}{dx}. \tag{97}$$

Exercise 3. Verify that

$$\frac{d}{dx}(\tanh v) = \operatorname{sech}^2 v \cdot \frac{dv}{dx}. \tag{98}$$

Exercise 4. Verify that

$$\frac{d}{dx}(\coth v) = -\operatorname{csch}^2 v \cdot \frac{dv}{dx}. \tag{99}$$

Exercise 5. Verify that

$$\frac{d}{dx}(\operatorname{sech} v) = -\operatorname{sech} v \tanh v \cdot \frac{dv}{dx}. \tag{100}$$

Exercise 6. Verify that

$$\frac{d}{dx}(\operatorname{csch} v) = -\operatorname{csch} v \coth v \cdot \frac{dv}{dx}. \tag{101}$$

Problems

1. Prove the following identities:

(*a*) $\cosh^2 x - \sinh^2 x = 1.$

(*b*) $1 - \tanh^2 x = \operatorname{sech}^2 x.$

(*c*) $\coth^2 x - 1 = \operatorname{csch}^2 x.$

(*d*) $\sinh (x \pm y) = \sinh x \cdot \cosh y \pm \cosh x \cdot \sinh y.$

(*e*) $\cosh (x \pm y) = \cosh x \cdot \cosh y \pm \sinh x \cdot \sinh y.$

(*f*) $\sinh 2x = 2 \sinh x \cdot \cosh x.$

(*g*) $\cosh 2x = \cosh^2 x + \sinh^2 x.$

(*h*) $\sinh^{-1} x = \log (x + \sqrt{x^2 + 1}).$

(*i*) $\tanh^{-1} x = \frac{1}{2} \log \frac{1 + x}{1 - x}.$

2. Prove each of the following:

(*a*) $\frac{d}{dx}(\sinh^{-1} v) = \frac{1}{\sqrt{1 + v^2}} \cdot \frac{dv}{dx}.$

(*b*) $\frac{d}{dx}(\cosh^{-1} v) = \frac{1}{\sqrt{v^2 - 1}} \cdot \frac{dv}{dx}.$

(*c*) $\frac{d}{dx}(\tanh^{-1} v) = \frac{1}{1 - v^2} \cdot \frac{dv}{dx}.$

3. (*a*) Find $\frac{dy}{dx}$ if $y = \tan^{-1} (\sinh v)$ and v is a function of x. [This function is called the Gudermannian of v and is usually denoted by $gd(v)$.]

Ans. $\operatorname{sech} v \cdot \frac{dv}{dx}.$

(*b*) Prove that $gd(x) = 2 \tan^{-1} (e^x) - \pi/2.$

4. Find $\frac{d^2y}{dx^2}$ if $y = \cosh x^2$, and state its value for $x = 0$.

5. Find $\frac{d^2y}{dx^2}$ if $y = \tanh (e^x)$, and state its value for $x = 0$.

6. Find $\frac{d}{dx}[\sinh^{-1} (3 - x) + \cosh^{-1} (3 - x)].$

7. Find $\frac{d^2y}{dx^2}$ if $y = \tanh^{-1} \left[\frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}\right].$

8. Show that $x = \cosh t$, and $y = \sinh t$ are the parametric equations of an equilateral hyperbola. (Hence the name *hyperbolic*, attached to these functions. What reason is there for calling the trigonometric functions *circular*, as is sometimes done?)

9. Show that if a tangent is drawn at any point of the curve $y = a \cosh \frac{x}{a}$, the length of the perpendicular drawn to the tangent from the projection of the point of contact upon the x-axis is constant.

56. Implicit Functions. A relation between x and y embodied in an equality, as $f(x,y) = 0$, ordinarily defines y as one or more functions of x. For example, the equation

$$x^2 + 3xy - y^2 + 5 = 0$$

defines y as one or the other of the two functions

$$\frac{3x \pm \sqrt{13x^2 + 20}}{2}.$$

In such a case we say that the equation defines y *implicitly* as a function of x. That a relation $f(x,y) = 0$ may fail to define y as a function of x, is manifest in the instance of

$$(x - y)(x + y) + y^2 - 5x + 6 = 0.$$

The conditions under which $f(x,y) = 0$ defines y as a function of x may be found in any book on advanced calculus.

Proceeding with the above example, we now propose to find dy/dx directly from the relation given. It is neither necessary nor desirable first to solve this equation for y. Instead, we proceed as follows: equate the derivatives of the two members with respect to x, obtaining in the case first mentioned

$$2x + 3\left(x \cdot \frac{dy}{dx} + y\right) - 2y \cdot \frac{dy}{dx} = 0.$$

Solving for dy/dx, the quantity that we are after, we obtain

$$\frac{dy}{dx} = \frac{2x + 3y}{2y - 3x}.$$

To find the second derivative of y with respect to x, we differentiate the last relation, obtaining

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{(2y - 3x)(2 + 3y') - (2x + 3y)(2y' - 3)}{(2y - 3x)^2}\\ &= \frac{13y - 13xy'}{(2y - 3x)^2}\\ &= \frac{13y - 13x \cdot \dfrac{2x + 3y}{2y - 3x}}{(2y - 3x)^2}\\ &= \frac{26y^2 - 78xy - 26x^2}{(2y - 3x)^3}.\end{aligned}$$

The numerator, incidentally, may be simplified to

$$(26)(y^2 - 3xy - x^2),$$

which, in view of the relation holding between x and y, *viz.*,

$$x^2 + 3xy - y^2 = -5,$$

reduces to $26(5) = 130$, and we thus have

$$\frac{d^2y}{dx^2} = \frac{130}{(2y - 3x)^3}.$$

Problems

1. Find dy/dx and d^2y/dx^2 in each of the following cases. If a value of x is indicated find the derivatives at that particular point.

(*a*) $x^2 + y^2 = 25$, $x = 3$. *Ans.* $y' = \pm 3/4$, $y'' = \pm 25/64$.

(*b*) $3xy - 2y^2 - 2x = 2$, $x = 4$.
Ans. $y' = -1/8$, $y'' = 13/128$; $y' = 13/8$, $y'' = -13/128$.

(*c*) $x^2 - 3xy + 4y^2 = 8$.
Ans. $y' = \dfrac{3y - 2x}{8y - 3x}$, $y'' = -\dfrac{112}{(8y - 3x)^3}$.

(*d*) $xy = 30$.

(*e*) $x^3 + y^3 - 3xy = 0$.

(*f*) $e^{x+y} + 2x + 3y = 0$, $x = 1$, $y = -1$.
Ans. $y' = -3/4$, $y'' = -1/64$.

(*g*) $\log\sqrt{x^2 + y^2} = \dfrac{1}{2}\tan^{-1}\dfrac{y}{x}$. *Ans.* $y' = \dfrac{2x + y}{x - 2y}$.

(*h*) $x - e^{\frac{y-x}{x}} = 0$. *Ans.* $y' = \dfrac{y}{x} + 1$.

2. (*a*) Is dy/dx defined at $x = 0$ if $y^2 = x^3$? If so, find its value.

HINT: If defined, the derivative is $\lim\limits_{x\to 0} \dfrac{y - 0}{x - 0}$.

(*b*) Examine whether d^2y/dx^2 is defined at $x = 0$.

3. (*a*) If $x^{2/3} + y^{2/3} = a^{2/3}$, find dy/dx and state for what points it is undefined. Find lim (dy/dx) as the point (x,y) approaches these points along the curve.

(*b*) Find d^2y/dx^2. For what points is it undefined?

4. Find the equation of the line tangent to the curve at the given point.

(a) $x^2 + 4y^2 = 8$, $(2, -1)$. *Ans.* $x - 2y = 4$.

(b) $x^{1/2} + y^{1/2} = 11$, $(64,9)$. *Ans.* $3x + 8y = 264$.

(c) $4x^2 - 6y^2 = 9$, (x_1,y_1).

(d) $x^2 = 12y - y^3$, $(-4,2)$.

(e) $x^3 - 2x^2y + xy^2 = 48$, $(3, -1)$. *Ans.* $5x - 3y = 18$.

(f) $\tan^{-1}(xy) + \dfrac{y}{x} = \dfrac{4 + \pi}{4}$, $(1,1)$. *Ans.* $x - 3y + 2 = 0$.

5. Show that the parabolas $y^2 = 8(x + 2)$ and $y^2 = -8(x - 2)$ cross at right angles.

6. Find the angle at which the line $y = x$ cuts the conic

$$x^2 + 2xy + 4y^2 = 63. \qquad \textit{Ans.} \quad \tan^{-1} \tfrac{2}{3}.$$

7. Prove the following:

(a) For the tangent drawn at any point of the curve

$$x^{1/2} + y^{1/2} = a^{1/2},$$

the sum of the intercepts on the coordinate axes is a constant.

(b) For the tangent drawn at any point of the curve

$$x^{2/3} + y^{2/3} = a^{2/3},$$

that part of it included between the coordinate axes is constant in length.

(c) The distance from the origin to the tangent drawn at any point of the curve $x^2 + y^2 = ay$, is equal in length to the ordinate of that point.

8. Find the equation of the line tangent to the curve $3x^2 - 4y^2 = 12$ and passing through the point $(7,6)$. *Ans.* $x - y = 1$, $13x - 15y = 1$.

9. Find the equation of the line tangent to the curve

$$x^2 + 4y^2 - 4x - 8y + 3 = 0$$

and passing through the point $(-1,3)$.

Ans. $x + 4y = 11$, $11x + 4y = 1$.

10. Find the equation of the line tangent to the conic $3x^2 - xy + y^2 = 9$ and perpendicular to the line $8y = 16 - 5x$. *Ans.* $8x - 5y = \pm 18$.

11. Find at what point of the curve $x + \sqrt{xy} + y = 1$, the tangent is parallel to the x-axis. *Ans.* $(1,0)$.

57. Parametric Equations. If x and y are two variables dependent upon a third variable, t, such that

$$\begin{cases} x = f(t), \\ y = g(t), \end{cases}$$

then a value t_0 of t produces corresponding values, x_0 and y_0, of x and y. By pairing these values of x and y, for a set of values of t, we establish a correspondence between x and y, whereby we may regard y as a function of x or x as a function

of y. To find D_xy, in such a case, note that by virtue of Theorem 6 of page 177

$$D_xy = D_ty \cdot D_xt,$$

or, since

$$D_xt = \frac{1}{D_tx},$$

as we know from Theorem 7,

$$D_xy = \frac{D_ty}{D_tx}.$$

Illustration. Given the equations $\begin{cases} x = t^2 + t \\ y = t^3 - t^2 \end{cases}$

we have

$$D_tx = 2t + 1, \qquad D_ty = 3t^2 - 2t$$

and, hence

$$D_xy = \frac{3t^2 - 2t}{2t + 1}.$$

To find $D_x{}^2y$, the second derivative of y with respect to x, we have

$$D_x{}^2y = D_x(D_xy) = D_t(D_xy) \cdot D_xt = \frac{D_t(D_xy)}{D_tx}.$$

Continuing the above illustration, we have

$$D_t(D_xy) = \frac{d}{dt}\left[\frac{3t^2 - 2t}{2t + 1}\right] = \frac{6t^2 + 6t - 2}{(2t + 1)^2},$$

and, as before

$$D_tx = 2t + 1.$$

Hence

$$D_x{}^2y = \frac{6t^2 + 6t - 2}{(2t + 1)^3}.$$

Problems

1. Find D_xy and $D_x{}^2y$ in each case.

(a) $\begin{cases} y = 4 \sin \theta, \\ x = 4 \cos \theta. \end{cases}$ (d) $\begin{cases} y = a(1 - \cos \theta), \\ x = a(\theta - \sin \theta). \end{cases}$

Ans. $D_xy = -\operatorname{ctn} \theta,\ D_x{}^2y = -\dfrac{\csc^3 \theta}{4}.$

(b) $\begin{cases} y = 5 \sec \theta, \\ x = 4 \tan \theta. \end{cases}$ (e) $\begin{cases} y = t^3 + 2t^2, \\ x = t^3 - t. \end{cases}$

(c) $\begin{cases} y = 2t^2 - 3t, \\ x = t^3. \end{cases}$ (f) $\begin{cases} y = e^t \sin t, \\ x = e^t \cos t. \end{cases}$

Ans. $D_x{}^2y = \dfrac{6 - 4t}{9t^5}.$ *Ans.* $D_x{}^2y = \dfrac{2}{e^t(\cos t - \sin t)^3}.$

2. If $y = t^4 - t^2 + 1$ and $x = 2t^4 + t$ are the parametric equations of a curve, find the coordinates of all points at which a line tangent to the curve is parallel to one of the coordinate axes.

3. Find the equation of the line tangent to the curve $y = 5 \cos \theta$, $x = 2 \sin \theta$, at the point defined by $\theta = \pi/3$. *Ans.* $5x\sqrt{3} + 2y = 20$.

4. Find the equations of the horizontal and vertical tangents to the curves below and state the coordinates of the points of contact in each case.

(a) $\begin{cases} y = 1 - 4s^2, \\ x = 4s^2 - 4s. \end{cases}$ *Ans.* $y = 1$, $x = -1$; $(0,1)$, $(-1,0)$.

(b) $\begin{cases} y = 3 \cos \theta - 4 \sin \theta + 6, \\ x = \sin \theta + \cos \theta + 1. \end{cases}$

(c) $\begin{cases} y = \dfrac{3at^2}{1 + t^3}, \\ x = \dfrac{3at}{1 + t^3}. \end{cases}$ *Ans.* $y = 0$, $y = a\sqrt[3]{4}$, $x = a\sqrt[3]{4}$.

CHAPTER IX

APPLICATIONS OF DERIVATIVES

58. Maximum and Minimum Values of Functions. Let $f(x)$ and its derivative, $f'(x)$, be continuous at $x = a$, and let the value of $f'(a)$ be p, a positive number. From the definition of the derivative as

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x},$$

it follows that there is a neighborhood of a such that, if $x = a + \Delta x$ is in that neighborhood, the value of

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

is a positive number. For, we can choose a positive number ϵ, such that $p - \epsilon$ (and, of course, $p + \epsilon$) is still positive, and corresponding to this ϵ there will exist a δ such that whenever $|\Delta x| < \delta$ then $\dfrac{f(a + \Delta x) - f(a)}{\Delta x}$ will satisfy the inequalities

$$p - \epsilon < \frac{f(a + \Delta x) - f(a)}{\Delta x} < p + \epsilon$$

and hence be positive. Thus, in that interval, $f(a + \Delta x) - f(a)$ is positive when Δx is positive; likewise, the two are negative simultaneously. In other words, $f(a + \Delta x) < f(a)$ for $a + \Delta x < a$ and $f(a + \Delta x) > f(a)$ for $a + \Delta x > a$. The function, $f(x)$, thus increases or decreases from $f(a)$ according as x increases or decreases from a. We embody this conclusion in

Theorem 1. *If $f'(a) > 0$, there exists an interval about $x = a$ in which $f(x)$ increases from $f(a)$ as x increases from a, and $f(x)$ decreases from $f(a)$ as x decreases from a.*

If, now, it is given that there is some interval about $x = a$ throughout which $f(x)$ increases as x increases, while $f(x)$ and $f'(x)$ are continuous at $x = a$, then $f(a + \Delta x) - f(a)$ and Δx are

of like signs, and their quotient

$$\frac{f(a + \Delta x) - f(a)}{\Delta x}$$

takes on a sequence of positive values as $\Delta x \to 0$. Hence the limit of the above fraction as $\Delta x \to 0$, *i.e.*, $f'(a)$, must be a positive number or zero. We have thus proved

Theorem 2. *If there exists an interval about $x = a$ in which $f(x)$ increases or decreases from $f(a)$ according as x increases or decreases from a, then $f'(a)$ is positive or zero, provided it exists.*

The case of the negative derivative may be discussed similarly, and the conclusions are stated in the following two theorems.

Theorem 3. *If $f'(b) < 0$, there exists an interval about $x = b$ in which $f(x)$ decreases from $f(b)$ as x increases from b, and $f(x)$ increases from $f(b)$ as x decreases from b.*

Theorem 4. *If there exists an interval about $x = b$ in which $f(x)$ decreases from $f(b)$ as x increases from b, and $f(x)$ increases from $f(b)$ as x decreases from b, then $f'(b)$ is negative or zero, provided it exists.*

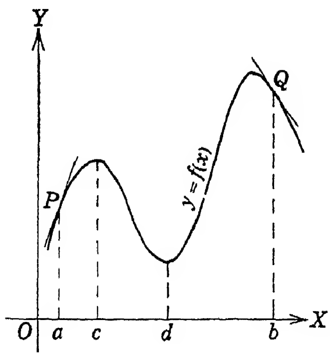

FIG. 102.

The geometrical significance will be made apparent to the student by a glance at Fig. 102, the graph of the equation $y = f(x)$. The positive value of the slope of the tangent at P, and hence of the derivative at $x = a$, is associated with the rising of the curve at P, *i.e.*, with y increasing as x increases, while the negative value of the slope of the tangent at Q, and hence of the derivative at $x = b$, is associated with the falling of the curve at Q, *i.e.*, with y decreasing as x increases.

We are now in a position to treat the following problem. Given a function of x, continuous in the interval $a \leqq x \leqq b$, to find for what values of x, if any, the function takes on maximum values and, likewise, for what values of x, if any, the function takes on minimum values.

To say that a function $f(x)$, continuous at $x = c$, takes on a maximum value at $x = c$, means this: there exists an interval about $x = c$, such that whenever $x = c + \Delta x \neq c$ is in that

interval then $f(c) > f(c + \Delta x)$. In other words, $f(x)$ increases as $x \to c$ from either side. Similarly, to say that $f(x)$ takes on a minimum value at $x = d$, amounts to saying that there exists an interval about $x = d$ such that whenever $x = d + \Delta x \neq d$ is in that interval, $f(d) < f(d + \Delta x)$. In other words $f(x)$ decreases as $x \to d$ from either side.*

The solution of the problem proposed above now rests, in the light of the preceding four theorems, on the following:

Theorem 5. *If, in a certain neighborhood of $x = c$, $f'(x) > 0$ for $x < c$ and $f'(x) < 0$ for $x > c$, $f(x)$ has a maximum value at $x = c$. Conversely, for $f(x)$ to have a maximum value at $x = c$, $f'(x)$ must be positive for $x < c$ and negative for $x > c$ in some neighborhood of $x = c$ [the continuity of $f(x)$ at $x = c$ and the existence of $f'(x)$ in a deleted neighborhood of $x = c$ being understood].*

Theorem 6. *If, in a certain neighborhood of $x = d$, $f'(x) < 0$ for $x < d$ and $f'(x) > 0$ for $x > d$, $f(x)$ has a minimum value at $x = d$. Conversely, for $f(x)$ to have a minimum value at $x = d$, $f'(x)$ must be negative for $x < d$ and positive for $x > d$ in some neighborhood of $x = d$ [the continuity of $f(x)$ at $x = d$ and the existence of $f'(x)$ in a deleted neighborhood of $x = d$ being understood].*

Our search for maximum and minimum values of a function is aided by the property of continuous functions stated as Property 1 in Sec. 47, from which we conclude that if $f'(x)$ is continuous in the interval $a \leqq x \leqq b$ and takes on both positive and negative values in that interval, then there is at least one value of x for which it vanishes (becomes zero). This suggests, as the first step in finding maximum and minimum values of a function (turning points of the corresponding curve, in geometrical language), setting the derivative of the function equal to zero and solving the resulting equation for x. For every value of x thus obtained, *i.e.*, for every root of the equation $f'(x) = 0$, we test the sign of $f'(x)$, in a suitable interval, for values of x less than that root and values of x greater than that root. If $f'(x)$ changes sign when x crosses one of these roots we draw the conclusion in accordance with Theorems 5 and 6.

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* Evidently, the maximum and minimum values of $f(x)$, as defined here, are relative and not absolute. That is, to say that $f(x)$ takes on a maximum value at $x = c$, does not mean that its value at $x = c$ is the greatest it ever takes on; nor does a minimum for $f(x)$ at $x = d$ mean that its value at $x = d$ is the least it ever takes on. A glance at Fig. 102 will make that clear.

Illustration

To test the function $4x^5 - 5x^4 - \frac{40}{3}x^3 + 2$ for maximum and minimum values, we proceed as follows:

$$\begin{aligned} f(x) &= 4x^5 - 5x^4 - \tfrac{40}{3}x^3 + 2 \\ f'(x) &= 20x^4 - 20x^3 - 40x^2 \\ &= 20x^2(x^2 - x - 2) \\ &= 20x^2(x + 1)(x - 2). \end{aligned}$$

$$f'(x) = 0 \qquad \text{at} \qquad x = -1, \qquad x = 0, \qquad x = 2.$$

To study the sign of $f'(x)$ in neighborhoods of $x = -1$, $x = 0$, and $x = 2$, it is well to consider the sign of each factor of $f'(x)$ in those neighborhoods. A scheme like the one below will be found helpful.

	$20x^2$	$x + 1$	$x - 2$	$f'(x)$
$x < -1$	+	−	−	+
$-1 < x < 0$	+	+	−	−
$0 < x < 2$	+	+	−	−
$x > 2$	+	+	+	+

It is thus seen (Theorems 5 and 6) that at $x = -1$ the function has a maximum value, at $x = 0$ the function has neither a maximum nor a minimum value, and at $x = 2$ the function has a minimum value.

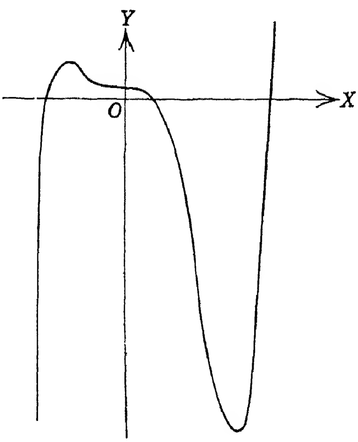

FIG. 103.

The values of the function, at its maximum and minimum (critical values, as they are sometimes called) are, evidently $f(-1)$ and $f(2)$, *i.e.*,

$$\tfrac{19}{3} \qquad \text{and} \qquad -\tfrac{170}{3}.$$

If it should be desired now to plot the corresponding curve, *i.e.*, the curve $y = 4x^5 - 5x^4 - \frac{40}{3}x^3 + 2$, the foregoing information will prove useful. It yields the following conclusions: the tangents at $(-1, \frac{19}{3})$, $(0,2)$, and $(2, -\frac{170}{3})$ are parallel to OX; the curve rises for $x < -1$, falls for

$$-1 < x < 2$$

and rises for $x > 2$. The general form of the graph is evidently as shown in Fig. 103.

Exercise 1. How many real roots does the equation

$$4x^5 - 5x^4 - \tfrac{40}{3}x^3 + 2 = 0$$

possess? How many complex roots? What can you state about the values of the real roots?

Let us now inquire to what extent the hypothesis of $f'(x)$ being continuous at $x = a$ is indispensable, for a critical value to exist at $x = a$. Consider, for example, the function

$$y = x^{2/3},$$

for which the derivative is

$$\frac{dy}{dx} = \frac{2}{3}x^{-1/3},$$

and is undefined, and consequently discontinuous, at $x = 0$. However, for $x < 0$, $dy/dx < 0$ and for $x > 0$, $dy/dx > 0$. The function, which is defined and continuous at $x = 0$, thus decreases, as x increases, for all negative values of x, and increases, as x increases, for all positive values of x. This is all that is essential for the function to possess a minimum value at $x = 0$. To be sure, Property 1, Sec. 47, cannot now be invoked, as it was in the discussion above, to insure the vanishing of the derivative for some value of x. But the essential condition, once more, is the change in sign of the derivative. If that condition is met (as it is in our illustration at $x = 0$), a critical value of the function is present.

The experience with the illustration just displayed suggests that in case the derivative becomes undefined for some value of x, but the function itself is continuous there, an examination of that value of x for a possible critical value of the function should be made, in accordance with Theorems 5 and 6.

Problems

1. Examine each of the following functions for critical values. Use the results of that examination to sketch the corresponding curve.

(a) $y = x^2 - 4x$. *Ans.* Min. at $(2, -4)$.
(b) $y = 2x^2 - 6x + 1$.
(c) $y = 6x - x^2 - 3$.
(d) $y = 3x + 5$.
(e) $y = 2x^3 - 9x^2 + 2$.
Ans. Max. at $(0,2)$, min. at $(3, -25)$.
(f) $y = x^3 - 3x^2 - 9x$.

(g) $y = 3x^4 + 4x^3 - 36x^2 + 7$.
Ans. Min. at $(-3, -182)$, $(2, -57)$; max. at $(0,7)$.

(h) $3y = 4x^3 - 9x^2 - 12x + 17$.

(i) $12y = 3x^4 - 16x^3 + 24x^2 - 5$. *Ans.* Min. at $(0, -5/12)$.

(j) $y = (x - 3)^3 + 1$.

(k) $y = x^4 + 192x^2 - 800x$.

(l) $20y = 4x^5 + 5x^4 - 40x^3 + 9$.

(m) $y = \dfrac{2}{1 + x^2}$.

(n) $y = (x - 2)^4(x + 3)^3$.
Ans. Max. at $x = -6/7$, min. at $x = 2$.

(o) $y = x^2(x - 2)^{1/3}$.

(p) $y = (x + 1)\sqrt{-x}$. *Ans.* Max. at $x = -1/3$.

(q) $y = \sqrt{\dfrac{x - 1}{3 - x}}$. *Ans.* Min. at $x = 1$.

(r) $y = \dfrac{x - 1}{x^2 - 2x + 2}$.

(s) $y = 1 + |1 - x|$.*

(t) $y = \tan x$.

(u) $y = 4x^2 + 2x + e^{-2x}$. *Ans.* Min. at $x = 0$.

(v) $y = e^{-\frac{x^3}{12}}$.

(w) $y = \cosh x$.

(x) $y = xe^x$.

(y) $y = \log (1 + x^2)$.

(z) $y = x \log x$.

2. Examine the function $(x - a)^n$, where n is a positive integer, for critical values.

3. Show that the quadratic function $ax^2 + 2bx + c$ $(a \neq 0)$ has a maximum or minimum value, depending on the sign of a, but not both.

4. Given the curve $y = 2ax^3 + 3bx^2 + 6cx - d$, show that it has one maximum and one minimum point (*i.e.*, one point of maximum ordinate and one point of minimum ordinate) if $b^2 - 4ac > 0$, but neither if

$$b^2 - 4ac \leqq 0.$$

5. Find the dimensions of the rectangle of largest area which can be inscribed in a given circle of radius r.

KEY: The quantity to be made a maximum must be expressed as a function of a suitable independent variable. Evidently, if we call one of the dimensions x, the other will be $\sqrt{4r^2 - x^2}$, and the area A is given as a function of x by

$$A = x\sqrt{4r^2 - x^2}.$$

To find what value of x will cause A to have a maximum value, we find first

$$\frac{dA}{dx} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = \frac{2(r\sqrt{2} - x)(r\sqrt{2} + x)}{\sqrt{4r^2 - x^2}}.$$

* HINT: $y = 2 - x$ for $x < 1$ and $y = x$ for $x > 1$.

This derivative vanishes for $x = r\sqrt{2}$ and changes sign from positive to negative when x changes from less than $r\sqrt{2}$ to greater than $r\sqrt{2}$. Hence, $x = r\sqrt{2}$ is one of the dimensions sought. The other dimension, *i.e.*, $\sqrt{4r^2 - x^2}$, is found also to be $r\sqrt{2}$, and the figure is a square. Why do we not employ also the value $x = -r\sqrt{2}$ which likewise causes the vanishing of the derivative?

ALTERNATIVE METHOD: Let us call one dimension of the rectangle x and the other dimension y. The area A is given by the equation

$$A = xy,$$

where the product xy is a function of one independent variable, say x, in view of the equality

$$x^2 + y^2 = 4r^2.$$

Differentiating with respect to x, we have

$$y + x\frac{dy}{dx} = 0 \text{ (zero because } A \text{ is to be a maximum)},$$

$$2x + 2y\frac{dy}{dx} = 0 \text{ (zero because } r \text{ is constant)}.$$

Eliminating dy/dx between these equations, we find that

$$x^2 = y^2$$

or

$$x = y.$$

Hence the figure is a square and its sides are easily found to be $r\sqrt{2}$.

6. A rectangular field which is to contain 1800 sq. yd. is fenced off along the bank of a straight river. If no fence is needed along the river, what must be the dimensions of the field to use the least amount of fencing material? *Ans.* 60 yd., 30 yd.

HINT: If the dimension parallel to the river is x yd., the other dimension is $1800/x$, and the length of the fence is

$$l = x + \frac{3600}{x}.$$

7. A rectangular box is made by cutting out squares from the corners of a rectangular sheet of metal 6 in. by 16 in. and bending up the sides. What size square should be cut out to produce a box of maximum volume? *Ans.* of side $\frac{4}{3}$ in.

8. Find the minimum distance from the point (8,0) to the parabola $y^2 = 12x$. *Ans.* $2\sqrt{15}$.

9. Find the minimum distance from the line $3x - 4y = 15$ to the parabola $x^2 = 4y$. *Ans.* $\frac{51}{20}$.

10. The range of a projectile is given by $\dfrac{v_0^2 \sin 2\varphi}{g}$, where v_0 is the initial velocity, g is the acceleration of gravity, and φ is the angle which the gun

makes with the horizontal. If v_0 and g are constants, find the angle φ which produces the maximum range.

11. A piece of wire is cut into two pieces. One piece is bent into an equilateral triangle and the other is bent into a square. What should be the ratio of their lengths in order that the combined area of the two figures should be a minimum? *Ans.* $9:4\sqrt{3}$.

12. From a piece of wire 20 ft. long a frame is made for a drum-shaped lamp shade consisting of two equal circles, two diametral wires at the top, and four vertical spacers from bottom to top. What should be the dimensions of the shade to have maximum cubical contents?

13. A page of print is to have 54 sq. in. of printed area, a margin of 2 in. at the bottom, and a margin of 1 in. at the sides and top. What are the dimensions of the smallest sheet of paper which will accommodate?

14. A rectangular field has a fence entirely around it and two other fences crossing it parallel to the ends. What should be the ratio of the two dimensions of the field in order that it may require the least fence for a given total area? *Ans.* 2:1.

15. A closed tin can is to be made of given capacity and minimum material. What should be the relation between its height and the diameter of its base? *Ans.* Height = diameter.

16. A vessel is cylindrical in form and open at the top. Find the relation between its dimensions in order that it may have the greatest capacity for a given amount of material. *Ans.* Height = radius.

17. If the stiffness of a beam varies as the breadth and the cube of the depth, find the dimensions of the beam of greatest stiffness which can be cut from a log 16 in. in diameter. *Ans.* Breadth = 8 in.

18. A man is in a boat 6 miles from a straight shore and wants to reach a point A, on the shore, 12 miles from the nearest point on the shore. If he can row 4 miles an hour and walk 5 miles an hour, where should he land in order to reach A in the least time? *Ans.* 4 miles from A.

19. The sum of two positive numbers is 15. If the product of one of the numbers by the square of the other is to be a maximum, what are the numbers? *Ans.* 5,10.

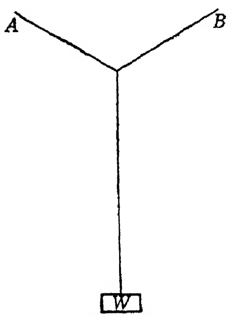

FIG. 104.

20. What size sector should be cut from a circle in order that the remainder will form the surface of a cone of maximum volume?

21. A weight is to be held 8 ft. below the horizontal line AB by a Y-shaped wire. If A and B are 6 ft. apart, what is the shortest total length of wire which can be used? *Ans.* $8 + 3\sqrt{3}$ ft.

22. A tablet 6 ft. in height is mounted on a wall with its lower edge 4 ft. above the eye of an observer. How far back from the wall should the observer stand in order that the tablet subtend the greatest possible vertical angle at his eye? *Ans.* $\sqrt{40}$ ft.

23. A man is at S, a distance of a miles out in the water from a straight shore and wishes to reach a point, P, a distance of b miles inland from the shore and not opposite his location in the water. If his rate on water is r_1

miles per hour and his rate on land is r_2 miles per hour and he makes the trip from S to P in the least possible time, landing at O, show that the angles ω and λ of Fig. 105 are connected by the relation

$$\frac{\sin \omega}{\sin \lambda} = \frac{r_1}{r_2}.$$

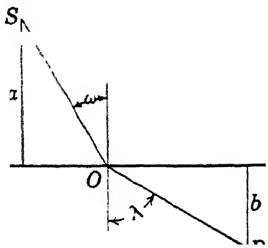

FIG. 105.

24. (*a*) A steamer burns $a + bx^3$ tons of fuel per hour in running x miles per hour relative to the water. Find the speed at which the least amount of fuel is used in traveling a given distance in still water.

(*b*) If the number of tons of fuel burned per hour in (*a*) is 50 when the speed is 8 miles per hour and 31.5 when the speed is 6 miles per hour (both in still water), find the most economical speed in traveling against a current $\frac{4}{3}$ miles per hour. *Ans.* 6 miles per hour.

25. A direct current of I amp. is driven through a resistance of $\frac{2}{15}$ ohm by a battery of 24 dry cells, arranged in x blocks of cells in parallel, each block consisting of y cells in series. Find the number in series, *i.e.*, find y, in order that the current be a maximum, if the internal resistance of each cell is 0.05 ohm. HINT: Let e = the e.m.f. of each cell. Then the total e.m.f. of the battery is $E = ey$. The internal resistance of one block of y cells is $5y/100$. The total internal resistance of the battery is the reciprocal of the sum of the reciprocals of the resistances of the blocks, or

$$\frac{1}{\frac{100}{5y} + \frac{100}{5y} + \cdots + \frac{100}{5y}}$$

Since the number of blocks is x this reduces to $y/20x$. The total resistance (sum of internal and external) is

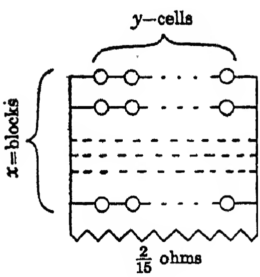

FIG. 106.

$$R = \frac{2}{15} + \frac{y}{20x}.$$

From Ohm's law, $E = RI$, we have

$$I = \frac{ey}{\frac{2}{15} + \frac{y}{20x}}$$

the quantity to be maximized. Also, since there are 24 cells in all, we have

$$xy = 24.$$

After obtaining the answer, compare the total internal resistance with the given external resistance. *Ans.* $y = 8$.

26. If a generator of direct current has an e.m.f. of E volts and an internal resistance of r ohms, what external resistance R will consume the most power? HINT: The total resistance is $r + R$ and, hence, by Ohm's law, the

current is

$$I = \frac{E}{r + R},$$

while the power is the resistance times the square of the current, or

$$P = \frac{E^2R}{(r + R)^2}.$$

Here P is to be a maximum and E and r are constants. *Ans.* $R = r$.

27. (*a*) Find the largest possible area for an isosceles triangle in which each of the two equal sides is 10 in. long.

(*b*) Find the largest possible area for a triangle inscribed in a circle, one side of the triangle being a chord 10 ft. long and cutting off a 70° arc.

28. Determine the dimensions of the right circular cylinder of greatest volume that can be inscribed in a given sphere of radius R.

29. Determine the dimensions of the right circular cone of least volume that can be circumscribed about a given sphere of radius R.

30. Find the coordinates of the vertices of the rectangle of greatest area which has two vertices on the x-axis and two on the curve $y = e^{-x^2}$.

31. Tangents are drawn to the ellipse of semiaxes a and b. Show that the least length of any tangent, intercepted by the axes of the ellipse, is $a + b$.

32. Find the equation of that tangent to the curve $y = x^3 - 3x^2 + 2x$ which has the least slope.

33. Show that the shortest distance between a curve and a fixed point external to it is measured along a normal to the curve which passes through the given point.

34. A wall 8 ft. high is 3 ft., $4\frac{1}{2}$ in. from a house. Find the length of the shortest ladder that will reach from the ground to the house when leaning over the wall. *Ans.* $15\frac{5}{8}$ ft.

35. A light is placed directly above the center of a circular plot of ground of radius 20 ft. What must be its height above the ground in order that the edge may get a maximum illumination, given that the intensity of light at any point of the edge is proportional to the cosine of the angle of incidence (*i.e.*, the angle between the ray of light and the vertical) and inversely proportional to the square of the distance from the source of light?

36. Rectangles are drawn with two vertices on the right-hand branch of the hyperbola $x^2 - 3y^2 = 36$ and two on the chord joining the points (12,6) and (12, −6). If they are made to revolve about the x-axis, find the vertices of the rectangle that will produce the maximum volume.

37. A manufacturer offers to deliver to a dealer 300 radio sets at $90.00 a set and to reduce the price per set on the entire lot by 25¢ for each additional set above 300. Find the amount of the largest possible transaction, in dollars, between the manufacturer and the dealer under these conditions.

38. A window is in the form of a rectangle surmounted by a semicircle. If its perimeter is 30 ft., find its dimensions so that it may admit the maximum amount of light.

59. Derived Curves; Concavity; Points of Inflection. In Fig. 107 are shown the graphs of three equations, $y = f(x)$ $y = f'(x)$ and $y = f''(x)$. The curve $y = f'(x)$ is known as the *first derived curve* of the curve $y = f(x)$. The first derived curve, $y = f''(x)$, of the curve $y = f'(x)$, is called the *second derived curve* of the original curve, $y = f(x)$.

From the relationship among these curves it follows that the turning points B, D, and F of the curve $y = f(x)$ correspond to (*i.e.*, have the same abscissa as) the points B', D', and F' where the first derived curve, $y = f'(x)$, crosses the x-axis. Likewise, the turning points C' and E' of the curve $y = f'(x)$ correspond to the points C'' and E'' where the second derived curve crosses the x-axis.

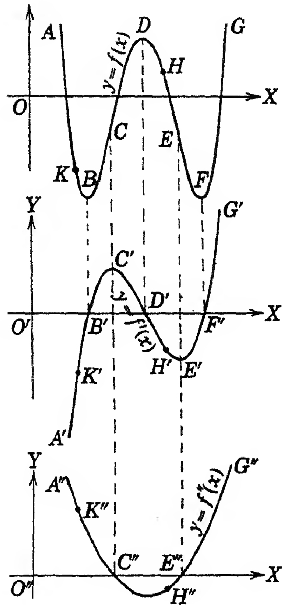

FIG. 107.

We inquire, now, into the significance, upon the original curve, $y = f(x)$, of the sign of $f''(x)$. Let K'' be any point on the curve $y = f''(x)$ at which $f''(x)$ is positive. Then the tangent line to the curve $y = f'(x)$ at the corresponding point K' has a positive slope. Consequently, by Theorem 1 of the preceding section, $f'(x)$ increases as x increases in some neighborhood of its value at K'. Therefore, in some neighborhood of the corresponding point K on the original curve, $y = f(x)$, the slope of the tangent line increases as x increases. In other words, in some neighborhood of K, the tangent line turns counterclockwise as x increases and, hence, the curve is concave upward. Similarly, if H'' is a point of the curve $y = f''(x)$ at which $f''(x)$ is negative, and H' and H are the corresponding points of the curves $y = f'(x)$ and $y = f(x)$, then, in some neighborhood of H', $f'(x)$ decreases as x increases, by Theorem 3. Hence, in some neighborhood of H, the tangent line to $y = f(x)$ turns clockwise as x increases and the curve is concave downward.

The two conclusions drawn from the foregoing arguments may be summarized briefly as follows:

If $f''(x)$ is positive at $x = k$ the curve $y = f(x)$ is concave upward in some neighborhood of the point $[k,f(k)]$.

If $f''(x)$ is negative at $x = h$ the curve $y = f(x)$ is concave downward in some neighborhood of the point $[h,f(h)]$.

At the point C of Fig. 107 the second derivative, $f''(x)$, changes sign, as may be seen by a glance at the corresponding point C'' of the second derived curve. Hence, as a point moving on the curve $y = f(x)$ crosses this point the curve changes from concave upward to concave downward. Similarly, at the point E the curve changes from concave downward to concave upward. A point, such as C or E, at which a curve $y = f(x)$ changes from concave upward to concave downward or vice versa, is called a point of *inflection*. To find a point of inflection of a curve $y = f(x)$, we need only find a point at which $f(x)$ is defined, $f'(x)$ is defined, or infinite, and $f''(x)$ is either zero or undefined and at which $f''(x)$ changes its sign.

Note carefully that the significant property of a point of inflection is that the second derivative changes sign at it. The mere fact that the second derivative vanishes at a point does not mark the point as a point of inflection. Thus, for both $y = x^3$ and $y = x^4$, the second derivative is zero at the origin. For the former, indeed, the origin is a point of inflection, since $6x$, the value of y'', changes sign there. For the latter case, the value, $12x^2$, of y'' does not change sign at the origin, and the origin is not a point of inflection. We may express this state of affairs by saying that in an interval where $f(x)$, $f'(x)$ and $f''(x)$ are continuous, the condition $f''(a) = 0$ is a necessary but not a sufficient condition for $x = a$ to be a point of inflection. The student has met a similar condition, *i.e.*, one that is necessary but not sufficient, in studying the behavior of the first derivative, *viz.*, in an interval in which $f(x)$ and $f'(x)$ are continuous the condition $f'(a) = 0$ is necessary but not sufficient for $f(a)$ to be a maximum or minimum value of $f(x)$. The condition $f'(a) = 0$ does not of itself make certain of either a maximum or a minimum value of $f(x)$ at that point. Here, again, the significant property is the change in sign of the derivative.

Exercise 1. Prove: If $f(x)$, $f'(x)$ and $f''(x)$ are continuous in a neighborhood of $x = a$, $f'(a) = 0$, and $f''(a) < 0$, then $f(x)$ has a maximum value at $x = a$.

Exercise 2. Prove: If $f(x)$, $f'(x)$, and $f''(x)$ are continuous in a neighborhood of $x = a$, $f'(a) = 0$, and $f''(a) > 0$, then $f(x)$ has a minimum value at $x = a$.

Note that Exercises 1 and 2 suggest a test for maximum and minimum values of a function, alternative to the test already employed. To illustrate, if $f(x) = x^4 - 2x^2 + 3$, then $f'(x) = 4x^3 - 4x$ and $f''(x) = 12x^2 - 4$. The roots of the equation $f'(x) = 0$ are $x = -1, 0, 1$. The values of $f'(x)$ and $f''(x)$ and the conclusion drawn for each of these points appear in the table

	$f'(x)$	$f''(x)$	$f(x)$
$x = -1$	0	8	Minimum by Exercise 2
$x = 0$	0	-4	Maximum by Exercise 1
$x = 1$	0	8	Minimum by Exercise 2

The test fails, however, for a value of x which makes both the first and second derivatives vanish. In other words, from the conditions $f'(a) = 0$, $f''(a) = 0$ nothing can be concluded as to whether $f(x)$ has a maximum or minimum value or neither at $x = a$. Observe again the behavior, in that respect, of the functions x^3 and x^4, for both of which the first and second derivatives vanish at the origin. While one of them has a minimum value at the origin, the other has no maximum or minimum points.

Exercise 3. Prove that the tangent to a curve at a point of inflection crosses the curve at that point. HINT: Note that a curve concave upward lies above any of its tangents while a curve concave downward lies below any of its tangents.

Problems

1. Reproduce each of these curves on squared paper to larger size and construct the first and second derived curves.

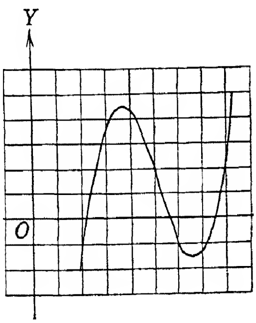

FIG. 108*a*.

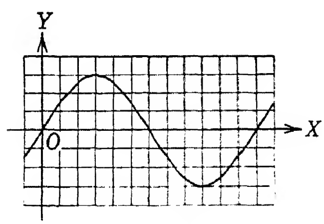

FIG. 108*b*.

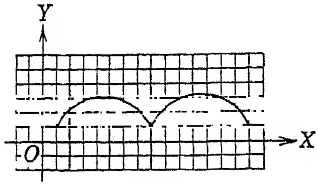

FIG. 108c.

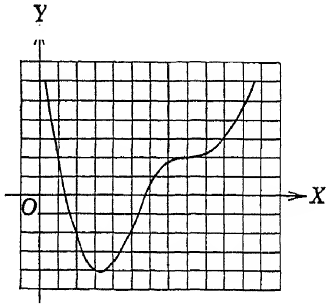

FIG. 108d.

2. By taking as ordinate, at the point whose abscissa is x, the area bounded by the line $y = 2$, the x-axis, the y-axis, and the line segment from $(x,0)$ to $(x,2)$, plot a curve. Find its first derived curve. Prove that the derived curve is the original line $y = 2$

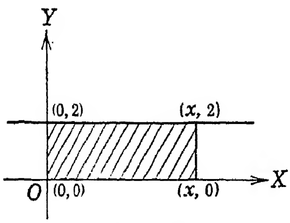

FIG. 109.

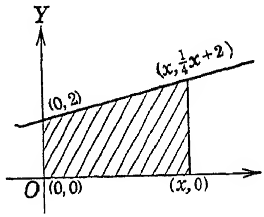

FIG. 110.

3. By taking as ordinate, at the point whose abscissa is x, the area of the trapezoid bounded by the axes, the line $y = \frac{1}{4}x + 2$ and the line from $(x,0)$ to $(x, \frac{1}{4}x + 2)$, plot a curve. Find its equation and prove that the derived curve is the given line $y = \frac{1}{4}x + 2$.

4. Find the points of maximum and minimum ordinate, the points of inflection, intervals of upward and downward concavity, and sketch the curve for each of the following equations:

(a) $3y = 4x^3 - 9x^2 - 12x + 17$.

(b) $y = (x - 3)^3 + 1$.

(c) $y = x^4 + 192x^2 - 800x$.

(d) $y = \tan x$.

(e) $y = e^{-x^3/12}$.

(f) $y = \dfrac{2}{1 + x^2}$.

(g) $y = \sinh x$.

(h) $y = \sqrt{\dfrac{x - 1}{3 - x}}$.

(i) $y = (x + 1)\sqrt{-x}$.

(j) $y = \dfrac{x - 1}{x^2 - 2x + 2}$.

Ans. (j) Points of inflection at $x = 1 - \sqrt{3}$, $x = 1$, $x = 1 + \sqrt{3}$.

(k) $y = \dfrac{e^x}{x}$.

(l) $y = x^2e^{-x}$.

(m) $y = 10^x + 10^{-x}$.

(n) $y = \sin 2x + \cos 2x$.

(o) $y = \sin x + 4 \cos x$.

(p) $y = \sin 2x + \cos 4x$.

5. Show that the curve $y = \dfrac{2 - x}{x^2 + 4}$ has three points of inflection which lie on a straight line.

6. Find the equation of the inflectional tangent (*i.e.*, tangent at the point of inflection) for the curve $x^3 - xy = 1$. *Ans.* $y = 3x - 3$.

60. Curvature. Suppose the length of the curve C, defined by the equation $y = f(x)$, from a fixed point A to a fixed point P_0 be designated by s_0, and the length from A to a variable point P designated by s. Then the length of the arc P_0P is the difference $|s - s_0|$. If the lines tangent to the curve C at the points P_0 and P have the inclinations θ_0 and θ, the difference $|\theta - \theta_0|$ is the angle between those tangents. The difference quotient

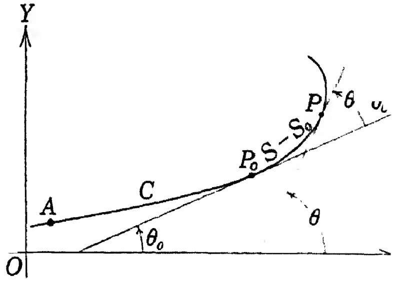

FIG. 111.

$$\frac{|s - s_0|}{\theta -}$$

represents the average change in arc length, s, per unit change of angle θ. If we designate by R the limit of this fraction as the point P approaches the point P_0 along C, we have

$$\begin{aligned}
R &= \lim_{s \to s_0} \left|\frac{s - s_0}{\theta - \theta_0}\right| \\
&= \lim_{x \to x_0} \left|\frac{\dfrac{s - s_0}{x - x_0}}{\dfrac{\theta - \theta_0}{x - x_0}}\right| \\
&= \left|\frac{\lim\limits_{x \to x_0} \dfrac{s - s_0}{x - x_0}}{\lim\limits_{x \to x_0} \dfrac{\theta - \theta_0}{x - x_0}}\right| \\
&= \left|\frac{ds/dx}{d\theta/dx}\right| = \left|\frac{ds}{d\theta}\right|.
\end{aligned}$$

In case the curve C is an arc of a circle, as in Fig. 112, the normals at P_0 and P meet at the center Q, and the angle between

them is $|\theta - \theta_0|$. But, on a circle, arc/angle = radius, so that, in that case

$$\left|\frac{s - s_0}{\theta - \theta_0}\right| = r,$$

a constant, and the quantity R as defined above is equal to the radius of the circle. It is therefore natural, in the case of any curve, to speak of R as the *radius of curvature* of the curve at the point.

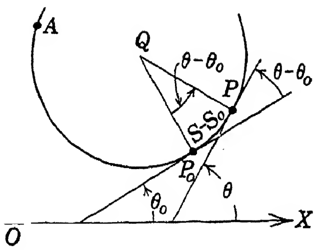

FIG. 112.

To evaluate the quantities $|ds/dx|$ and $|d\theta/dx|$, recall that ds is the increment which s would have taken on if the rate of change of s per unit change in x had become constant, *i.e.*, if the curve had become a straight line.

Hence

$$\overline{ds}^2 = \overline{dx}^2 + \overline{dy}^2 \tag{102}$$

and

$$= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

The derivative $d\theta/dx$ can be evaluated directly from the relation

$$\theta = \tan^{-1}\left(\frac{dy}{dx}\right),$$

whence

$$\frac{d\theta}{dx} = \frac{d^2y/dx^2}{1 + (dy/dx)^2}$$

and

$$R = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} \tag{103}$$

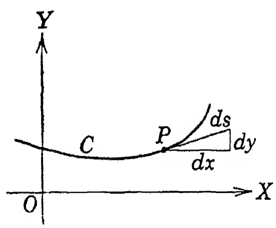

FIG. 113.

where the derivatives dy/dx and d^2y/dx^2 are to be evaluated at the point where the radius of curvature is desired.

It is customary to define the *curvature* of a curve at a point as $K = 1/R$, where R is the radius of curvature, that is

$$K = \frac{|d^2y/dx^2|}{[1 + (dy/dx)^2]^{3/2}}. \tag{104}$$

Exercise 1. Prove formula (102) from the fact that $\lim\limits_{\text{arc}\to 0}\left(\dfrac{\text{chord}}{\text{arc}}\right) = 1$. HINT: Call the end points of the arc (x,y) and $(x + \Delta x,\ y + \Delta y)$. Then $(\text{chord})^2 = \overline{\Delta x}^2 + \overline{\Delta y}^2$. Divide through by Δs^2, where Δs is the length of the arc, and pass to the limit.

Problems

1. Find the curvature of the line $y = mx + b$.

2. Find the radius of curvature of the curve $y = 4x^2$ at the point $(0,0)$ and at the point whose abscissa is x. *Ans.* $\frac{1}{8}$, $\dfrac{(1 + 64x^2)^{3/2}}{8}$.

3. Find an expression for the radius of curvature of the hyperbola $xy = a^2$ at the point whose abscissa is x, and find the coordinates of the point at which it is least. *Ans.* (a,a), $(-a,-a)$.

4. Find the greatest and least radius of curvature for the ellipse

$$x = a\cos\theta, \qquad y = b\sin\theta, \qquad \text{where } a > b > 0.$$

5. Find the radius of curvature of the following curves at the point indicated:

(a) $x = t^3 - 3t$, $y = 3t^2$ at $t = 0$. *Ans.* $\frac{3}{2}$.

(b) $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ at (x,y).

(c) $y = e^x$ at $x = 0$. *Ans.* $2\sqrt{2}$.

(d) $x^3 + y^3 = 6xy$ at $(3,3)$. *Ans.* $3\sqrt{2}/8$.

(e) $x^{2/3} + y^{2/3} = a^{2/3}$ at (x,y). *Ans.* $3(axy)^{1/3}$.

(f) $y = \csc x$ at (x,y).

(g) $y = \cosh\dfrac{x}{a}$ at (x,y).

(h) $y = \sin x$ at the points where y has maximum or minimum values.

6. For what points on the curve $8y = x^2$ does the radius of curvature have the value 32? *Ans.* $(\pm 4\sqrt{3}, 6)$.

7. Find R and K for the curve $y = x^2 - 4x + 6$.

8. Find R and K for the curve $x^{1/2} + y^{1/2} = a^{1/2}$ at $(a/4, a/4)$ and at (x,y).

9. If a curve is represented by $x = f(t)$, $y = g(t)$, derive an expression for the radius of curvature in terms of $f'(t)$, $g'(t)$, $f''(t)$ and $g''(t)$.

Ans. $\dfrac{[(f'(t))^2 + (g'(t))^2]^{3/2}}{f'(t)\cdot g''(t) - g'(t)\cdot f''(t)}$.

61. Circle of Curvature; Evolutes. *The circle of curvature* at a point P of a curve is defined as the circle tangent to the curve at P, with center on the concave side of the curve and radius equal to the radius of curvature of the curve at P. The center of the circle of curvature is called the *center of curvature*.

Let us find the coordinates α and β of C, the center of curvature at P, in terms of x, y, and the derivatives at (x,y). To that end,

we employ the two equations:

$$(x - \alpha)^2 + (y - \beta)^2 = R^2,$$
$$y - \beta = -\frac{1}{y'}(x - \alpha),$$

the reason for which is quite obvious. Eliminating $(x - \alpha)$ from the two equations, we obtain

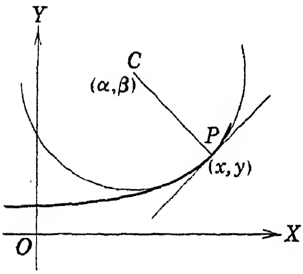

FIG. 114.

$$y'^2(y - \beta)^2 + (y - \beta)^2 = R^2,$$

or

$$|y - \beta| = \frac{R}{\sqrt{1 + y'^2}},$$

and, since $R = (1 + y'^2)^{3/2}/|y''|$,

$$|y - \beta| = \frac{1 + y'^2}{|y''|}.$$

Note now that if the curve is concave upward in the neighborhood of P, as in Fig. 114, $y - \beta$ is negative, while y'' is positive. Hence,

$$y - \beta = -\frac{1 + y'^2}{y''},$$

and

$$\beta = y + \frac{1 + y'^2}{y''}.$$

Let the student examine the case of a point P in whose neighborhood the curve is concave downward, and satisfy himself that the above expression for β still holds. Substituting now in the second of the pair of equations employed above, we obtain

$$\alpha = x - \frac{y'(1 + y'^2)}{y''}.$$

The *evolute of a curve* is defined as the locus of its centers of curvature. In other words, if a given curve is the locus of the points (x,y), its evolute is the locus of the points (α,β) treated above. Its equation is found by eliminating x and y from the equation, $y = f(x)$, of the given curve and the above equations for α and β.

Exercise 1. Prove that the normal to the curve at P is tangent to the evolute at the corresponding point C. HINT: Show that $\frac{d\beta}{d\alpha} = \frac{d\beta/dx}{d\alpha/dx} = -\frac{1}{y'}$.

We now prove an important property of the evolute, which we state as follows: If P_1 and P_2 are two points on a curve and C_1 and C_2 are the corresponding points on the evolute, *then the length of the evolute between C_1 and C_2 is equal to the difference between the radii of curvature at P_1 and P_2*, provided the radius of curvature is continuous on the arc P_1P_2.

Proof: From the relation $(x - \alpha)^2 + (y - \beta)^2 = R^2$, we obtain, differentiating with respect to x,

$$(x - \alpha)\left(1 - \frac{d\alpha}{dx}\right) + (y - \beta)\left(y' - \frac{d\beta}{dx}\right) = R\frac{dR}{dx}. \qquad (A)$$

Again, by $(y - \beta)/(x - \alpha) = -1/y'$ (since CP is normal to the given curve), we have

$$(x - \alpha) + (y - \beta)y' = 0,$$

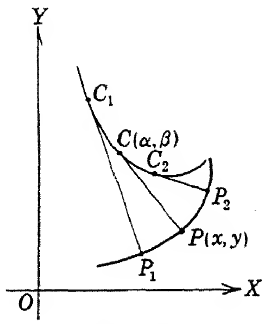

FIG. 115.

and, in view of this, (A) becomes

$$(x - \alpha)\frac{d\alpha}{dx} + (y - \beta)\frac{d\beta}{dx} = -R\frac{dR}{dx}. \qquad (B)$$

Employing the relation $\frac{y - \beta}{x - \alpha} = \frac{d\beta}{d\alpha}$ (since each member of this measures the slope of the line tangent to the evolute), or $\frac{d\alpha/dx}{x - \alpha} = \frac{d\beta/dx}{y - \beta}$, we set each of the last two ratios equal to u and obtain $d\alpha/dx = u(x - \alpha)$, $d\beta/dx = u(y - \beta)$, whence Eq. (B) takes the form

$$u(x - \alpha)^2 + u(y - \beta)^2 = -R\frac{dR}{dx}.$$

Solving for u we obtain

$$u = \frac{-R\frac{dR}{dx}}{(x - \alpha)^2 + (y - \beta)^2} = -\frac{\frac{dR}{dx}}{R}.$$

We now have (from the values of u),

$$\frac{d\alpha}{dx} \qquad \frac{-(x-\alpha)\dfrac{dR}{dx}}{R},$$

$$\frac{\quad}{dx} \qquad \frac{-(y-\beta)\dfrac{dR}{dx}}{R}.$$

Now call s the length of the evolute measured from C_1 in the direction of C_2. Then, by (102) and the above relations, we have

$$\left(\frac{ds}{dx}\right)^2 = \left(\frac{dR}{dx}\right)^2$$

$$\frac{ds}{dx} = \pm\frac{dR}{dx}.$$

Since $\left|\frac{dR}{dx}\right|$ the rate of change of s, relative to x, is always equal to the rate of change of R, relative to x. Hence, the total change in s over the arc C_1C_2 of the evolute equals the total change in R as P traces the arc P_1P_2, or $|R_1 - R_2| = \text{arc } C_1C_2$.

Problems

1. Find the coordinates of the center of curvature for each of the curves at the given points.

(a) $y = 8x^2$, $(0,0)$, $(-1,8)$. *Ans.* $(0,\frac{1}{16})$, $(256,\frac{385}{16})$.

(b) $y = x^3 + 2x^2 - 4x + 5$, $(0,5)$, $(1,4)$.

(c) $xy = 18$, $(-3,-6)$. *Ans.* $(-\frac{21}{2},-\frac{39}{4})$.

(d) $y = \cos x$.

(e) $(x-1)^2 + (y+2)^2 = 5$ at any point. *Ans.* $(1,-2)$.

2. Find the equation of the evolute of the curve $y^2 = 8x$. Draw the curve and its evolute.

KEY: Find

$$\alpha = x + \frac{y^2+16}{\quad} = 3x + 4, \qquad -\frac{y^3}{16},$$

whence

$$x = \frac{\alpha - 4}{\quad}, \qquad = (-16\beta)^{1/3}, \qquad y^2 = 8x$$

give

$$(16\beta)^{2/3} = \frac{8(\alpha-4)}{\quad}.$$

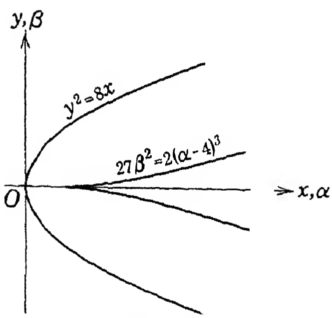

FIG. 116.

or

$$\beta^2 = \tfrac{2}{27}(\alpha - 4)^3$$

3. Find the equation, or parametric equations, of the evolute for each of the following curves. Draw the curve and its evolute.

(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$ *Ans.* $(a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3}.$

(b) $xy = a^2.$ *Ans.* $(\alpha + \beta)^{2/3} - (\alpha - \beta)^{2/3} = (4a)^{2/3}.$

(c) $x^{2/3} + y^{2/3} = a^{2/3}.$ *Ans.* $(\alpha + \beta)^{2/3} + (\alpha - \beta)^{2/3} = 2a^{2/3}.$

(d) $y = \cosh x.$

(e) $\begin{cases} x = 3t, \\ y = t^2 + 2. \end{cases}$ *Ans.* $\alpha = -\dfrac{4t^3}{3}.$ $\beta = 3t^2 + 1\tfrac{3}{2}.$

(f) $\begin{cases} x = 5 \cos t, \\ y = 5 \sin t. \end{cases}$

(g) $\begin{cases} x = 18t, \\ y = 9t^3. \end{cases}$ *Ans.* $\alpha = 9t - \tfrac{81}{4}t^5$ $\beta = \dfrac{45}{2}t^3 + \dfrac{6}{t}.$

(h) $\begin{cases} x = 2 \cos t + \cos 2t, \\ y = 2 \sin t + \sin 2t. \end{cases}$

(i) $\begin{cases} x = 4 \sec t, \\ y = 3 \tan t. \end{cases}$ *Ans.* $(4\alpha)^{2/3} - (3\beta)^{2/3} = 25^{2/3}.$

4. Find the length of the evolute between two of its points C_1 and C_2 that correspond to the points P_1 and P_2 on the curves below:

(a) $y = x^2$, P_1 is $(0,0)$, P_2 is $(2,4)$. *Ans.* $\dfrac{17\sqrt{17} - 1}{}$

(b) $y = \sin x$, P_1 is $(\pi/6, \tfrac{1}{2})$, P_2 is $(\pi/2, 1)$. *Ans.* $\dfrac{7\sqrt{7} - 4}{}$

(c) $x = 2t + 1$, $y = t^2 - 1$, P_1 is $(1, -1)$, P_2 is $(3, 0)$. *Ans.* $4\sqrt{2} - 2.$

5. If $x = f(t)$ and $y = g(t)$ are the equations of a curve, find formulas for α and β, the coordinates of the center of curvature, in terms of $f(t)$, $g(t)$, $f'(t)$, $g'(t)$, $f''(t)$, and $g''(t)$.

6. Show that the evolute of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$ is another cycloid.

7. Prove that y'', computed for the circle of curvature at a point P of a given curve, has the same value as y'' computed for the curve at P.

8. Verify that the curve $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$ is an involute of the circle $x^2 + y^2 = a^2$. NOTE. A curve A is said to be an *involute* of a curve B if B is the evolute of A.

62. Mean Value Theorem. We shall treat in this section a set of very useful theorems on derivatives, the first of which is known as

Rolle's Theorem. *If $f(x)$ and $f'(x)$ are continuous in the interval $a \leqq x \leqq b$ and $f(a) = f(b) = 0$, then there exists at least one constant ξ, $a < \xi < b$, such that $f'(\xi) = 0$.*

Proof: If $f'(x) = 0$ for every x on the interval $a \leqq x \leqq b$, the theorem obviously holds. If $f'(x)$ is not everywhere zero,

it must take on both positive and negative values between $x = a$ and $x = b$. For, if $f'(x) > 0$ in the entire interval, then, by the preceding text (Theorem 1, Sec. 58) $f(a) < f(b)$, contrary to hypothesis, and if $f'(x) < 0$ in the entire interval, then (Theorem 3) $f(a) > f(b)$, also contrary to hypothesis. Since, in this case $f'(x)$ takes on both positive and negative values in the interval and is continuous, it takes on the value zero, by Property 1 of Sec. 47. If ξ is the value of x_1 called for in that property we have $f'(\xi) = 0$, and Rolle's theorem is established.

As a corollary to Rolle's theorem we may easily establish:

The Theorem of the Mean. *If $f(x)$ and $f'(x)$ are continuous on the interval $a \leqq x \leqq b$, there exists at least one constant ξ such that $f(b) = f(a) + (b - a)f'(\xi)$.*

Proof. Consider the function

$$\varphi(x) = f(x) - f(a) - (x - a)\frac{f(b) - f(a)}{b - a}.$$

Obviously, $\varphi(x)$ is continuous and has a continuous derivative in the interval from a to b. Also

$$\varphi(a) = f(a) - f(a) - (a - a)\frac{f(b) - f(a)}{b - a} = 0$$

and

$$\varphi(b) = f(b) - f(a) - (b - a)\frac{f(b) - f(a)}{b - a} = 0.$$

Hence, Rolle's theorem applies, and there exists at least one constant ξ, between a and b such that $\varphi'(\xi) = 0$. Now

$$\varphi'(x) = f'(x) - \frac{f(b) - f(a)}{b - a},$$

and

$$0 = f'(\xi) - \frac{f(b) - f(a)}{b - a}$$

This reduces to

$$f(b) = f(a) + (b - a)f'(\xi). \qquad Q.E.D.$$

This theorem is true for any value of b, say x, such that $f(x)$ and $f'(x)$ are continuous in the interval from a to x, so we may

regard x as a variable, subject to these restrictions, and write

$$f(x) = f(a) + (x - a)f'(\xi), \qquad a < \xi < x. \tag{105}$$

It may also be written in the forms

$$f(a + h) = f(a) + hf'(a + \theta h), \qquad 0 < \theta < 1,$$

and

$$f(x + \Delta x) = f(x) + \Delta x f'(x + \theta \Delta x), \qquad 0 < \theta < 1.$$

Exercise 1. Show that the function $\varphi(x)$, employed in the proof of the theorem of the mean, represents the directed distance QP, where P is a point on the curve $y = f(x)$, with abscissa x satisfying $a \leqq x \leqq b$, and Q is a point with the same abscissa and lying on the secant line connecting the points $[a,f(a)]$ and $[b,f(b)]$.

Exercise 2. Show that at the point on the curve $y = f(x)$, whose abscissa is the quantity ξ called for in the theorem of the mean, the tangent to the curve is parallel to the secant line through the points $[a,f(a)]$ and $[b,f(b)]$.

Exercise 3. Prove that if $g(x), f(x), g'(x)$, and $f'(x)$ are continuous in the interval $a \leqq x \leqq b$ and $f(b) \neq f(a)$, then there exists a constant ξ such that

$$\frac{g(b) - g(a)}{f(b) - f(a)} = \frac{g'(\xi)}{f'(\xi)}, \qquad (a < \xi < b).$$

HINT: Apply Rolle's theorem to the function

$$\varphi(x) = g(x) - g(a) - [f(x) - f(a)]\frac{g(b) - g(a)}{f(b) - f(a)}.$$

NOTE: The formula of this exercise is called Cauchy's formula.

Exercise 4. Under the hypothesis that $f''(x)$, as well as $f(x)$ and $f'(x)$, are continuous in the interval $a \leqq x \leqq b$, prove that there exists at least one constant ξ such that

$$f(b) = f(a) + f'(a) \cdot (b - a) + f''(\xi)\frac{(b - a)^2}{2}, \qquad a < \xi < b.$$

HINT: Apply Rolle's theorem to the function

$$\varphi(x) = f(b) - f(x) - (b - x)f'(x) - \frac{(b - x)^2}{(b - a)^2}[f(b) - f(a) - (b - a)f'(a)]$$

and solve the resulting equality for $f(b)$.

Exercise 5. Under the hypothesis that $f(x), f'(x), f''(x), \ldots, f^{(n+1)}(x)$ are all continuous in the interval $a \leqq x \leqq b$, prove that there exists at least one constant ξ such that

$$f(b) = f(a) + (b - a)f'(a) + \frac{(b - a)^2 f''(a)}{2!} + \frac{(b - a)^3 f'''(a)}{3!} + \cdots + \frac{(b - a)^n f^{(n)}(a)}{n!} + \frac{(b - a)^{n+1} f^{(n+1)}(\xi)}{(n + 1)!}, \qquad a < \xi < b.$$

Hint: Apply Rolle's theorem to the function

$$\varphi(x) = f(b) - f(x) - (b - x)f'(x) - \frac{(b-x)^2}{2!}f''x - \cdots - \frac{(b-x)^n}{n!}f^{(n)}(x)$$
$$- \frac{(b-x)^{n+1}}{(b-a)^{n+1}}\left[f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f'(a) \cdots - \frac{(b-a)^n}{n!}f^{(n)}(a)\right].$$

Note: The formula proved in this exercise is known as Taylor's formula. Note also that the formula of Exercise 4 is Taylor's formula with $n = 1$ and the Theorem of the Mean is Taylor's formula with $n = 0$.

Problems

1. Find the value of ξ called for in Rolle's theorem for the given function in each case.

(a) $f(x) = x^2 - 3x$. *Ans.* $3/2$.

(b) $f(x) = x^3 + x^2 - 2x$.

2. Verify the theorem of the mean for $f(x) = A + Bx$, and show that the choice of ξ is immaterial.

3. Find the value of ξ involved in the theorem of the mean for

$$f(x) = 5 - 3x + 4x^2, \qquad a = 0, \qquad b = 2. \qquad \textit{Ans.}\ 1.$$

4. Find the value of ξ involved in Cauchy's formula (Exercise 3),

(a) for $f(x) = \cos x$, $g(x) = \sin x$, $a = \pi/6$, $b = \pi/2$;

Ans. $\pi/3$.

(b) for $f(x) = \log(x + 1)$, $g(x) = (x + 1)^2$, $a = 0$, $b = e - 1$.

5. Find the point on the curve $y = x^2 + 3x$ at which the tangent is parallel to the secant drawn through the points of the curve where $x = 1$ and $x = 3$. *Ans.* (2,10).

6. Find the point on the curve $y = e^x$ at which the tangent is parallel to the secant drawn through the points of the curve where $x = 0$ and $x = 1$.

7. Show that the theorem of the mean does not apply to $f(x) = x^{2/3}$, with $a = -1$ and $b = 2$. Explain why.

8. (a) Apply Taylor's formula (Exercise 5) to show that an approximate value of sin 5° is $\frac{\pi}{36} - \frac{1}{6}\left(\frac{\pi}{36}\right)^3$, with an error not exceeding $\frac{1}{120}\left(\frac{\pi}{36}\right)^5$.

Hint: Set $a = 0$, $b = \pi/36$ (radian measure of 5°) and use $n = 4$ in Taylor's formula.

(b) What is the limit of error in (a) if Taylor's formula is employed with $n = 5$?

(c) Employ $n = 1$ in Taylor's formula for sin 5°. By comparing the result with the value given for sin 5° in a five-place table, find the value of ξ involved in the formula.

9. Derive, by the theorem of the mean and by Taylor's formula with $n = 1$ and $n = 2$, successive values of $\log_{10}(n + 1)$ as

$$\log_{10}(n + 1) = \log_{10} n \cdot \frac{0.4343}{\xi_1} \qquad (n < \xi_1 < n + 1),$$

$$\log_{10}(n+1) = \log_{10} n + 0.4343 \quad \frac{0.4343}{2\xi_2^2} \qquad (n < \xi_2 < n+1),$$

$$\log_{10}(n+1) = \log_{10} n + \frac{0.4343}{n} - \frac{0.4343}{2n^2} + \frac{0.4343}{3\xi_3^3} \qquad (n < \xi_3 < n+1),$$

where 0.4343 is the (approximate) value of $\log_{10} e$. Set $n = 100$ and, by means of a table, find ξ_1, ξ_2, and ξ_3.

10. Express $\sqrt{1-x}$ as a polynomial in x of degrees 2 and 3 successively and estimate the error in each case if $x = 0.3$.

KEY: Apply Taylor's formula with $n = 2$ and $n = 3$ successively and obtain

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{1}{16}(1-\xi_1)^{-5/2} \cdot x^3, \qquad (0 < \xi_1 < x),$$

$$\sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5}{128}(1-\xi_2)^{-7/2} \cdot x^4, \qquad (0 < \xi_2 < x).$$

Hence $\sqrt{0.7} = 1 - \frac{0.3}{2} - \frac{(0.3)^2}{8}$, with an error not exceeding $\frac{(0.3)^3}{16(0.7)^{5/2}}$ and $\sqrt{0.7} = 1 - \frac{0.3}{2} - \frac{(0.3)^2}{8} - \frac{(0.3)^3}{16}$; with an error not exceeding $\frac{5(0.3)}{128\,(0.7)^{7/2}}$.

11. Express $\sqrt{1+x}$ as polynomials of degrees 2 and 3 successively, and by means of them compute $\sqrt{0.6}$. Estimate the error in using each polynomial.

63. Polynomial Approximation. Given a function $f(x)$ which, along with its first n derivatives, $f(x), f'(x), \ldots, f^{(n)}(x)$, is continuous at $x = x_0$, consider the problem of finding a polynomial $P(x)$ of degree n in x such that

$$\begin{aligned} P(x_0) &= f(x_0), \\ P'(x_0) &= f'(x_0), \\ P''(x_0) &= f''(x_0), \\ &\cdots\cdots\cdots \\ P^{(n)}(x_0) &= f^{(n)}(x_0). \end{aligned}$$

If we examine the problem first with $n = 1$, we may let the desired polynomial have the form

$$P(x) = Bx + C$$

whence

$$P'(x) = B.$$

Substituting x_0 for x and imposing the conditions of the problem, *viz.*, $P(x_0) = f(x_0)$ and $P'(x_0) = f'(x_0)$, we have

$$\begin{aligned} Bx_0 + C &= f(x_0), \\ B &= f'(x_0). \end{aligned}$$

Solving these for B and C, we have

$$\begin{aligned} B &= f'(x_0), \\ C &= f(x_0) - x_0 f'(x_0), \end{aligned}$$

and the polynomial takes the form

$$\begin{aligned} P(x) &= xf'(x_0) + f(x_0) - x_0 f'(x_0) \\ &= f(x_0) + (x - x_0)f'(x_0). \end{aligned}$$

The form taken by the solution to our problem for the case $n = 1$, suggests that, for the general problem, we express the polynomial sought as a polynomial in the quantity $x - x_0$. Hence, let

$$P(x) = A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + \cdots + A_n(x - x_0)^n.$$

whence, by differentiating with respect to x, we obtain

$$\begin{aligned} P'(x) &= A_1 + 2A_2(x - x_0) + 3A_3(x - x_0)^2 + \cdots \\ &\qquad + nA_n(x - x_0)^{n-1}, \\ P''(x) &= 2A_2 + 3!A_3(x - x_0) + 4 \cdot 3A_4(x - x_0)^2 + \cdots \\ &\qquad + n(n - 1)A_n(x - x_0)^{n-2}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ P^{(n-1)}(x) &= (n - 1)!A_{n-1} + n!A_n(x - x_0). \\ P^{(n)}(x) &= n!A_n \end{aligned}$$

Substituting $x = x_0$ in these relations, we obtain

$$\begin{aligned} P(x_0) &= A_0, \\ P'(x_0) &= A_1, \\ P''(x_0) &= 2!A_2, \\ &\cdots\cdots\cdots \\ P^{(n-1)}(x_0) &= (n - 1)!A_{n-1}, \\ P^{(n)}(x_0) &= n!A_n. \end{aligned}$$

From these, and the conditions of the problem, we obtain

$$A_0 = f(x_0), \qquad A_1 = f'(x_0), \qquad A_2 = \frac{f''(x_0)}{2!}$$

$$A_n = \frac{f^{(n)}(x_0)}{n!},$$

and our problem has a unique solution furnished by the polynomial

$$f(x_0) + (x - x_0)f'(x_0) + (x - x_0)^2\frac{f''(x_0)}{2!} + \cdot \qquad + (x - x_0)^n\frac{f^{(n)}(x_0)}{n!}$$

For example, if $f(x) = \cos x - \sin x$, we have $f(0) = 1$ and

$$\begin{aligned}
f'(x) &= -\sin x - \cos x, & f'(0) &= -1,\\
f''(x) &= -\cos x + \sin x, & f''(0) &= -1,\\
f'''(x) &= \sin x + \cos x, & f'''(0) &= 1,\\
f^{iv}(x) &= \cos x - \sin x, & f^{iv}(0) &= 1\\
f^{(n)}(x) &= f^{(n-4)}(x), & f^{(n)}(0) &= f^{(n-4)}(0), \qquad n > 4,
\end{aligned}$$

and the polynomial approximations to $\cos x - \sin x$ at $x_0 = 0$ are

$$y = 1 - x \qquad \text{of degree 1,}$$

$$y = 1 - x - \frac{x^2}{2!} \qquad \text{of degree 2,}$$

$$y = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} \qquad \text{of degree 3,}$$

$$y = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \qquad \text{of degree 4,}$$

$$y = 1 - x - \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \qquad \text{of degree 5, etc.}$$

In Fig. 117 are shown the graphs of the function $\cos x - \sin x$ and the graphs of the polynomials $1 - x$, $1 - x - \frac{x^2}{2!}$, and

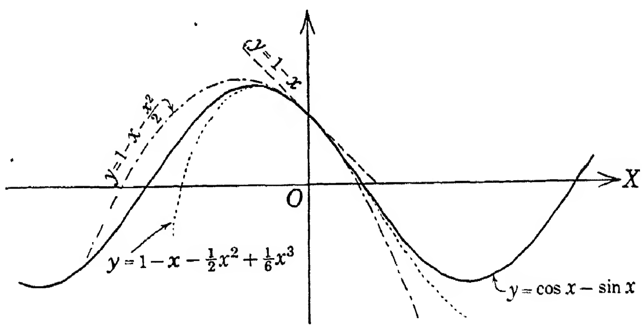

FIG. 117.

$1 - x - \frac{x^2}{2!} + \frac{x^3}{3!}$ superimposed on the same system of coordinates.

To appraise the accuracy of this method of approximation, let its error be represented by the constant $\frac{h^{n+1}E}{(n+1)!}$ when

$x - x_0$ has the value h. We thus have the exact equation

$$f(x) = f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) + \frac{h^{n+1}E}{(n+1)!},$$

whence we obtain

$$f(x_0 + h) - f(x_0) - hf'(x_0) - \frac{h^2}{2!}f''(x_0) - \quad - \frac{h^n}{n!}f^{(n)}(x_0) - \frac{h^{n+1}E}{(n+1)!} = 0.$$

Note that the left-hand member of this last equation is $\varphi(0)$, where

$$\varphi(z) = f(x_0 + h) - f(x_0 + z) - (h - z)f'(x_0 + z) - \frac{(h-z)^2}{2!}f''(x_0 + z) - \cdots - \frac{(h-z)^n}{n!}f^{(n)}(x_0 + z) - \frac{(h-z)^{n+1}E}{(n+1)!},$$

so that $\varphi(0) = 0$. By substitution we see that $\varphi(h) = 0$ and, since $f(x)$ and its first $(n + 1)$ derivatives are continuous, $\varphi(z)$ is continuous and satisfies the hypotheses of Rolle's theorem with $a = 0$ and $b = h$. Hence $\varphi'(\zeta) = 0$, where ζ is some constant between 0 and h. Differentiating $\varphi(z)$, we have

$$\begin{aligned}\varphi'(z) = {} & -f'(x_0 + z) \\ & + f'(x_0 + z) - (h - z)f''(x_0 + z) \\ & + (h - z)f''(x_0 + z) - \frac{(h-z)^2}{2!}f'''(x_0 + z) \\ + \quad & + \frac{(h-z)^{n-2}}{(n-2)!}f^{(n-1)}(x_0 + z) \\ & - \frac{(h-z)^{n-1}}{(n-1)!}f^{(n)}(x_0 + z) \\ & + \frac{(h-z)^{n-1}}{(n-1)!}f^{(n)}(x_0 + z) - \frac{(h-z)^n}{n!}f^{(n+1)}(x_0 + z) \\ & + \frac{(h-z)^n E}{n!} \\ & \frac{(h-z)^n}{n!}[E - f^{(n+1)}(x_0 + z)].\end{aligned}$$

But $\varphi'(\zeta) = 0$, where $\zeta \neq h$, hence $E - f^{(n+1)}(x_0 + \zeta) = 0$, or

$$E = f^{(n+1)}(x_0 + \zeta).$$

We conclude that, for any function $f(x)$ which is continuous and has $n + 1$ continuous derivatives at values of x between and including x_0 and $x_0 + h$, we may write

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \qquad + \frac{h^n}{n!}f^{(n)}(x_0) + \frac{h^{n+1}}{(n+1)!}f^{(n+1)}(x_0 + \zeta)$$

where ζ is between 0 and h. But, since this holds for any value of h subject to the above conditions as to continuity, we may write $x_0 + h$ as x, h as $x - x_0$, and $x_0 + \zeta$ as ζ, giving

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!}f^{(n)}(x_0) + \frac{(x - x_0)^{n+1}}{(n+1)!}f^{(n+1)}(\zeta)$$

and this is Taylor's formula, as displayed in Section 62, with a replaced by x_0 and b replaced by x.

Exercise 1. Given a function $f(x)$, prove that in order that a polynomial $A + Bx + Cx^2 + Dx^3$ and its first derivative have respectively the values $f(0)$ and $f'(0)$ at $x = 0$, A must have the value $f(0)$ and B must have the value $f'(0)$.

Exercise 2. Prove that in order that the second derivative of the polynomial of Exercise 1 have the value $f''(0)$ at $x = 0$, C must equal $\frac{1}{2}f''(0)$.

Problems

1. Show that if $f(x)$ is taken as $A + Bx + Cx^2 + Dx^3$, then Taylor's formula with $n = 3$ and $x_0 = 0$ reproduces $f(x)$.

2. Use Taylor's formula to express the polynomial $x^3 - 5x^2 + 13x - 7$ in the form $A(x - 2)^3 + B(x - 2)^2 + C(x - 2) + D$. HINT: Take $x_0 = 2$.

Ans. $(x - 2)^3 + (x - 2)^2 + 5(x - 2) + 7$.

3. Express $x^4 - 6x^3 + 10x^2 + 12x - 5$ as a polynomial in $(x + 1)$ by using Taylor's formula with $x_0 = -1$.

4. Express each of the following functions approximately by a polynomial in x of the degree indicated and discuss the error involved. HINT: Use Taylor's formula with $x_0 = 0$.

(*a*) e^x, degree 6.
(*b*) $\sin x$, degree 9.
(*c*) $\cos x$, degree 10.
(*d*) $\tan x$, degree 7.
(*e*) $\sec x$, degree 6.
(*f*) $\dfrac{1}{x - 1}$, degree 8.
(*g*) $\sinh x$, degree 9.
(*h*) $\cosh x$, degree 8.

Ans. (*a*) $1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \dfrac{x^5}{5!} + \dfrac{x^6}{6!}$.

5. Express $\log x$ as a polynomial in powers of $(x - 1)$. What can you say of the approximation to $\log x$ by a polynomial in x?

6. Compute $\sin 61°$ correct to five decimal places. HINT: Express the function $\sin x$ approximately as a polynomial in $\left(x - \frac{\pi}{3}\right)$ and take

$$x = \frac{\pi}{3} + \frac{\pi}{180} = \frac{\pi}{3} + 0.017453,$$

the radian measure of 61°.

7. For how large an angle may we safely use $1 - \frac{1}{2}x^2$ as an approximation for $\cos x$ if the error must be less than 0.00005?

Ans. $x < 0.1861$ radians.

8. Compute the value of $\log_e 2.81828$ to four decimal places. HINT: Express $\log_e (e + x)$ approximately as a polynomial in x and set $x = 0.1$.

9. Find $\log_{10} 998$ to five decimal places.

10. Verify the approximate formula $\log (10 + x) = 2.303 + \frac{x}{10}$. Calculate the natural logarithms of 9.5, 9.9, 10.1 and 10.5 and compare with the values listed in a table. Compare the value for $\log 15$.

11. Verify the approximate formula $\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2} - (0.8660)x$ and compare the value of $\cos 61°$ obtained this way with the value listed in a table.

12. Draw, on the same axes, the graphs of $\cos x$ and its polynomial (in x) approximations of degrees 0, 2, and 4.

13. Approximate $\frac{1}{2 - x}$ by a polynomial and use it to compute the reciprocal of 1.98 to five significant figures.

14. Prove, in the case of the approximation to $\cos x$ by a polynomial in x, that the error term $\frac{x^{n+1}f^{(n+1)}(\xi)}{(n + 1)!}$, $0 < \xi < x$, can be made as small as we please, for any given value of x, by taking n sufficiently large.

15. Prove the factor theorem of algebra, *i.e.*, if $f(x)$ is a polynomial of degree n and $f(a) = 0$, then $f(x)$ is divisible by $x - a$. HINT: $f^{(n+1)}(x) \equiv 0$.*

16. Prove that if the equation $f(x) = 0$, where $f(x)$ is a polynomial in x, has $x = a$ for a root of multiplicity k, then $f'(x)$ has $x = a$ for a root of multiplicity $k - 1$. HINT: $f(x)$ is divisible by $(x - a)^k$, but by no higher power of $x - a$.

64. Indeterminate Forms. The forms $0/0$, ∞/∞, 0^0, 1^∞, $0 \cdot \infty$, and $\infty \pm \infty$ are indeterminate in the sense that we can exhibit a function $F(x)$ which takes on one of these forms at $x = a$ and for which $\lim_{x \to a} F(x)$ has any preassigned value. If,

---

* By writing $F(x) \equiv G(x)$ we mean that $F(x)$ is identically equal to $G(x)$. That is, $F(x)$ equals $G(x)$ for every value of x.

however, the function is of the form $\frac{f(x)}{g(x)}$, or can be so written, we may frequently evaluate $\lim_{x\to a}\frac{f(x)}{g(x)}$ by the use of

L'Hospital's Rule. *If* $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = b$ *where* $b = 0$ *or* $b = \infty$, *and* a *is any finite constant or* ∞, *then*

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \lim_{x\to a}\left(\frac{f'(x)}{g'(x)}\right),$$

provided the derivatives and limits exist.

Illustration 1. In applying L'Hospital's rule to $\lim_{x\to 0}\left(\frac{\tan x}{3x}\right)$ we obtain

$$\lim_{x\to 0}\left(\frac{\tan x}{3x}\right) = \lim_{x\to 0}\left(\frac{\sec^2 x}{3}\right) = \frac{1}{3}.$$

Illustration 2. The function xe^{-x} assumes the form $0\cdot\infty$ as $x\to+\infty$. To find its limit, let us write it as x/e^x and use L'Hospital's rule. We thus obtain

$$\lim_{x\to+\infty}\left(\frac{x}{e^x}\right) = \lim_{x\to+\infty}\left(\frac{1}{e^x}\right) = 0.$$

Illustration 3. To find $\lim_{x\to 0}\left[\frac{1}{x^2} - \frac{1}{x^2\sec x}\right]$, we have

$$\begin{aligned}\lim_{x\to 0}\left(\frac{1}{x^2} - \frac{1}{x^2\sec x}\right) &= \lim_{x\to 0}\left(\frac{\sec x - 1}{x^2\sec x}\right)\\ &= \lim_{x\to 0}\left(\frac{\sec x\tan x}{x^2\sec x\tan x + 2x\sec x}\right)\\ &= \lim_{x\to 0}\left(\frac{\tan x}{x^2\tan x + 2x}\right).\end{aligned}$$

This limit is still indeterminate, so we apply the rule again, obtaining

$$\lim_{\to 0}\left(\frac{\sec^2 x}{x^2\sec^2 x + 2x\tan x + 2}\right) = \frac{1}{2}.$$

We shall now give a proof of L'Hospital's rule for the case in which a is finite and $b = 0$ and $g'(a)$ is finite and not zero. From these restrictions and the hypotheses of the rule we have two functions $f(x)$ and $g(x)$, such that $f(a) = g(a) = 0$, having derivatives $f'(x)$ and $g'(x)$ in the neighborhood of $x = a$, such that $g'(a)$ is finite and not zero. It follows that

$$\begin{aligned}
\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) &= \lim_{x\to a}\left(\frac{f(x)-f(a)}{g(x)-g(a)}\right) \\
&= \lim\left(\frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}\right) \\
&= \frac{\lim\limits_{x\to a}\left(\dfrac{f(x)-f(a)}{x-a}\right)}{\lim\limits_{x\to a}\left(\dfrac{g(x)-g(a)}{x-a}\right)} \\
&= \frac{f'(a)}{g'(a)}.
\end{aligned}$$

A complete proof of L'Hospital's rule is too difficult for the beginning student and should be deferred until he is taking a second course in calculus.

Exercise 1. Prove L'Hospital's rule for the case $f(a) = g(a) = 0$ and $f'(a)/g'(a)$ is defined, by using Cauchy's formula (Exercise 3, page 221).

HINT: If $a < \xi < x$ and $x \to a$, then $\xi \to a$.

Exercise 2. If $f(a) = f'(a) = g(a) = g'(a) = 0$ but $f''(a)/g''(a)$ is defined, prove that

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{f''(a)}{g''(a)}.$$

HINT: Apply Cauchy's formula first to the functions $f(x)$ and $g(x)$ and then to $f'(x)$ and $g'(x)$.

Problems

1. Find the following limits:

(*a*) $\lim\limits_{\theta\to 0}\left(\dfrac{1-\cos\theta}{\sin\theta}\right).$ *Ans.* 0.

(*b*) $\lim\limits_{\theta\to 0}\left(\dfrac{\theta-\sin\theta}{\tan^3\theta}\right).$ *Ans.* $\frac{1}{6}$.

(*c*) $\lim\left(\dfrac{1-\sin 2x}{x-\frac{\pi}{4}}\right)$ Ans. 0.

(*d*) $\lim\limits_{x\to 0}\left(\dfrac{1-\cos^2 x}{4x^2}\right).$ *Ans.* $\frac{1}{4}$.

(*e*) $\lim\limits_{y\to 0}\left(\dfrac{1-\tan y-\sec y}{\tan y-\sec y+1}\right).$ *Ans.* -1.

(*f*) $\lim\limits_{x\to 1}\left(\dfrac{(10)^{(x-1)}-2^{(x-1)}}{x-1}\right).$ *Ans.* log 5.

(g) $\lim\limits_{x \to \frac{\pi}{2}} \left(\dfrac{\cos^2 x}{(2x - \pi)^2} \right)$. *Ans.* $\frac{1}{4}$.

(h) $\lim\limits_{x \to 3} \left(\dfrac{e^x - e^3}{x - 3} \right)$.

(i) $\lim\limits_{x \to 5} \left(\dfrac{\log (5/x)}{x - 5} \right)$.

(j) $\lim\limits_{x \to \pi}$

$1 - \sin \dfrac{x}{2}$

(k) $\lim\limits_{x \to \infty}$ | *Ans.* 0.

HINT: Write it as

$\sqrt{x + 1} + \sqrt{x}$

(l) $\lim\limits_{x \to 0} [\cot x(1 - \cos x)]$. *Ans.* 0.

(m) $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\log (\cos x)}{\log (\tan x)}$. *Ans.* -1.

(n) $\lim\limits_{x \to 0} [\csc^2 x - x^{-2}]$. *Ans.* $\frac{1}{3}$.

(o) $\lim\limits_{x \to 0} (\sin x)^{\tan x}$, $x > 0$. *Ans.* 1.

HINT: Write $y = (\sin x)^{\tan x}$ and take the logarithm of both sides.

(p) $\lim\limits_{x \to 0} (x + 1)^{\cot x}$. *Ans.* e.

(q) $\lim\limits_{x \to \infty} [x(a^{1/x} - 1)]$.

(r) $\lim\limits_{x \to 0} \left(\dfrac{\tan x - x}{x - \sin x} \right)$. *Ans.* 2.

65. Infinitesimals. A variable which has the limit zero is called an *infinitesimal.* If two infinitesimals are functionally related so that the limit of their ratio can be studied, we have the following definition:

If u and v are two infinitesimals, then

(a) *u and v are said to be infinitesimals of the same order if* $\lim\limits_{u,v \to 0} (u/v) = k$, *where k is a finite quantity not zero;*

(b) *u is said to be an infinitesimal of higher order than v if* $\lim\limits_{u,v \to 0} (u/v) = 0$.

Thus, if in a certain process a variable x is made to approach zero (*i.e.*, becomes an infinitesimal), then $\sin x$ and $1 - \cos x$ also approach zero and, hence, become infinitesimals. Now, by results previously obtained,

$$\lim_{x \to 0} \left(\frac{\sin x}{x} \right) = 1, \qquad \text{while} \qquad \lim_{x \to 0} \left(\frac{1 - \cos x}{x} \right) = 0.$$

We say, then, that $\sin x$ is an infinitesimal of the same order as x, while $1 - \cos x$ is an infinitesimal of higher order than x.

In particular, if

$$\lim_{u,v\to 0}\left(\frac{u}{v}\right) = \lim_{u,v\to 0}\left(\frac{u}{v^2}\right) = \cdot \cdot \cdot = \lim_{u,v\to 0}\left(\frac{u}{v^{n-1}}\right) = 0,$$

while

$$\lim_{u,v\to 0}\left(\frac{u}{v^n}\right) = k,$$

where k is a finite quantity not zero, u is said to be an infinitesimal of the nth order relative to v.

Thus, $\lim\limits_{x\to 0}\left(\dfrac{1 - \cos x}{x}\right) = 0$, while $\lim\limits_{x\to 0}\left(\dfrac{1 - \cos x}{x^2}\right) = \dfrac{1}{2}$, so that the quantity $1 - \cos x$ is said to be an infinitesimal of the second order relative to (or with respect to) x.

In subsequent work in integral calculus we shall have use for a theorem concerning the limit of a sum of positive infinitesimals, known as

Duhamel's Theorem. *If $u_1, u_2, \cdot \cdot \cdot, u_n$ are a set of n positive infinitesimals which increase in number as each approaches zero and such that*

$$\lim_{\substack{u_i\to 0\\ n\to\infty}} (u_1 + u_2 + \cdot \cdot \cdot + u_n)$$

exists and equals S and if, further, $v_1, v_2, \cdot \cdot \cdot, v_n$ are a set of positive infinitesimals so related to the u's that, given any positive number α, we can take m sufficiently large so that, for every $n > m$, each of the quantities $\left|\dfrac{v_1 - u_1}{u_1}\right|, \left|\dfrac{v_2 - u_2}{u_2}\right|, \cdot \cdot \cdot, \left|\dfrac{v_n - u_n}{u_n}\right|$ is less than α, then

$$\lim_{\substack{v_i\to 0\\ n\to\infty}} (v_1 + v_2 + \cdot \cdot \cdot + v_n)$$

also exists and equals S.

Proof: Let $v_i = u_i + \epsilon_i u_i$ ($i = 1, 2, 3, \cdot \cdot \cdot, n$) (in other words, $v_1 = u_1 + \epsilon_1 u_1$, $v_2 = u_2 + \epsilon_2 u_2, \cdot \cdot \cdot$). Then

$$\frac{v_i - u_i}{u_i} = \epsilon_i$$

and, for sufficiently large values of n,

$$-\alpha \leqq \epsilon_i \leqq \alpha \qquad (i = 1, 2, 3, \cdots, n)$$

Hence,

$$-\alpha u_i \leqq \epsilon_i u_i \leqq \alpha u_i \qquad (i = 1, 2, 3, \cdots, n).$$

(The multiplication preserves the signs in the inequality, because u_i, by hypothesis, is positive, for every i.) It follows that

$$\sum_{i=1}^{n} -\alpha u_i \leqq \sum_{i=1}^{n} \epsilon_i u_i \leqq \sum_{i=1}^{n} \alpha u_i$$

or

$$-\alpha \sum_{i=1}^{n} u_i \leqq \sum_{i=1}^{n} \epsilon_i u_i \leqq \alpha \sum_{i=1}^{n} u_i,$$

whence

$$-\alpha \lim_{\substack{n \to \infty \\ u_i \to 0}} \sum_{i=1}^{n} u_i \leqq \lim_{\substack{n \to \infty \\ u_i \to 0}} \sum_{i=1}^{n} \epsilon_i u_i \leqq \alpha \lim_{\substack{n \to \infty \\ u_i \to 0}} \sum_{i=1}^{n} u_i,$$

and

$$-\alpha S \leqq \lim_{\substack{n \to \infty \\ u_i \to 0}} \sum_{i=1}^{n} \epsilon_i u_i \leqq \alpha S.$$

The conclusion is that

$$\lim_{\substack{n \to \infty \\ u_i \to 0}} \sum_{i=1}^{n} \epsilon_i u_i = 0,$$

and since

$$\sum_{i=1}^{n} v_i = \sum_{i=1}^{n} u_i + \sum_{i=1}^{n} \epsilon_i u_i,$$

we finally have

$$\lim_{\substack{n \to \infty \\ v_i \to 0}} \sum_{i=1}^{n} v_i = \lim_{\substack{n \to \infty \\ u_i \to 0}} \sum_{i=1}^{n} u_i, \qquad Q.E.D.$$

Problems

1. Show that $\tan x$ is an infinitesimal of the same order as $\sin x$. (Both are infinitesimals as $x \to 0$.)
2. Show that $x - \sin x$ is an infinitesimal of the third order relative to x.

3. For each pair of infinitesimals given below, find the order of the first with respect to the second.

(*a*) $\log(x + 1)$, x. *Ans.* First. (*d*) $x^3 + 1$, $x + 1$. (As $x \to -1$).
(*b*) $\tan x - \sin x$, $\sin x$. *Ans.* Third. (*e*) the volume of a cube and its edge.
(*c*) $\sin 2x$, $\sin x$. *Ans.* First. (*f*) $x \sin x$, $\tan x$. *Ans.* Second.

4. Show that $\sqrt{x}$ is an infinitesimal of lower order with respect to x. (An infinitesimal u is said to be of lower order than an infinitesimal v if v is of higher order than u.)

5. Show that the sum of two positive infinitesimals of the same order, is another infinitesimal of that same order. Can the same be said of their difference?

6. Show that in a circle the difference between an infinitesimal arc and its chord is an infinitesimal of the third order relative to either the arc or the chord.

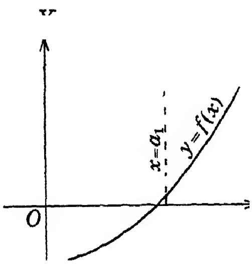

FIG. 118.

66. Newton's Method of Approximation. Suppose we wish to find values of x for which $f(x) = 0$, where $f(x)$ is a function which vanishes for at least one value of x and has continuous derivatives of orders 1 and 2 in the neighborhood of its vanishing point. By drawing the graph of the function and observing where it crosses the x-axis, we can obtain a first approximation to the solution. Let this approximation be $x = a_1$. Then, by Taylor's formula, we have

$$f(x) = f(a_1) + (x - a_1)f'(a_1) + \frac{(x - a_1)^2}{2}f''(\xi_1),$$

where ξ_1 is between a_1 and x. If a_1 is sufficiently near to the correct solution, we may neglect the last term of the right-hand side, and, since we are trying to solve the equation $f(x) = 0$, write

$$f(a_1) + (x - a_1)f'(a_1) = 0,$$

or

$$x = a_1 - \frac{f(a_1)}{f'(a_1)} = a_2,$$

a second approximation. If the first approximation, a_1, is sufficiently good, the second approximation a_2 will be better. From the second approximation we may at once form a third

$$x = a_2 - \frac{f(a_2)}{f'(a_2)} = a_3,$$

etc.

Illustration. To solve $\operatorname{ctn} x - 2x = 0$, we consult a table and find, for $x = 0.65$,

$$\operatorname{ctn} x = 1.315, \qquad 2x = 1.300, \qquad \operatorname{ctn} x - 2x = 0.015.$$

Hence, we take $a_1 = 0.65$ and obtain

$$\begin{aligned} a_2 &= 0.65 - \frac{0.015}{-\csc^2(0.65) - 2} \\ &= 0.65 + \frac{0.015}{4.730} \\ &= 0.653 \end{aligned}$$

Checking, we find

$$\operatorname{ctn} a_2 = 1.307, \qquad 2a_2 = 1.306, \qquad \operatorname{ctn} a_2 - 2a_2 = 0.001$$

A repetition of the process gives $a_3 = 0.6532$, but the last place cannot be checked from a table showing only four significant figures, so we drop it and take as solution $x = 0.653$.

Exercise 1. Find an expression for the x-intercept of the tangent to the curve $y = f(x)$ at the point $[a_1, f(a_1)]$, and give a geometric interpretation of Newton's method.

Problems

1. Determine graphically the number of distinct roots of the following equations and in each case approximate that one which is least in absolute value but different from zero.

(a) $x^3 + 6x^2 + 11x - 8 = 0$. *Ans.* 0.55.
(b) $3 \sin x - x = 0$. *Ans.* 2.279.
(c) $\cos(x/4) + x = 3$.
(d) $1 + x + \tan x = 0$.
(e) $5 \cos x + 2x = 0$.
(f) $e^{-x} - \operatorname{ctn} x = 0$. *Ans.* 1.306.

2. Find the maximum and minimum values of the following functions.

(a) $2e^x + (x + 2)^2$.
(b) $6 \cos x + (x - 1)^3$.
(c) $x \sin 3x$, $0 < x < 2\pi/3$.
(d) $x \cos x$, $0 < x < 2\pi$.
(e) $x^2 - x \log_{10} x - 8x$.

67. Rectilinear Motion; Rotation.

We have already seen that if $s = f(t)$, where t measures the time after a given instant and s measures the distance on a straight line, from a given point on the line, $ds/dt = f'(t)$, measures the velocity, v, and

$$d^2s/dt^2 = f''(t),$$

measures the acceleration, j. For example, assuming that a

body falls from rest under a constant force of gravity, to find how far it will fall in t seconds, note that if $s = F(t)$ is the result sought then

$$F''(t) = g = \text{constant}.$$

Hence $F(t)$ is a function of t whose second derivative is the constant g. This is true of the functions

$$s = F(t) = \frac{g}{2}t^2 + c_1 t + c_2,$$

where c_1 and c_2 are constants. For, by differentiation,

$$v = F'(t) = gt + c_1$$

and

$$j = F''(t) = g.$$

In order that the distance and velocity be zero when $t = 0$, we must have $c_1 = c_2 = 0$, or the motion is given by

$$s = \tfrac{1}{2}gt^2.$$

Of especial interest is a type of rectilinear motion known as *simple harmonic motion*, ordinarily defined as the motion of a particle on a straight line having an acceleration directed toward a fixed point on the line, called the *center*, and proportional to the distance from that point. That is,

$$\frac{d^2s}{dt^2} = -k^2 s, \tag{106}$$

where we have used $-k^2$ as the constant of proportionality to direct the acceleration toward the center, or origin, *i.e.*, to give to the acceleration a sign opposite to that of s. This relation (106) is a differential equation which, fortunately, we can solve by inspection, since s is some function of t whose second derivative is $-k^2$ times the function itself. The experience of the student in differentiating trigonometric functions should enable him to pick out $\sin kt$ and $\cos kt$ as two functions having that property. He may readily show that the function $c_1 \sin kt + c_2 \cos kt$ also has that property, regardless of the values of the constants c_1 and c_2. This function, in fact, is the most general one that satisfies Eq. (106). Hence, the most general equation

for simple harmonic motion is

$$s = c_1 \sin kt + c_2 \cos kt$$

if $s = 0$ is its center. By putting one of the constants c_1, and c_2 equal to A and the other equal to zero, we see that the equations

$$s = A \sin kt,$$
$$s = A \cos kt,$$

where A is some constant, also represent simple harmonic motion.

Exercise 1. Show that the period in the case of simple harmonic motion represented by any one of the equations above is equal to $2\pi/k$. (By the *period* is meant the time interval between two successive instances when s, v, and j all take on the same values.)

A type of motion which is very closely analogous to rectilinear motion is *circular motion* or the motion of a particle in a circle of given radius r. We may designate the position of such a particle by an angle θ which the radius to the moving point P makes with a fixed radius. We have, then,

$$\theta = f(t).$$

By the instantaneous *angular velocity* ω at the time t is meant the derivative of θ with respect to t, or

$$\omega = \frac{d\theta}{dt} = f'(t).$$

The length of arc, s, traveled over in time t is found to be

$$s = r\theta = rf(t),$$

the velocity in the path is

$$v = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\omega = rf'(t),$$

and the acceleration, in the path, is

$$j = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha = rf''(t),$$

where α designates the *angular acceleration*.

Problems

1. A body is shot vertically upward against a constant force of gravity. If the equation of the motion is $s = -16t^2 + 320t$, find (*a*) the initial velocity, (*b*) the height to which it will rise, (*c*) its velocity upon returning to the starting point. What is its acceleration? Explain the sign.

Ans. (*a*) 320; (*b*) 1600; (*c*) -320.

2. A body falls according to the law $s = 32(t + e^{-t} - 1)$. Find the values of s, v, and j when $t = 0$ and their limits as $t \to +\infty$.

3. A body moves according to the law $s = t^3 - 6t^2 + 9t + 4$. Find the maximum and minimum values of s, v, and j. *Ans.* Min. $v = -3$.

4. A particle moves on the x-axis according to the law $x = 6 - 6 \sin 4t$. Show that the motion is simple harmonic. Find the center, the period, and the acceleration.

5. Describe the motion and state the force acting on a body of mass 10 lb. moving according to the law $s = 5 \sin \frac{t}{4}$, where t is in seconds and s is in feet. Find the period. HINT: Force = mass × acceleration.

6. A body moves on a straight line according to the law $s = 10(\sin \pi t)e^{-t/10}$ Find its velocity and acceleration as $t \to +\infty$. Is it periodic? (Such motion is called *damped harmonic motion.*)

7. A belt of negligible thickness runs over a pulley which is revolving 200 r.p.m. What is the velocity of the belt if the pulley is 1 ft. in diameter?

8. A particle moves on a circle with a constant linear velocity of 4π units per second. Find the angular velocity if the radius of the circle is 2 units.

9. A point moves with constant angular velocity on a circle of radius r. Prove that the projection of the point upon a diameter moves in simple harmonic motion.

10. A spoke of a wheel turns according to the law $\theta = \sin \frac{\pi t}{2}$. Find the angular velocity and angular acceleration.

11. Show that the motion on a straight line represented by

$$s = 5 \sin \left(\frac{\pi}{3}t + \frac{\pi}{6}\right)$$

is simple harmonic. Find the period. *Ans.* Period = 6.

12. A particle moves in simple harmonic motion. Find the equation of the motion, given that when $t = 0$, $s = 3$, $v = -8$, and $j = -12$. HINT: Assume $s = c_1 \sin kt + c_2 \cos kt$, and determine c_1, c_2, and k.

Ans. $s = -4 \sin 2t + 3 \cos 2t$.

13. A particle moves in simple harmonic motion. Find the equation of the motion given that at the beginning of the motion $s = 2$, $v = 3$, and after a quarter of a period $j = -9$. *Ans.* $s = \sin 3t + 2 \cos 3t$.

68. Curvilinear Motion. Let the position of a point P (x,y) at the time t be given by the equations

$$x = f(t),$$
$$y = g(t),$$

$f(t)$ and $g(t)$ being continuous, at least in some interval. Let s measure the distance along the path from a fixed point of it. Then, if v represents the *velocity in the path*, at the time t, v_x, the *x-component of velocity*, and v_y, the *y-component of velocity*, we shall understand that v, v_x, and v_y are defined by the equations

$$v = \frac{ds}{dt},$$
$$v_x = \frac{dx}{dt},$$
$$v_y = \frac{dy}{dt}.$$

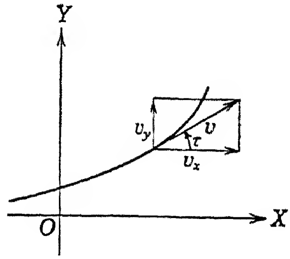

FIG. 119.

We have already shown (page 214) that $ds = \sqrt{(dx)^2 + (dy)^2}$, so we find at once that

$$v = \frac{\sqrt{(dx)^2 + (dy)^2}}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(v_x)^2 + (v_y)^2}.$$

The particle moves on the curve whose equation is defined parametrically by the equations $x = f(t)$, $y = g(t)$, and, hence, the direction of the motion, at any instant, is that of the line tangent to the curve at the point determined by the parameter t. Therefore, if τ represents the angle from the x-axis to the tangent line, we shall speak of τ as the *direction* of the velocity. From this definition we have

$$\tan \tau = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$$

and

$$\sin \tau = g'(t)/\sqrt{[f'(t)]^2 + [g'(t)]^2}.$$

The geometrical significance of v, v_x, v_y, and τ can be seen in Fig. 119.

Similarly, let us define the *components* of acceleration parallel to the axes as

$$j_x = x\text{-}component\ of\ acceleration = \frac{d^2x}{dt^2} = f''(t)$$

and

$$j_y = y\text{-}component\ \ of\ \ acceleration = \frac{d^2y}{dt^2} = g''(t)$$

and the *resultant acceleration as*

$$j = \sqrt{(j_x)^2 + (j_y)^2} = \sqrt{\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2} = \sqrt{[f''(t)]^2 + [g''(t)]^2}.$$

The direction, ω, of the resultant acceleration is given by

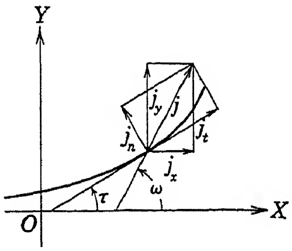

FIG. 120.

$$\tan\omega = \frac{d^2y/dt^2}{d^2x/dt^2} = \frac{g''(t)}{f''(t)}$$

and

$$\sin\omega = g''(t)/\sqrt{[f''(t)]^2 + [g''(t)]^2}.$$

By the components of acceleration, j_t along the tangent to the curve, and j_n along the normal to the curve, are understood two components in those directions, whose resultant is coincident with j, the resultant of j_x and j_y. Referring to Fig. 120, we see that

$$\frac{j_t}{j} = \cos(\omega - \tau) = \cos\omega\cos\tau + \sin\omega\sin\tau,$$

and

$$\frac{j_n}{j} = \sin(\omega - \tau) = \sin\omega\cos\tau - \cos\omega\sin\tau.$$

Now

$$\cos\tau = \frac{v_x}{v} = \frac{f'(t)}{\sqrt{[f'(t)]^2 + [g'(t)]^2}},$$

$$\sin\tau = \frac{v_y}{v} = \frac{g'(t)}{\sqrt{[f'(t)]^2 + [g'(t)]^2}},$$

$$\cos\omega = \frac{j_x}{j} = \frac{f''(t)}{\sqrt{[f''(t)]^2 + [g''(t)]^2}},$$

$$\sin\omega = \frac{j_y}{j} = \frac{g''(t)}{\sqrt{[f''(t)]^2 + [g''(t)]^2}},$$

from which we calculate the results

$$j_t = \frac{f'(t)f''(t) + g'(t)g''(t)}{\sqrt{[f'(t)]^2 + [g'(t)]^2}} = \frac{v_xj_x + v_yj_y}{v}$$

and

$$j_n = \frac{f'(t)g''(t) - g'(t)f''(t)}{\sqrt{[f'(t)]^2 + [g'(t)]^2}} = \frac{v_x j_y - v_y j_x}{v}.$$

Exercise 1. Prove that $|j_n| = v^2/R$, where R is the radius of curvature. HINT: Use the result of Prob. 9, page 215.

Illustration. A point (x,y) moves according to the law

$$\begin{cases} x = t^2, \\ y = 2t, \end{cases} \qquad t \geqq 0.$$

At $t = 0$, the point is at the origin, and as t increases, the point moves out on the half parabola $y = 2\sqrt{x}$. The components of velocity are

$$v_x = 2t,$$
$$v_y = 2.$$

The resultant velocity, or velocity in the path, is

$$v = 2\sqrt{1 + t^2},$$

and its direction, τ is given by

$$\tau = \tan^{-1}\left(\frac{1}{t}\right).$$

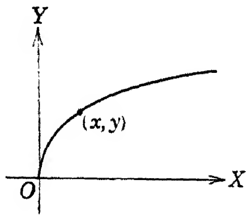

FIG. 121.

The acceleration and its components are given by the equations

$$j_x = 2, \qquad j_y = 0, \qquad j = \sqrt{(j_x)^2 + (j_y)^2} = 2,$$
$$j_t = \frac{v_x j_x + v_y j_y}{v} = \frac{4t}{2\sqrt{1 + t^2}} = \frac{2t}{\sqrt{1 + t^2}},$$
$$j_n = \frac{v_x j_y - v_y j_x}{v} = \frac{-4}{2\sqrt{1 + t^2}} = \frac{-2}{\sqrt{1 + t^2}},$$

and the direction of the resultant acceleration is $\omega = \tan^{-1}(j_y/j_x) = 0$.

These results may now be examined in detail for assigned values of t. The values at $t = 0$, $t = 2$ and their limits as $t \to +\infty$ are displayed below in tabular form.

	x	y	v_x	v_y	v	τ	j_x	j_y	j	ω	j_t	j_n
$t = t$	t^2	$2t$	$2t$	2	$2\sqrt{1 + t^2}$	$\tan^{-1}\frac{1}{t}$	2	0	2	0	$\frac{2t}{\sqrt{1 + t^2}}$	$\frac{-2}{\sqrt{1 + t^2}}$
$t = 0$	0	0	0	2	2	$\pi/2$	2	0	2	0	0	-2
$t = 2$	4	4	4	2	$2\sqrt{5}$	$\tan^{-1}\frac{1}{2}$	2	0	2	0	$4/\sqrt{5}$	$-2/\sqrt{5}$
$t \to +\infty$	$+\infty$	$+\infty$	$+\infty$	2	$+\infty$	0	2	0	2	0	2	0

We note that v_y, j_x, j_y, j, and ω are constants, independent of time.

Exercise 2. Prove that $j_t = dv/dt$.

Problems

1. A point (x,y) moves according to the law $x = t^2$, $y = t^3$, where x and y are measured in feet and t is measured in seconds. Find the path, the velocity and its components, and the acceleration and its components, and exhibit the results in a table for t general, $t = 0$, $t = 1$, and their limits as $t \to +\infty$. At what value of t is the resultant acceleration equal to g, the acceleration of gravity, if we take $g = 32$? *Ans.* $\sqrt{255}/3$.

2. A particle moves along the line $3x - 4y = 0$ so that $j = 10$, a constant, while at $t = 0$, the particle is at the origin and its velocity is zero. Express x and y in terms of t and find v_x, v_y, v, τ, j_x, j_y, j_t, j_n and ω. HINT: Let $x = 4f(t)$ and $y = 3f(t)$ and, by setting $j = 10$, find the function $f(t)$. *Ans.* $x = \pm 4t^2$, $y = \pm 3t^2$.

3. A particle moves along the curve $y = x^3 - x^2 + 1$ in such a way that the x-component of its velocity is 3. Find v, j, and j_t. *Ans.* $j = 54x - 18$.

4. A particle moves on the line $2x + y = 2$ with a velocity of 5. Find v_x, v_y, and j. *Ans.* $v_x = \pm\sqrt{5}$, $v_y = \mp 2\sqrt{5}$

5. A particle moves to the right on the upper half of the parabola

$$y^2 + 3x = 3$$

with a velocity of 3 units per second. Find v_x, v_y, j_x, j_y, j_t, and j_n as it passes the point where $y = 2$.

6. A body moves on a circle with uniform angular velocity. Prove that $j_t = 0$, $j =$ a constant, and that j is always directed toward the center.

7. If a point moves on the cubic curve $y = x^3 - 3x^2 + 6x + 2$, find the value of x for which $v_y = D_t (\tan \tau)$. *Ans.* 2.

8. A point moves according to the law $x = 1 + \cos \frac{\pi t}{2}$, $y = 1 - \cos \frac{\pi t}{2}$. Draw the path and make a table showing values of x and y and the components, resultants and directions of velocity and acceleration when $t = 0, 1, 2, 3$, and 4.

9. A particle moves on a circle of radius 4 in. If at a certain instant its velocity is 6 in. per sec. and is increasing at the rate of $\sqrt{40}$ in./sec.^{2} find its acceleration. *Ans.* 11.

69. Related Rates. An endless variety of problems arise in applications of mathematics in which one variable is a function of another and both are functions of time. No special theory is needed for such problems, the general scheme being that such relations as

$$x = f(t),$$
$$y = g(t)$$

are either given or implied and, therefore, y is, in general, a function of x. If this relationship is expressed as

$$y = F(x),$$

we have, upon differentiating with respect to t,

$$\frac{dy}{dt} = F'(x)\frac{dx}{dt}.$$

Illustration 1. Water is running into a conical reservoir, 15 ft. deep and 8 ft. in diameter at the top, at the rate of 1 cu. ft./min. At what rate is the level rising when the water is 5 ft. deep? What is the depth when the level is rising $9/32\pi$ ft./min.?

Solution. If h is the depth and r the radius of the circular water surface, then

$$r:4 = h:15$$

or

$$r = \frac{4h}{15}.$$

The volume of the cone of water, $\frac{\pi r^2 h}{3}$, therefore reduces to

$$V = \frac{16\pi h^3}{675}.$$

Then

$$\frac{dV}{dt} = \frac{16\pi h^2}{225}\cdot\frac{dh}{dt}.$$

As $dV/dt = 1$, this gives

$$\frac{dh}{dt} = \frac{225}{16\pi h^2}.$$

When $h = 5$, we obtain

$$\frac{dh}{dt} = \frac{9}{16\pi},$$

so, at that instant, the level is rising $9/16\pi$ ft./min. When $dh/dt = 9/32\pi$, we have

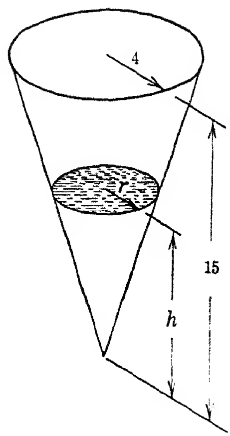

FIG. 122.

$$\frac{9}{32\pi} = \frac{225}{16\pi h^2},$$

or $h = 5\sqrt{2}$ ft.

Illustration 2. A kite is 300 ft. high and there are 500 ft. of line out. If the kite travels horizontally away from the flyer at the rate of 6 m.p.h., how fast is the line being paid out?

Solution. Here the student should heed a general warning. The amount of line out and the height of the kite are given at the outset. If the student

follows his first impulse he may label both these distances as constants. Then all derivatives are zero and no motion takes place. Rather he must recognize that one of these quantities is actually constant, *viz.*, the 300 ft. of height, while the 500 ft. of line is an instantaneous value of a variable. The problem is easily seen to reduce to

$$x^2 + 90{,}000 = l^2$$

and thence to

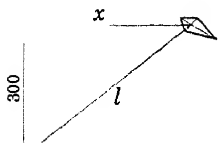

FIG. 123.

$$2x \cdot \frac{dx}{dt} = 2l \cdot \frac{dl}{dt}$$

or

$$\frac{dl}{dt} = \frac{x}{l} \cdot \frac{dx}{dt}.$$

Since $dx/dt = 6(5280)$, and at the stated instant, $l = 500$ and $x = 400$, the instantaneous value of dl/dt becomes $(400/500)(6)(5280)$. Hence the line is being paid out at the rate $(24/5)(5280)$ ft./hr., or $24/5$ mi./hr.

Problems

1. A rubber spherical balloon is being inflated at the rate of 35 cu. in./min. How fast is the surface increasing when the diameter is 14 in.?

Ans. 10 sq. in./min.

2. A rope is tied at a point A, passes through a pulley B supporting a weight, and over a stationary pulley C which is 10 ft. from A and on the same level. If the free end of the rope is being pulled down at a steady rate of 1 ft./min., how fast is the weight rising when it is 12 ft. lower than the point A?

3. A stone thrown into still water sends out concentric circular waves. If the radius of the outer wave increases at the rate of 3 ft./sec., how fast is the area included growing when the diameter is 12 ft.?

4. A man 6 ft. tall walks on a straight path which is 30 ft. from the point on the ground directly under a light 24 ft. above the ground. If the man walks 6 ft./sec., how fast is his shadow lengthening when the man is 72 ft. from the point on the path nearest to the light? *Ans.* $24/13$ ft./sec.

5. An aeroplane beacon revolving 5 r.p.m. is 100 rods from the nearest point on a straight road. How fast is the lighted spot on the pavement moving down the road when it is 240 rods from the point on the road nearest the beacon? *Ans.* 6760π rods/min.

6. The top of a ladder 15 ft. long slides down a vertical wall as the foot is pulled out at the rate of 1 ft./min. How fast is the top descending when it is 12 ft. above the ground? Find the direction of the motion of the midpoint of the ladder and its velocity at that instant.

7. An ellipse has a constant area. How fast is the focal width (latus rectum) changing when its semimajor axis is 13 units and its semiminor axis is 10 units, if the major axis is increasing 1 unit per minute? The area of an ellipse is π times the product of its semiaxes.

8. Sand is falling from a spout upon a conical pile at the rate of 18 cu. ft./min. If the slant height of the cone is always $5/4$ of its altitude, how fast is the radius of the base increasing when its value is 6 ft.?

9. The sides of an equilateral triangle are changing at the constant rate of 3 in./min. each. How fast is the area changing when the sides are 20 in long? *Ans.* $30\sqrt{3}$ sq. in./min.

10. A stone is dropped from a bridge at a point 104 ft. above the surface of the water and falls according to the law $s = 16t^2$, where s is in feet and t is in seconds. A motor boat was directly under the stone when it was dropped but traveling 40 ft./sec. What is the rate of change of the distance from the stone to the boat when $t = \frac{1}{2}$? When the stone hits the water? *Ans.* $20\sqrt{26}/13$ ft./sec., 40 ft./sec.

11. An observation balloon is at rest 2 miles above a highway. A truck passes along the highway at the rate of 40 m.p.h. How fast is the distance from the truck to the balloon changing when that distance is $2\frac{1}{2}$ miles?

12. A ship leaves a port and travels due east at the rate of 20 m.p.h. Twenty-four hours later another ship leaves the same port, traveling south-east at the rate of 15 m.p.h. How fast are they separating when the first ship has gone 600 miles?

13. As a point whose polar coordinates are (r,θ) runs around the cardioid $r = 8(1 - \cos\theta)$ the angular velocity is $\omega = d\theta/dt = \frac{1}{10}$ radian per minute. What is the rate of change of r when $\theta = \pi/6$? When $\theta = 3\pi/2$?

14. As an angle increases from 0° to 90°, the cosine of the angle decreases at the steady rate of 0.1 unit per minute. What is the rate of decrease of the cotangent of the angle when the angle is 30°? Is that rate increasing or decreasing as the angle increases? *Ans.* $\frac{4}{5}$ unit per minute.

15. Two points start out at the same time from (2,0) and move toward the y-axis, one along the upper half of the parabola $y^2 + 2x = 4$ and the other along the upper half of the parabola $4y^2 + 2x = 4$. If $v_x = -1/5$ for the first and $-2/5$ for the second, how fast is the distance between them changing when the second point crosses the y-axis?

16. One man starts out from a point A on a circle and walks along the circumference at the rate of 5 ft./sec. Another man starts at the same time from the center O and walks along the radius OB perpendicular to the diameter through A. They meet at B, on the circumference, the first man having walked one quarter of the circumference. At what rate is the distance between the men changing when the first man has walked half of the arc AB, if the radius of the circle is 50 ft.?

17. Two vertices of a triangle are (5,0) and (0,3). The third vertex moves on the line $y = 2x$ with a velocity of $\sqrt{20}$ units per second. At what rate is the area changing? *Ans.* 13 units per second.

18. A point traces the curve $y = 4x^2$ with a velocity of 4 units per second. At what rate is the inclination of the line joining it to the focus changing as the point passes through the origin? *Ans.* 64 radians per second.

70. Polar Coordinates. When the equation of a curve is stated in polar coordinates, in the form $r = f(\theta)$, we may wish to find the angle ψ from the radius vector of the point (r,θ) to

the tangent drawn at that point. To find that angle, let τ be the inclination of the tangent to the polar axis and note, from Fig. 124, that

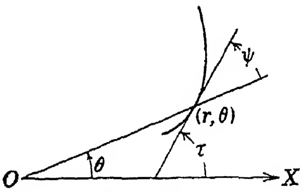

FIG. 124.

$$\psi = \tau - \theta$$

and hence

$$\tan \psi = \frac{\tan \tau - \tan \theta}{1 + \tan \tau \tan \theta}.$$

If we change over to rectangular coordinates and avail ourselves of the equalities $x = r \cos \theta$, $y = r \sin \theta$, we obtain

$$\frac{dx}{d\theta} = r \cdot \frac{d(\cos \theta)}{d\theta} + \cos \theta \cdot \frac{dr}{d\theta} = -r \sin \theta + \cos \theta \cdot \frac{dr}{d\theta},$$
$$\frac{dy}{d\theta} = r \cdot \frac{d(\sin \theta)}{d\theta} + \sin \theta \cdot \frac{dr}{d\theta} = r \cos \theta + \sin \theta \cdot \frac{dr}{d\theta}.$$

Exercise 1. Using the last relations above, prove that

$$\tan \psi = \frac{r}{dr/d\theta}.$$

HINT: $\tan \tau = \dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}.$

Exercise 2. Prove: $ds = \sqrt{r^2(d\theta)^2 + (dr)^2}$.
HINT: $ds = \sqrt{(dx)^2 + (dy)^2}$; $dx = -r \sin \theta d\theta + \cos \theta dr$;

$$dy = r \cos \theta d\theta + \sin \theta dr.$$

Exercise 3. Prove: $K = \left| \dfrac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r \cdot \dfrac{d^2r}{d\theta^2}}{[r^2 + (dr/d\theta)^2]^{3/2}} \right|$, where K denotes the curvature at a point. HINT: $K = \left|\dfrac{d\tau}{ds}\right| = \left|\dfrac{d\tau/d\theta}{ds/d\theta}\right|$; $\tau = \psi + \theta$; $\psi = \tan^{-1}\left(\dfrac{r}{dr/d\theta}\right)$.

Exercise 4. Obtain the differential of arc by using the property that $\lim\limits_{\text{arc}\to 0} (\text{chord}/\text{arc}) = 1$.

HINT: Call the end points of the arc (r,θ) and $(r + \Delta r, \theta + \Delta\theta)$. By the law of cosines,

$$(\text{chord})^2 = r^2 + (r + \Delta r)^2 - 2r(r + \Delta r) \cos \Delta\theta.$$

This reduces to

$$(\text{chord})^2 = 4r^2 \sin^2 \frac{\Delta\theta}{2} + 4r\Delta r \sin^2 \frac{\Delta\theta}{2} + (\Delta r)^2,$$

and use $ds/d\theta = \lim\limits_{\Delta\theta\to 0} (\Delta s/\Delta\theta) = \lim\limits_{\Delta\theta\to 0} (\text{chord}/\Delta\theta)$.

Problems

1. Find $\tan \psi$ for the following curves. Find in each case the points on the curve where the radius vector is perpendicular to the tangent.

(a) $r = \cos 3\theta$. *Ans.* $-\frac{1}{3} \operatorname{ctn} 3\theta$.

(b) $r^2 = 2 \sin 3\theta$. *Ans.* $\frac{2}{3} \tan 3\theta$.

(c) $r = 2a(1 - \sin \theta)$.

(d) $r = a$.

(e) $r = \dfrac{2a}{1 + \cos \theta}$.

(f) $r = 3 \sin \theta - 4 \cos \theta$.

(g) $r = ce^{a\theta}$.

2. Find the angles between the following pairs of curves at their points of intersection:

(a) $\begin{cases} r = 2 \cos \theta, \\ r = 2 \sin \theta. \end{cases}$ *Ans.* $90°$.

(b) $\begin{cases} r = 6 \cos \theta, \\ r = 3. \end{cases}$ *Ans.* $60°$.

(c) $\begin{cases} r = 1 + \sin \theta, \\ r = 5 - 3 \sin \theta. \end{cases}$ *Ans.* $0°$.

3. Find the curvature for the following curves. Compute for the points indicated.

(a) $r = \theta$, (π,π). *Ans.* $\dfrac{\pi^2 + 2}{(\pi^2 + 1)^{3/2}}$.

(b) $r = a$.

(c) $r^2 = \cos 2\theta$, $(1,0)$. *Ans.* 3.

(d) $r = 1 + \sin \theta$, $(2,\pi/2)$.

(e) $r = a(1 - 2 \cos \theta)$, $(a, \pi/2)$.

4. If p is the length of the perpendicular from the pole upon the tangent at (r,θ), prove $dp/dr = rK$, where K is the curvature at (r,θ).

CHAPTER X

INDEFINITE INTEGRALS

71. Notation for an Antiderivative. In his work thus far the student has been supplied with a function, as $f(x)$, of which he has been required to take the derivative. The notation for the result has been varied among the symbols $f'(x)$, $\frac{d}{dx}f(x)$, and $D_xf(x)$. The inverse of the process of differentiation is known as *integration*. That is, if we begin with a given function, say $F(x)$, and find a function, $f(x)$, whose derivative, with respect to x, is $F(x)$, we have performed, upon $F(x)$, an integration. The operation is also called a *quadrature* by many writers.

The process of integration, or taking of quadratures, is fundamentally more difficult than differentiation, because it is an inverse process. Thus, to find $D_x(x^2)$ we take the definition

$$D_x(x^2) = \lim_{u \to x} \frac{u^2 - x^2}{u - x}$$

and proceed directly to evaluate the limit. To integrate x^2, however, we must find a function $f(x)$ such that

$$\lim_{u \to x} \frac{f(u) - f(x)}{u - x} = x^2,$$

and no direct process for finding this function is at hand.

Success in integration depends, very largely, upon the experience of the student in taking derivatives, augmented by the accumulated experience of others. Thus, in the above case, it may occur at once, to a student who has studied differential calculus, that the derivative of $\frac{1}{3}x^3$ with respect to x, is x^2 and that $\frac{1}{3}x^3$ is, therefore, acceptable as the *integral* of x^2. But his experience would probably fail to supply him with a function whose derivative is $\log x$. To find such a function he must depend upon formulas and methods of integration which have come down from the experience of others.

In the last analysis, proof that a given quadrature has been correctly performed, is effected by differentiation. Thus, by differentiating the function $x \log x - x$ with respect to x, we obtain $\log x$, thus proving that an integral of $\log x$ is that function.

The notation for quadratures in common use is of the form

$$\int f(x)dx$$

where $f(x)$ is the function to be integrated. The symbol $\int$ is called an *integral sign* and the differential, $f(x)\,dx$, is called the *integrand*. Historically, $\int$ comes from the letter S for *sum*, and the connection is made clear in the next chapter. The symbol dx is the ordinary *differential* x. Its immediate use is to indicate x as the variable of integration. To portray the above problems and their solutions in this notation, we write

$$\int x^2\,dx = \tfrac{1}{3}x^3,$$

and

$$\int \log x\,dx = x \log x - x.$$

72. The Constant of Integration. Since the derivative of a constant is zero, we can write

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C,$$

where C is any constant whatever, and is called the *constant of integration*. Any quadrature $\int f(x)\,dx$, when performed, demands this arbitrary constant. In special circumstances, however, we may desire that the resulting function have an assigned value at some chosen value of the independent variable, and accomplish the effect by appropriate choice of the constant of integration. For example, let it be known that a body falls from rest at the time $t = 2$ and has a constant acceleration g. To find the distance, s, through which it has fallen at the time, t, as a function of t, we have

$$s = f(t),$$
$$v = \frac{ds}{dt} = f'(t),$$

where v represents the velocity, and

$$g = \frac{dv}{dt}.$$

Since v is a function of t whose derivative, with respect to t, is the constant g, we write

$$v = \int g \, dt.$$

From our experience in differentiation we see that this gives

$$v = gt + C_1,$$

where C_1 is a constant. However, since the body is at rest when $t = 2$, we have

$$0 = 2g + C_1,$$

or

$$C_1 = -2g,$$

and

$$v = gt - 2g.$$

The remainder of the problem is left to the student as Exercise 1.

Exercise 1. Prove that the expression sought, for s in terms of t, in the above example, is $s = \frac{1}{2}gt^2 - 2gt + 2g$. HINT: Express s as $\int (gt - 2g)dt$ and chose the constant of integration so that $s = 0$ when $t = 2$.

Exercise 2. Show that $\int 2 \sin x \cos x \, dx$ can be written either as

$$\sin^2 x + C$$

or as $-\cos^2 x + K$, where C and K are arbitrary constants.

Exercise 3. Show that the two expressions for $\int 2 \sin x \cos x \, dx$ given in Exercise 2 are identical if $K = C + 1$.

The integral, $\int 2 \sin x \cos x \, dx$, of Exercise 2 above could evidently be written in two different ways, but the student who has examined Exercise 3 has seen that the two expressions differ only by a constant. The situation illustrated by these exercises is general and may be stated in the form of

Theorem 1. *If $\int F(x)dx = f(x)$ and $\int F(x)dx = g(x)$, then $f(x) - g(x)$ is identically equal to some constant.*

The proof of this theorem is embodied in the following two exercises, noting, from the hypotheses of the theorem, that

$$\frac{d}{dx} f(x) = F(x), \qquad \frac{d}{dx} g(x) = F(x). \tag{107}$$

Exercise 4. If we define $\varphi(x)$ by the equation

$$\varphi(x) = f(x) - g(x)$$

and Eqs. (107) hold, prove that $\varphi'(x) = 0$ for every value of x.

Exercise 5. By use of Rolle's theorem prove that $\varphi(x)$ is equal to a constant. HINT: $\varphi(x) = \varphi(a) + (x - a)\varphi'(x_1)$ where x_1 is between x and a.

Exercise 6. Prove that $\int af(x)dx = a\int f(x)dx$ if a is a constant.

Exercise 7. Prove that $\int[f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$.

Problems

1. Verify the following integral formulas by differentiation:

(a) $\displaystyle\int \tan x\, dx = -\log \cos x + C.$

(b) $\displaystyle\int \operatorname{ctn} x\, dx = \log \sin x + C.$

(c) $\displaystyle\int \sec x\, dx = \log (\sec x + \tan x) + C.$

(d) $\displaystyle\int \csc x\, dx = \log \tan \frac{x}{2} + C.$

(e) $\displaystyle\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \frac{x - a}{x + a} + C.$

(f) $\displaystyle\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \log (x + \sqrt{x^2 \pm a^2}) + C.$

2. Write down the derivatives, with respect to x, of the functions x^n, e^x, $\log x$, $\sin x$, $\cos x$, $\tan x$, $\operatorname{ctn} x$, $\sec x$, $\csc x$, $\sin^{-1} \frac{x}{a}$, $\tan^{-1} \frac{x}{a}$, and $\sec^{-1} \frac{x}{a}$ and use the results to evaluate the following integrals:

(a) $\displaystyle\int x^m\, dx,\ m \neq -1.$

(b) $\displaystyle\int \frac{1}{x}\, dx.$

(c) $\displaystyle\int e^x\, dx.$

(d) $\displaystyle\int \sin x\, dx.$

(e) $\displaystyle\int \cos x\, dx.$

(f) $\displaystyle\int \sec^2 x\, dx.$

(g) $\displaystyle\int \csc^2 x\, dx.$

(h) $\displaystyle\int \sec x \tan x\, dx.$

(i) $\displaystyle\int \csc x \operatorname{ctn} x\, dx.$

(j) $\displaystyle\int \frac{dx}{x^2 + a^2}.$

(k) $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}}.$

(l) $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}.$

3. Prove, by illustration, that, in general,

$$\int f(x)g(x)dx \neq f(x)\int g(x)dx.$$

4. Verify that $\int \tan x \sec^2 x\, dx = \frac{1}{2} \tan^2 x + C$ and also

$$\int \tan x \sec^2 x\, dx = \tfrac{1}{2} \sec^2 x + C.$$

Explain the apparent ambiguity.

5. If $\frac{d}{dx}F(x) = \tan x$ and $F(0) = 1$, find $F(x)$. [See Prob. 1(a).]

6. If $\frac{d}{dx}F(x) = \frac{1}{x^2 - 4}$ and $F(3) = \frac{1}{4}$, find $F(x)$. [See Prob. 1(e).]

73. Formulas of Integration. To facilitate integration the student should become as familiar as possible with the following

formulas which he may readily verify and which have appeared, in some form, as problems in the preceding section.

1. $\int u^n \, du = \frac{u^{n+1}}{n+1} + C \ (n \neq -1).$

2. $\int \frac{du}{u} = \log u + C.$

3. $\int e^u \, du = e^u + C.$

4. $\int a^u \, du = \frac{a^u}{\log a} + C. \quad \begin{pmatrix} a > 0 \\ a \neq 1 \end{pmatrix}$

5. $\int \sin u \, du = -\cos u + C.$

6. $\int \cos u \, du = \sin u + C.$

7. $\int \tan u \, du = \log \sec u + C.$

8. $\int \operatorname{ctn} u \, du = \log \sin u + C.$

9. $\int \sec^2 u \, du = \tan u + C.$

10. $\int \csc^2 u \, du = -\operatorname{ctn} u + C.$

11. $\int \sec u \tan u \, du = \sec u + C.$

12. $\int \csc u \operatorname{ctn} u \, du = -\csc u + C.$

13. $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C.$

14. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C.$

15. $\int \frac{du}{\sqrt{u^2 \pm a^2}} = \log (u + \sqrt{u^2 \pm a^2}) + C.$

16. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \frac{u}{a} + C.$

17. $\int \sec u \, du = \log (\sec u + \tan u) + C.$

18. $\int \csc u \, du = -\log (\csc u + \operatorname{ctn} u) + C.$

19. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \left(\frac{u - a}{u + a}\right) + C$ if $\frac{u - a}{u + a}$ is positive.

20. $\int \frac{du}{u^2 - a^2} = \frac{1}{2a} \log \left(\frac{a - u}{a + u}\right) + C$ if $\frac{a - u}{a + u}$ is positive.

21. $\int \sinh u\, du = \cosh u + C.$

22. $\int \cosh u\, du = \sinh u + C.$

23. $\int \operatorname{sech}^2 u\, du = \tanh u + C.$

24. $\int \operatorname{csch}^2 u\, du = -\coth u + C.$

25. $\int \tanh u \operatorname{sech} u\, du = -\operatorname{sech} u + C.$

26. $\int \coth u \operatorname{csch} u\, du = -\operatorname{csch} u + C.$

An integral not immediately of any one of these forms may frequently be put into such form by a substitution. For example, consider the integral

$$\int \sin^6 x \cos x\, dx.$$

This is not included in the above list, but if we make the substitution,

$$\sin x = u,$$

and, hence,

$$\cos x\, dx = du,$$

we obtain, in place of the above integral, the new one,

$$\int u^6\, du,$$

which is of the form of formula 1 of the foregoing list and has the value

$$\tfrac{1}{7}u^7 + C.$$

Hence we write

$$\int \sin^6 x \cos x\, dx = \int (\sin x)^6 d(\sin x) = \tfrac{1}{7} \sin^7 x + C.$$

This device of introducing a new variable is of very frequent use in integration. The underlying idea is to choose a new variable in such a way that, in terms of it, the integral assumes the form of one of the standard formulas in the above list of 26. Consider the two additional illustrations:

Illustration 1. To integrate $\int \frac{x\,dx}{4 - x^2}$, choose as a new variable, $u = 4 - x^2$, so that $du = -2x\,dx$, and $x\,dx = -\frac{1}{2}\,du$. The integral now becomes

$$\int \frac{x dx}{4 - x^2} = \int \frac{-\frac{1}{2}\,du}{u} = -\frac{1}{2}\int \frac{du}{u} \quad -\frac{1}{2}\log u + C = -\frac{1}{2}\log(4 - x^2) + C.$$

Illustration 2. In integrating $\int \frac{\sin 3x\,dx}{5 - \cos^2 3x}$, set $\cos 3x = u$, and, hence, have $-3 \sin 3x\,dx = du$, and $\sin 3x\,dx = -\frac{1}{3}\,du$. We thus obtain

$$\int \frac{\sin 3x\,dx}{5 - \cos^2 3x} = \int \frac{-\frac{1}{3}\,du}{5 - u^2} = \frac{1}{3}\int \frac{du}{u^2 - 5} = \frac{1}{3} \cdot \frac{1}{2\sqrt{5}} \cdot \log \frac{\sqrt{5} - u}{\sqrt{5} + u} + C$$

$$= \frac{1}{6\sqrt{5}} \log \frac{\sqrt{5} - \cos 3x}{\sqrt{5} + \cos 3x} + C.$$

Problems

1. Integrate the following:

(a) $\int x\sqrt{x}\,dx.$ *Ans.* $\frac{2}{5}x^2\sqrt{x} + C.$

(b) $\int \frac{dx}{x^2\sqrt{x}}.$ *Ans.* $-\frac{2}{3x\sqrt{x}} + C.$

(c) $\int (2x^2 + 1)^2 dx.$ *Ans.* $\frac{4x^5}{5} + \frac{4x^3}{3} + x + C.$

(d) $\int \left(5x^2 + 2 - \frac{3}{x}\right) dx.$

(e) $\int \sqrt{2x + 1}\,dx$ (Set $u = 2x + 1$). *Ans.* $\frac{1}{3}(2x + 1)^{3/2} + C.$

(f) $\int (3x^2 - 2)x\,dx.$ *Ans.* $\frac{3x^4}{4} - x^2 + C.$

(g) $\int (3x^2 - 2)^{3/2}\,x\,dx$, (Set $u = 3x^2 - 2$).

Ans. $\frac{1}{15}(3x^2 - 2)^{5/2} + C.$

(h) $\int (2x^3 + 3)dx.$

(i) $\int (2x^3 + 3)^2 dx.$

(j) $\int (2x^3 + 3)x^2\,dx.$ *Ans.* $\frac{(2x^3 + 3)^2}{12} + C.$

(k) $\int (2x^3 + 3)^{10}x^2\,dx.$ *Ans.* $\frac{(2x^3 + 3)^{11}}{66} + C.$

(l) $\int \cos^3 x \sin x\,dx.$ *Ans.* $-\frac{\cos^4 x}{} + C.$

(m) $\int \sin^3 2x \cos 2x\,dx.$

(n) $\int \tan^3 \frac{x}{2} \sec^2 \frac{x}{2}\,dx.$ *Ans.* $\frac{\tan^4 \frac{x}{2}}{2} + C.$

(o) $\int \frac{\log x}{x} dx.$ *Ans.* $\frac{(\log x)^2}{2} + C.$

(p) $\int \frac{\log^2 3x}{}$

(q) $\int \frac{x\,dx}{\sqrt{4 + x^2}}.$ *Ans.*

(r) $\int \sqrt[3]{\operatorname{ctn}\frac{x}{3}} \csc^2 \frac{x}{3} dx$. *Ans.* $-\frac{9}{4}\left(\operatorname{ctn}\frac{x}{3}\right)^{\frac{4}{3}} + C$.

2. Integrate the following:

(a) $\int \frac{dx}{x+2}$.

(b) $\int \frac{dx}{1-x}$. *Ans.* $-\log(1-x) + C$.

(c) $\int \frac{\cos 2x}{1-\sin 2x} dx$.

(d) $\int \frac{x\,dx}{4+x^2}$. *Ans.* $\frac{1}{2}\log(4+x^2) + C$.

(e) $\int \frac{x^3\,dx}{3-x^4}$.

(f) $\int \frac{\sec^2 \frac{x}{2} dx}{3 + \tan \frac{x}{2}}$. *Ans.* $2 \log \left(3 + \tan \frac{x}{2}\right) + C$.

(g) $\int \frac{(x^2+1)dx}{x^3+3x-1}$.

(h) $\int e^{2x}\,dx$. *Ans.* $\frac{1}{2}e^{2x} + C$.

(i) $\int e^{x^2-1}x\,dx$. *Ans.* $\frac{1}{2}e^{x^2-1} + C$.

(j) $\int e^{\sec x} \sec x \tan x\,dx$.

(k) $\int \frac{e^{\sin^{-1} x}}{\sqrt{1-x^2}} dx$.

(l) $\int \frac{e^{\tan^{-1} 2x}}{1+4x^2} dx$.

(m) $\int e^{2x^2-4x}(x-1)dx$. *Ans.* $\frac{1}{4}e^{2x^2-4x} + C$.

(n) $\int \frac{e^{3x}\,dx}{1+e^{3x}}$. *Ans.* $\frac{1}{3}\log(1+e^{3x}) + C$.

(o) $\int 10^{2x+1}\,dx$. *Ans.* $\frac{10^{2x+1}}{2\log 10} + C$.

(p) $\int 5^{\sec 3x} \sec 3x \tan 3x\,dx$.

3. Integrate the following:

(a) $\int \cos \frac{x}{2} dx$. *Ans.* $2 \sin \frac{x}{2} + C$.

(b) $\int \tan 4x\,dx$. *Ans.* $\frac{1}{4} \log \sec 4x + C$.

(c) $\int \sin \frac{x}{3} dx$.

(d) $\int \operatorname{ctn}(x+1)dx$.

(e) $\int \cos(2x + \pi)dx$. *Ans.* $\frac{1}{2}\sin(2x + \pi) + C$.

(f) $\int \sec(2 - x)dx$.

(g) $\int \csc\left(x + \frac{\pi}{2}\right)dx$.

(h) $\int \sec^2 2x\, dx$.

(i) $\int \csc^2(1 - x)dx$. *Ans.* $\operatorname{ctn}(1 - x) + C$.

(j) $\int \sec 2x \tan 2x\, dx$.

(k) $\int \csc\frac{x}{4} \operatorname{ctn}\frac{x}{4}dx$.

(l) $\int \frac{dx}{\csc 3x}$. *Ans.* $-\frac{1}{3}\cos 3x + C$.

(m) $\int \frac{dx}{\sin 2x}$.

(n) $\int \frac{dx}{\tan(2x + 1)}$. *Ans.* $\frac{1}{2}\log\sin(2x + 1) + C$.

(o) $\int \frac{\cos\sqrt{x}}{\sqrt{x}}dx$. HINT: Set $u = \sqrt{x}$. *Ans.* $2\sin\sqrt{x} + C$.

(p) $\int \tan^2 3x\, dx$. HINT: $\tan^2 3x = \sec^2 3x - 1$.

(q) $\int \operatorname{ctn}^2\frac{x}{2}\, dx$. *Ans.* $-2\operatorname{ctn}\frac{x}{2} - x + C$.

4. Integrate the following:

(a) $\int \frac{dx}{9 + x^2}$.

(b) $\int \frac{dx}{9 + 4x^2}$. *Ans.* $\frac{1}{6}\tan^{-1}\frac{2x}{3} + C$.

(c) $\int \frac{e^x dx}{4 + e^{2x}}$. *Ans.* $\frac{1}{2}\tan^{-1}\frac{e^x}{2} + C$.

(d) $\int \frac{dx}{x(1 + \log^2 2x)}$. HINT: Set $u = \log 2x$.

(e) $\int \frac{\sin x\, dx}{1 + \cos^2 x}$. *Ans.* $-\tan^{-1}(\cos x) + C$.

(f) $\int \frac{\sec^2\frac{x}{2}dx}{3 + \tan^2\frac{x}{2}}$.

(g) $\int \frac{dx}{1 - 9x^2}$.

(h) $\int \frac{\sin 3x\, dx}{4 - \cos^2 3x}$. *Ans.* $\frac{1}{12}\log\frac{2 - \cos 3x}{2 + \cos 3x} + C$.

(i) $\int \frac{dx}{x(5 - \log^2 3x)}$

(j) $\int \frac{\sec x \tan x \, dx}{\sqrt{4 + 9 \sec^2 x}}$.

(k) $\int \frac{dx}{\sqrt{4 - 9x^2}}$. *Ans.* $\frac{1}{3} \sin^{-1} \frac{3x}{2} + C$.

(l) $\int \frac{\csc 2x \operatorname{ctn} 2x \, dx}{\sqrt{9 - 4 \csc^2 2x}}$.

(m) $\int \quad dx$

(n) $\int \frac{}{\sin 3x \sqrt{\sin^2 3x - \frac{1}{4}}}$ *Ans.* $\frac{2}{3} \sec^{-1} (2 \sin 3x) + C$.

5. Integrate the following:

(a) $\int \frac{dx}{x^2 - 4}$.

(b) $\int \frac{x \, dx}{x^2 - 4}$.

(c) $\int \frac{2 + 3x}{x^2 - 4} dx$ *Ans.* $\frac{1}{2}\left[\log \frac{x - 2}{x + 2} + 3 \log (x^2 - 4)\right] + C$.

(d) $\int \frac{dx}{\sqrt{x^2 - 4}}$

(e) $\int \frac{x \, dx}{\sqrt{x^2 - 4}}$ *Ans.* $\sqrt{x^2 - 4} + C$.

(f) $\int x e^{3x^2} \, dx$.

(g) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx$. *Ans.* $2e^{\sqrt{x}} + C$.

(h) $\int e^{\tan 2x} \sec^2 2x \, dx$.

(i) $\int \frac{dx}{x \log x \sqrt{\log^2 x - 1}}$. *Ans.* $\sec^{-1} \log x + C$.

(j) $\int \frac{\cos 3x \, dx}{\sin^2 3x - \frac{1}{9}}$.

(k) $\int \frac{e^x \cos e^x \, dx}{4 + \sin^2 e^x}$.

(l) $\int \cosh^2 \left(x + \frac{\pi}{6}\right) \sinh \left(x + \frac{\pi}{6}\right) dx$.

(m) $\int \frac{\operatorname{sech}^2 2x \, dx}{4 + \tanh^2 2x}$.

(n) $\int \frac{1 + \cosh 3x}{3x + \sinh 3x} dx$.

(o) $\int \frac{\log (\log x)}{x \log x} dx$.

(p) $\int \sin \frac{x}{3} \cos \frac{x}{6} dx$. *Ans.* $-4 \cos^3 \frac{x}{6} + C$.

6. Integrate the following:

(a) $\int \frac{x^2 + 2}{x + 1} dx$. HINT: In integrating an improper rational fraction,

reduce it first, by division, to the sum of a polynomial and a proper fraction. (A rational fraction is said to be proper when the degree of the numerator is less than that of the denominator; otherwise, improper.) In the given problem, the integral becomes $\int\left(x - 1 + \frac{3}{x+1}\right)dx$.

(b) $\int\frac{2x-3}{x+4}dx$. *Ans.* $2x - 11 \log (x + 4) + C$.

(c) $\int\frac{5-4x^2}{3+2x}dx$. *Ans.* $-x^2 + 3x - 2 \log (3 + 2x) + C$.

(d) $\int\frac{2x^3}{x^2-4}dx$. *Ans.* $x^2 + 4 \log (x^2 - 4) + C$.

(e) $\int\frac{x^4-1}{4x^2-1}dx$.

(f) $\int\frac{e^{3x}dx}{e^{2x}+3}$. HINT: Set $u = e^x$.

(g) $\int\frac{\sin^2 x \cos x}{3 - 4\sin^2 x}dx$. *Ans.* $-\frac{\sin x}{4} - \frac{\sqrt{3}}{16}\log\frac{\sqrt{3} - 2\sin x}{\sqrt{3} + 2\sin x} + C$.

(h) $\int\frac{\tanh^2 x \operatorname{sech}^2 x}{1 - \tanh x}dx$.

(i) $\int\frac{(1+x^6)x^2}{2+x^3}dx$. *Ans.* $\frac{x^6}{6} - \frac{2}{3}x^3 + \frac{5}{3}\log (2 + x^3) + C$.

(j) $\int\frac{2 + \log^2 x}{x(1 - \log x)}dx$.

(k) $\int\frac{\sin^3 2x \cos 2x}{2 - \cos^2 2x}dx$.

7. Integrate the following:

(a) $\int\frac{dx}{2x^2 - x + 2}$. HINT: In integrating an expression of the type $\int\frac{dx}{ax^2+bx+c}$ or $\int\frac{dx}{\sqrt{ax^2+bx+c}}$, we may reduce it to one of the forms represented by formulas 13, 14, 15, 19, or 20 by completing the square in the denominator. To avoid irrationalities and fractions, it is well to multiply the trinomial by $4a$, if a is positive, or by $-4a$, if a is negative. Thus, the trinomial becomes

$$\frac{1}{4a}(4a^2x^2 + 4abx + 4ac) = \frac{1}{4a}[(2ax + b)^2 + (4ac - b^2)]$$

or

$$-\frac{1}{4a}(-4a^2x^2 - 4abx - 4ac) = -\frac{1}{4a}[(b^2 - 4ac) - (2ax + b)^2].$$

In the given problem, the integral becomes

$$\int\frac{dx}{\frac{1}{8}[(4x-1)^2 + 15]} = \int\frac{8\,dx}{(4x-1)^2 + 15}.$$

Set $u = 4x - 1$. *Ans.* $\frac{2}{\sqrt{15}}\tan^{-1}\frac{4x-1}{\sqrt{15}} + C$.

(b) $\displaystyle\int \frac{dx}{3x^2 + x - 2}.$ *Ans.* $\frac{1}{5} \log \frac{3x - 2}{3x + 3} + C.$

(c) $\displaystyle\int \frac{dx}{x^2 + 2x - 2}.$ *Ans.* $\frac{1}{2\sqrt{3}} \log \frac{x + 1 - \sqrt{3}}{x + 1 + \sqrt{3}} + C.$

(d) $\displaystyle\int \frac{dx}{-2x^2 + x - 1}.$ *Ans.* $-\frac{2}{\sqrt{7}} \tan^{-1} \frac{4x - 1}{\sqrt{7}} + C.$

(e) $\displaystyle\int \frac{dx}{-3x^2 + 2x - 2}.$

(f) $\displaystyle\int \frac{dx}{4x^2 - x}.$ *Ans.* $\log \frac{4x - 1}{4x} + C.$

(g) $\displaystyle\int \frac{dx}{5x - 2x^2}.$ *Ans.* $-\frac{1}{5} \log \frac{2x - 5}{2x} + C.$

(h) $\displaystyle\int \frac{dx}{\sqrt{4x^2 - x + 2}}.$

(i) $\displaystyle\int \frac{dx}{\sqrt{5x^2 + 2x - 3}}.$

(j) $\displaystyle\int \frac{dx}{\sqrt{-x^2 + 3x + 2}}.$ *Ans.* $\sin^{-1} \frac{2x - 3}{\sqrt{17}} + C.$

(k) $\displaystyle\int \frac{dx}{\sqrt{5x^2 - x}}.$

(l) $\displaystyle\int \frac{dx}{\sqrt{-2x^2 + x}}.$

(m) $\displaystyle\int \frac{e^x\, dx}{e^{2x} - 5e^x + 1}.$ *Ans.* $\frac{1}{\sqrt{21}} \log \frac{2e^x - 5 - \sqrt{21}}{2e^x - 5 + \sqrt{21}} + C.$

8. Integrate the following:

(a) $\displaystyle\int \frac{\sqrt{2x + 1} + 2}{\sqrt[3]{2x + 1} - 1} dx.$ HINT: If the only irrationalities present in the integrand are rational powers of some linear function of the independent variable, say $(ax + b)^{m/n}$, $(ax + b)^{p/n}$, $(ax + b)^{q/n}$, . . . , the integrand is rationalized by setting $u = (ax + b)^{1/n}$, whence $u^n = ax + b$, $x = \frac{u^n - b}{a}$, $dx = \frac{nu^{n-1}\, du}{a}$. In the given problem the two irrational expressions may be written as $(2x + 1)^{3/6}$ and $(2x + 1)^{2/6}$, suggesting the substitution $u = (2x + 1)^{1/6}$, whence $u^6 = 2x + 1$, and

$$dx = 3u^5\, du.$$

The integral now becomes $\displaystyle\int \frac{u^3 + 2}{u^2 - 1} \cdot 3u^5\, du = 3\int \frac{u^8 + 2u^5}{u^2 - 1} du$, and the integrand is rational.

(b) $\displaystyle\int x\sqrt{1 - x}\, dx.$ *Ans.* $2\left[\frac{(1 - x)^{5/2}}{5} - \frac{(1 - x)^{3/2}}{3}\right] + C.$

(c) $\displaystyle\int x^2\sqrt[3]{3 - 2x}\, dx.$

Ans. $-\frac{3}{8}\left[\frac{9}{4}(3 - 2x)^{4/3} - \frac{6}{7}(3 - 2x)^{7/3} + \frac{1}{10}(3 - 2x)^{10/3}\right] + C.$

(d) $\displaystyle\int \frac{\sqrt{x}}{\sqrt[4]{x}+1}\,dx.$

(e) $\displaystyle\int \frac{\sqrt{x}}{2-\sqrt[3]{x}}\,dx.$

(f) $\displaystyle\int \frac{dx}{\sqrt{3+4x}\,(1+\sqrt{3+4x})}.$

(g) $\displaystyle\int \frac{dx}{\sqrt{x}\,(2+\sqrt[3]{x})}.$ *Ans.* $6\left(\sqrt[6]{x}-\sqrt{2}\tan^{-1}\dfrac{\sqrt[6]{x}}{\sqrt{2}}\right)+C.$

(h) $\displaystyle\int \frac{dx}{(4-5x)^{2/3}-(4-5x)^{3/4}}.$

(i) $\displaystyle\int \frac{\quad}{(3+2x)^{3/2}}\,dx.$

(j) $\displaystyle\int \frac{3+\sqrt{x+1}}{4+x}\,dx.$

(k) $\displaystyle\int \frac{dx}{x+2-2\sqrt{x+2}}.$

9. Integrate the following:

(a) $\displaystyle\int \frac{dx}{x^2\sqrt{4x^2-9}}.$ KEY: If the irrationality present in the integrand is of the form (a), $\sqrt{a^2x^2+b^2}$; (b), $\sqrt{b^2-a^2x^2}$; or (c), $\sqrt{a^2x^2-b^2}$, set, in case (a), $ax = b\tan\theta$; in case (b), $ax = b\sin\theta$; and, in case (c), $ax = b\sec\theta$. In the respective cases the radicals become $b\sec\theta$, $b\cos\theta$, and $b\tan\theta$, and the integrand becomes rational in terms of trigonometric functions. In the given problem we thus set $2x = 3\sec\theta$, whence

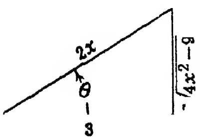

FIG. 125.

$$\sqrt{4x^2-9} = 3\tan\theta,$$

and $dx = \frac{3}{2}\sec\theta\tan\theta\,d\theta$, and the integral becomes

$$\int \frac{\frac{3}{2}\sec\theta\tan\theta\,d\theta}{\frac{9}{4}\sec^2\theta\cdot 3\tan\theta} = \frac{2}{9}\int\frac{d\theta}{\sec\theta} = \frac{2}{9}\int\cos\theta\,d\theta$$
$$= \tfrac{2}{9}\sin\theta + C.$$

Note that the relation between x and θ, *viz.*, $\sec\theta = 2x/3$, may be exhibited by the adjoining Fig. 125 and, hence, $\frac{2}{9}\sin\theta + C$, the value of the integral, is, in terms of x, equal to $\dfrac{2}{9}\cdot\dfrac{\sqrt{4x^2-9}}{2x} + C = \dfrac{\sqrt{4x^2-9}}{9x} + C.$

(b) $\displaystyle\int \frac{dx}{x\sqrt{9x^2+4}}.$ *Ans.* $-\dfrac{1}{2}\log\dfrac{2+\sqrt{9x^2+4}}{3x}+C.$

(c) $\displaystyle\int \frac{dx}{x^2\sqrt{4-x^2}}.$ *Ans.* $-\dfrac{\sqrt{4-x^2}}{4x}+C.$

(d) $\displaystyle\int \frac{dx}{(1-4x^2)^{3/2}}.$ *Ans.* $\dfrac{x}{\sqrt{1-4x^2}}+C.$

(e) $\displaystyle\int \frac{dx}{x^2\sqrt{3x^2 - 4}}$. *Ans.* $\displaystyle\frac{\sqrt{3x^2 - 4}}{4x} + C.$

(f) $\displaystyle\int \frac{dx}{(x^2 + 4)^{3/2}}$. *Ans.* $\displaystyle\frac{x}{4\sqrt{x^2 + 4}} + C.$

(g) $\displaystyle\int \frac{\sqrt{4x^2 - 5}}{x}dx.$

(h) $\displaystyle\int \frac{x\,dx}{\sqrt{4 - x^2}}$, by two methods.

(i) $\displaystyle\int \frac{x\,dx}{\sqrt{2 + 3x^2}}$, by two methods.

74. Integration of Trigonometric Functions. Certain products of powers of trigonometric functions are readily integrable when account is taken of some well-known identities, such as $\sin^2\theta + \cos^2\theta = 1$, $1 + \tan^2\theta = \sec^2\theta$, etc. In what follows we consider a number of special cases, some worked out in the text and others left to the student as exercises.

(a) *Integrals of the Form* $\int \cos^m u\,du$

(I) If m is an odd positive integer, write the integrand as $\cos^{m-1} u \cdot \cos u\,du$. The exponent, $m - 1$, is even, and the first factor, $\cos^{m-1} u$, is easily expressible as a polynomial in $\sin u$, while the second factor, $\cos u\,du$, is $d(\sin u)$. The integrand thus becomes $f(v)dv$, where $f(v)$ is a polynomial in v and $v = \sin u$. The integration is now immediate.

To illustrate, take the integral, $\int \cos^5 2x\,dx$. Following the plan described above, we write

$$\begin{aligned}
\int \cos^5 2x\,dx &= \int \cos^4 2x \cdot \cos 2x\,dx \\
&= \int (1 - \sin^2 2x)^2 \cdot \cos 2x\,dx \\
&= \int (1 - 2\sin^2 2x + \sin^4 2x) \cdot \cos 2x\,dx \\
&= \int \cos 2x\,dx - 2\int \sin^2 2x \cos 2x\,dx + \\
&\qquad\qquad \int \sin^4 2x \cos 2x\,dx \\
&= \frac{\sin 2x}{2} - \frac{\sin^3 2x}{3} + \frac{\sin^5 2x}{10} + C.
\end{aligned}$$

(II) If m is an even positive integer, make use of the formula $\cos^2 u = \dfrac{1 + \cos 2u}{2}$, and the integrand becomes

$$\left(\frac{1 + \cos 2u}{2}\right)^{m/2} du,$$

where $m/2$ is an integer. The coefficient of du is seen to be a polynomial in $\cos 2u$. The odd-powered terms of that polynomial are treated as in (I). The even-powered terms have exponents not greater than $m/2$ and, hence, smaller than the original even exponent, m. The repetition of the process just explained, applied to these new even powers, will yield still other odd powers, which may be integrated as in (I), and other even powers with still smaller exponents. Continuation of the process ultimately reduces all powers to integrable forms.

As an illustration of the process, consider the integral $\int \cos^6 x\,dx$. From the identity $\cos^2 x = \dfrac{1 + \cos 2x}{2}$, we write

$$\begin{aligned}
\int \cos^6 x\,dx &= \int \frac{(1 + \cos 2x)^3}{8}dx \\
&= \tfrac{1}{8}\int(1 + 3\cos 2x + 3\cos^2 2x + \cos^3 2x)\,dx \\
&= \frac{x}{8} + \frac{3\sin 2x}{16} + \frac{3}{8}\int \cos^2 2x\,dx + \frac{1}{8}\int \cos^3 2x\,dx \\
&= \frac{x}{8} + \frac{3\sin 2x}{16} + \frac{3}{16}\int(1 + \cos 4x)dx + \\
&\qquad \frac{1}{8}\int(1 - \sin^2 2x)\cos 2x\,dx \\
&= \frac{x}{8} + \frac{3\sin 2x}{16} + \frac{3x}{16} + \frac{3\sin 4x}{64} + \frac{\sin 2x}{16} - \\
&\qquad \frac{\sin^3 2x}{48} + C \\
&= \frac{5x}{16} + \frac{\sin 2x}{4} - \frac{\sin^3 2x}{48} + \frac{3\sin 4x}{64} + C.
\end{aligned}$$

(b) *Integrals of the Form* $\int \sin^m u\,du$

An integral of this form, where m is a positive integer, may be treated in a manner entirely analogous to the discussion in (a). The work is left to the student in the form of three exercises:

Exercise 1. Write out a description of the method of integrating $\int \sin^m u\,du$, where m is a positive odd integer, similar to the discussion in (I) under (a) above.

Exercise 2. If m is an even positive integer, write a description of a process which will express the integral $\int \sin^m u\,du$ in terms of similar integrals with smaller exponents.

Exercise 3. Illustrate the discussions made in Exercises 1 and 2 by completely integrating $\int \sin^8 x\,dx$.

(c) *Integrals of the Form* $\int \tan^m u\, du$

(I) If m is an odd positive integer, write the integrand as $\tan u(\sec^2 u - 1)^{\frac{m-1}{2}} du$. The second factor in the coefficient of du, when expanded, will be a polynomial in $\sec u$, of which the last term will be ± 1. Making use of the fact that $d(\sec u) = \tan u \sec u\, du$, we shall write each term, but the last one, in the expanded integrand in the form $(\sec u)^r d(\sec u)$, where r is some positive integer. The last term will integrate by the rule for $\int \tan u\, du$.

(II) If m is an even positive integer, express the integrand as $\tan^{m-2} u(\sec^2 u - 1)du = \tan^{m-2} u \cdot \sec^2 u\, du - \tan^{m-2} u\, du$ $= \tan^{m-2} u \cdot \sec^2 u\, du - \tan^{m-4} u \cdot \sec^2 u\, du + \tan^{m-4} u\, du$, etc. The process, thus continued, will lead, in the end, to a last term, $\tan^2 u\, du$, which can be rewritten as $(\sec^2 u - 1)du$ and integrated, while all the other terms encountered are of the form $\tan^r u \cdot \sec^2 u\, du$ and are also immediately integrable.

As an illustration of (c), with m odd, we present

$$\begin{aligned}
\int \tan^7 \pi x\, dx &= \int \tan^6 \pi x \cdot \tan \pi x\, dx = \int (\tan^2 \pi x)^3 \tan \pi x\, dx \\
&= \int (\sec^2 \pi x - 1)^3 \tan \pi x\, dx \\
&= \int (\sec^6 \pi x - 3\sec^4 \pi x + 3\sec^2 \pi x - 1) \tan \pi x\, dx \\
&= \int [(\sec^5 \pi x - 3\sec^3 \pi x + 3\sec \pi x) \sec \pi x \tan \pi x \\
&\qquad\qquad - \tan \pi x]dx \\
&= \frac{1}{\pi}\int (\sec^5 \pi x - 3\sec^3 \pi x + 3\sec \pi x)d(\sec \pi x) - \\
&\qquad\qquad \frac{1}{\pi}\int \tan \pi x\, d(\pi x) \\
&= \frac{\sec^6 \pi x}{6\pi} - \frac{3\sec^4 \pi x}{4\pi} + \frac{3\sec^2 \pi x}{2\pi} - \frac{\log \sec \pi x}{\pi} + C.
\end{aligned}$$

For an illustration of (c) with m even, take

$$\begin{aligned}
\int \tan^8 5x\, dx &= \int \tan^6 5x \cdot \tan^2 5x\, dx = \int \tan^6 5x(\sec^2 5x - 1)dx \\
&= \int [\tan^6 5x \sec^2 5x - \tan^6 5x]dx \\
&= \int [\tan^6 5x \sec^2 5x - \tan^4 5x(\sec^2 5x - 1)]dx \\
&= \int [\tan^6 5x \sec^2 5x - \tan^4 5x \sec^2 5x + \tan^4 5x]dx \\
&= \int [\tan^6 5x \sec^2 5x - \tan^4 5x \sec^2 5x + \\
&\qquad\qquad \tan^2 5x(\sec^2 5x - 1)]dx \\
&= \int [\tan^6 5x \sec^2 5x - \tan^4 5x \sec^2 5x + \\
&\qquad\qquad \tan^2 5x \sec^2 5x - \tan^2 5x]dx
\end{aligned}$$

$$= \int [\tan^6 5x \sec^2 5x - \tan^4 5x \sec^2 5x + \tan^2 5x \sec^2 5x - \sec^2 5x + 1]dx$$

$$= \frac{\tan^7 5x}{35} - \frac{\tan^5 5x}{25} + \frac{\tan^3 5x}{15} - \frac{\tan 5x}{5} + x + C.$$

(d) *Integrals of the Form $\int ctn^m u\, du$*

If m is a positive integer, an integral of the form $\int \operatorname{ctn}^m u\, du$ can be evaluated in a manner similar to the method employed for $\int \tan^m u\, du$. The details of discussion are left to the student in the form of exercises.

Exercise 4. Write out a description of the method of integrating

$$\int \operatorname{ctn}^m u\, du$$

if m is an odd positive integer.

Exercise 5. If m is an even positive integer, show how the integral, $\int \operatorname{ctn}^m u\, du$, can be expressed in terms of a similar integral in which the exponent of ctn u is $m - 2$.

Exercise 6. Illustrate the description given in Exercise 4 by completely integrating $\int \operatorname{ctn}^5 4x\, dx$.

Exercise 7. Illustrate Exercise 5 by expressing the integral, $\int \operatorname{ctn}^{12} 2x\, dx$, in terms of the integral $\int \operatorname{ctn}^{10} 2x\, dx$.

(e) *Integrals of the Form $\int \sin^m u \cos^n u\, du$*

Integrals of this form in which m and n are positive integers can be handled by methods very similar to those employed for cases (*a*) and (*b*) above. The details are left as exercises.

Exercise 8. If n is odd show how the integral $\int \sin^m u \cos^n u\, du$ can be put into the form $\int f(\sin u)d(\sin u)$, where $f(\sin u)$ is a polynomial in sin u. Illustrate by integrating completely $\int \sin^6 2x \cos^3 2x\, dx$.

Exercise 9. If m is an odd positive integer, show how the integral $\int \sin^m u \cos^n u\, du$ can be expressed in the form $\int f(\cos u)d(\cos u)$, where $f(\cos u)$ is a polynomial in cos u. Illustrate by integrating completely $\int \sin^5 3x \cos^2 3x\, dx$.

Exercise 10. If both m and n are even positive integers, show that the identities $\sin^2 u = \dfrac{1 - \cos 2u}{2}$ and $\cos^2 u = \dfrac{1 + \cos 2u}{2}$ will reduce the integral $\int \sin^m u \cos^n u\, du$ to a sum of integrals of the form $a \int \cos^r 2u\, du$, in each of which a is a constant and r is a positive integer or zero. Illustrate by integrating $\int \cos^4 x \sin^2 x\, dx$.

(f) *Integrals of the Form $\int \tan^m u \sec^n u\, du$*

Integrals of this form, in which both m and n are positive integers, are readily integrable if m is odd or if n is even. Details are left to the student in the following exercises.

Exercise 11. If m is an odd positive integer, show that the integral $\int \tan^m u \sec^n u\, du$ is equivalent to an integral of the form $\int f(\sec u) d(\sec u)$, where $f(\sec u)$ is a polynomial in $\sec u$. Illustrate by integrating

$$\int \tan^3 \theta \sec^5 \theta\, d\theta$$

completely.

Exercise 12. If n is an even positive integer, show that the integral $\int \tan^m u \sec^n u\, du$ is equivalent to an integral of the form $\int f(\tan u) d(\tan u)$, where $f(\tan u)$ is a polynomial in $\tan u$. Illustrate by evaluating

$$\int \tan^4 \pi x \sec^4 \pi x\, dx.$$

(g) *Integrals of the Form* $\int \operatorname{ctn}^m u \csc^n u\, du$

Integrals of this form are analogous to those of the form $\int \tan^m u \sec^n u\, du$ and may be handled by similar methods.

Exercise 13. Show how to integrate $\int \operatorname{ctn}^m u \csc^n u\, du$ if m is an odd positive integer. Illustrate with $\int \operatorname{ctn}^5 2\theta \csc^3 2\theta\, d\theta$.

Exercise 14. Show how to integrate $\int \operatorname{ctn}^m u \csc^n u\, du$ if n is an even positive integer.

(h) *Integrals Rational in Trigonometric Functions*

An integral of the form $\int F(u) du$, where $F(u)$ is a rational function of $\sin u$, $\cos u$, $\tan u$, $\operatorname{ctn} u$, $\sec u$, and $\csc u$, can be made to become an integral of the form $\int G(v) dv$ where $G(v)$ is a rational function of v, by the substitution $v = \tan u/2$. The following exercises are given to explain the method.

Exercise 15. Show that the substitution $v = \tan \frac{u}{2}$ leads to

$$\begin{aligned} \sin u &= \frac{2v}{1+v^2}, & \csc u &= \frac{1+v^2}{2v}, \\ \cos u &= \frac{1-v^2}{1+v^2}, & \sec u &= \frac{1+v^2}{1-v^2}, \\ \tan u &= \frac{2v}{1-v^2}, & \operatorname{ctn} u &= \frac{1-v^2}{2v}, \end{aligned}$$

$$du = \frac{2dv}{1+v^2}$$

Exercise 16. By employing the substitutions of Exercise 15, with $u = x$, show that

$$\int \frac{dx}{\sin x - 2\cos x - 2} = \int \frac{dv}{v-2} = \log (v-2) + C$$
$$= \log \left(\tan \frac{x}{2} - 2\right) + C.$$

Problems

Integrate each of the following:

1. $\int \cos^3 3x\,dx.$ *Ans.* $\dfrac{\sin 3x}{3} - \dfrac{\sin^3 3x}{9} + C.$

2. $\int \sin^3 \dfrac{x}{2}dx.$ *Ans.* $\dfrac{2}{3}\cos^3\dfrac{x}{2} - 2\cos\dfrac{x}{2} + C.$

3. $\int \sin^5 2x\,dx.$

4. $\int \sin^2 \dfrac{x}{3}dx.$ *Ans.* $\dfrac{x}{2} - \dfrac{3}{4}\sin\dfrac{2x}{3} + C.$

5. $\int \sin^4 x\,dx.$ *Ans.* $\dfrac{3x}{8} - \dfrac{\sin 2x}{4} + \dfrac{\sin 4x}{32} + C.$

6. $\int \cos^4 2x\,dx.$

7. $\int \tan^2 \dfrac{2x}{3}dx.$ *Ans.* $\dfrac{3}{2}\tan\dfrac{2x}{3} - x + C.$

8. $\int \operatorname{ctn}^3 x\,dx.$ *Ans.* $-\frac{1}{2}\operatorname{ctn}^2 x - \log\sin x + C.$

9. $\int \operatorname{ctn}^4 3x\,dx.$ *Ans.* $-\frac{1}{9}\operatorname{ctn}^3 3x + \frac{1}{3}\operatorname{ctn} 3x + x + C.$

10. $\int \tan^5 x\,dx.$ *Ans.* $\dfrac{\sec^4 x}{4} - \sec^2 x - \log\cos x + C.$

11. $\int \sec^4 x\,dx.$ *Ans.* $\dfrac{\tan^3 x}{3} + \tan x + C.$

12. $\int \csc^6 \dfrac{x}{3}dx.$ *Ans.* $-\dfrac{3}{5}\operatorname{ctn}^5\dfrac{x}{3} - 2\operatorname{ctn}^3\dfrac{x}{3} - 3\operatorname{ctn}\dfrac{x}{3}$

13. $\int \dfrac{dx}{\sec^3 2x}$

14. $\int \sin^2 2x\cos^3 2x\,dx.$ *Ans.* $\dfrac{\sin^3 2x\cos^2 2x}{10} + \dfrac{\sin^3 2x}{15} + C.$

15. $\int \sin^3 x\cos^4 x\,dx.$ *Ans.* $\dfrac{\cos^7 x}{7} - \dfrac{\cos^5 x}{5} + C.$

16. $\int \sin^4 x\cos^2 x\,dx.$

17. $\int \sin^2\dfrac{x}{4}\cos^2\dfrac{x}{4}dx.$

18. $\int \tan^3 x\sec^4 x\,dx.$ *Ans.* $\dfrac{\tan^6 x}{6} + \dfrac{\tan^4 x}{4} + C.$

19. $\int \tan^2 x\sec^8 x\,dx.$

20. $\int \tan^3 2x\sec^3 2x\,dx.$ *Ans.* $\dfrac{\sec^5 2x}{10} - \dfrac{\sec^3 2x}{6} + C.$

21. $\int \tan^5 3x\sec^3 3x\,dx.$

22. $\int \operatorname{ctn} x\csc^5 x\,dx.$

23. $\int \operatorname{ctn}^3 x \csc^4 x\, dx.$

24. $\int \sin x \cos 2x\, dx.$ *Ans.* $\cos x - \frac{2}{3}\cos^3 x + C.$

25. $\int \sin^2 2x \cos^3 x\, dx.$

26. $\int \frac{\cos^2 u}{\sin u} du.$ HINT: Write as $\int \frac{\cos^2 u \sin u\, du}{1 - \cos^2 u}.$

. $\int \frac{\sqrt{9 - 4x^2}}{x^2} dx.$ *Ans.* $-2\left(\sin^{-1}\frac{2x}{3} + \frac{\sqrt{9 - 4x^2}}{2x}\right) + C.$

28. $\int \frac{x^2 dx}{\sqrt{4 - 9x^2}}.$ *Ans.* $\frac{2}{27}\left(\sin^{-1}\frac{3x}{2} - \frac{3x\sqrt{4 - 9x^2}}{4}\right) + C.$

29. $\int \tan^3 \theta \cos^2 \theta\, d\theta.$ *Ans.* $-\frac{\sin^2 \theta}{2} - \log \cos \theta + C.$

30. $\int x^3\sqrt{4x^2 - 1}\, dx.$ *Ans.* $\frac{(4x^2 - 1)^{3/2}}{48} + \frac{(4x^2 - 1)^{5/2}}{80} + C.$

31. $\int \frac{dx}{\sin x - \cos x - 1}$ *Ans.* $\log\left(\tan\frac{x}{2} - 1\right) + C.$

32. $\int \frac{dx}{1 + 2\sin x}.$

33. $\int \frac{dx}{3 + \cos x}.$ *Ans.* $\frac{1}{\sqrt{2}}\tan^{-1}\frac{\tan\frac{x}{2}}{\sqrt{2}} + C.$

34. $\int \frac{\sin x\, dx}{9 - 2\sin^2 x}.$ *Ans.* $\frac{-\sqrt{2}}{2\sqrt{7}}\tan^{-1}\left(\frac{\sqrt{2}\cos x}{\sqrt{7}}\right) + C.$

35. $\int \frac{dx}{\tan x - \sin x}.$ *Ans.* $-\frac{1}{4}\operatorname{ctn}^2\frac{x}{2} - \frac{1}{2}\log\tan\frac{x}{2} + C.$

36. $\int \frac{dx}{2\operatorname{ctn} x + 3\csc x}$

75. Integration by Parts. A method of importance and very frequent use is the method of *integration by parts*, embodied in the formula

$$\int u\, dv = uv - \int v\, du. \tag{108}$$

The origin of this formula is the rule for obtaining the differential of a product. Thus, if $u(x)$ and $v(x)$ are two functions of x, then $d(uv) = u\,dv + v\,du$, whence

$$u\,dv = d(uv) - v\,du,$$

and

$$\int u\,dv = \int d(uv) - \int v\,du = uv - \int v\,du.$$

In practice, the method consists in identifying one factor of the integrand as a function $u(x)$, and the complmentary

factor as the differential of some other function, $v(x)$. It is then applied to advantage if, first, the function $v(x)$ is readily recognizable and, second, $v\,du$ is readily integrable or, at least, more so than the original integrand. No particular rule can be stated as to how to break up the integrand into factors u and dv. The student's experience in using this method will train him in applying it properly.

Illustration 1. $\int \log x\,dx$. The factors u and dv of the integrand may be selected as

$$u = \log x, \qquad dv = dx,$$

giving

$$du = \frac{dx}{x}, \qquad v = x.$$

(Note that if $dv = dx$, $v = x + C$, but we are contented with a particular value of v instead of the most general value. We shall, of course, append an arbitrary constant to the integral obtained in the end.) Applying (108), we obtain

$$\int \overbrace{\log x}^{u}\,\overbrace{dx}^{dv} = \overbrace{(\log x)}^{u} \cdot \overbrace{x}^{v} - \int \overbrace{x}^{v} \cdot \overbrace{\frac{dx}{x}}^{du},$$

or

$$\int \log x\,dx = x \log x - x + C.$$

Illustration 2. $\int xe^x\,dx$. Suppose, in this case, we pick out the factors of the integrand as

$$u = e^x, \qquad dv = x\,dx,$$

hence

$$du = e^x\,dx, \qquad v = \frac{x^2}{2}.$$

Applying (108), we obtain

$$\int xe^x\,dx = \frac{x^2}{2}e^x - \int \frac{x^2e^x\,dx}{2},$$

and it is seen that we now have an integral, $\frac{1}{2}\int x^2e^x\,dx$, on our hands which is more complicated than the one originally proposed. The trouble lay here in the choice of u and dv.

Let us now break up the integrand into factors u, dv as

$$u = x, \qquad dv = e^x\,dx,$$

hence

$$du = dx, \qquad v = e^x.$$

The formula now leads to

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The last integral is, evidently, an improvement on the original one. Its value is identified at once as e^x, and the result is

$$\int xe^x\,dx = xe^x - e^x + C.$$

Illustration 3. $\int e^x \cos x\,dx$. Let us choose u and dv as

$$u = e^x, \qquad dv = \cos x\,dx,$$

hence

$$du = e^x\,dx, \qquad v = \sin x.$$

By (108), we get

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The last integral, in turn, lends itself to integration by parts, and we break up the integrand into factors as

$$\begin{cases} u = e^x, \\ du = e^x\,dx, \end{cases} \qquad \begin{cases} dv = \sin x\,dx, \\ v = -\cos x, \end{cases}$$

whence

$$\int e^x \sin x\,dx = -e^x \cos x + \int e^x \cos x\,dx.$$

Substituting this into the preceding equality, we get

$$\begin{aligned} \int e^x \cos x\,dx &= e^x \sin x - (-e^x \cos x + \int e^x \cos x\,dx) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx. \end{aligned}$$

It may seem disconcerting that, after all our labor, we have arrived at an integral that is precisely the one we set out to evaluate in the first place. But we find its value at once by handling the last equality as an equation in one unknown, *viz.*, $\int e^x \cos x\,dx$, obtaining

$$\int e^x \cos x\,dx + \int e^x \cos x\,dx = e^x \sin x + e^x \cos x,$$

or

$$\int e^x \cos x\,dx = \tfrac{1}{2}(e^x \sin x + e^x \cos x).$$

Appending the arbitrary constant that appears in every integration, we finally have

$$\int e^x \cos x\,dx = \frac{e^x}{2}(\sin x + \cos x) + C.$$

Problems

1. Integrate each of the following:

(a) $\displaystyle\int x \log x\,dx$. *Ans.* $\dfrac{x^2}{4}(2\log x - 1) + C$.

(b) $\displaystyle\int x^2 \log x\,dx$. *Ans.* $\dfrac{x^3}{9}(3\log x - 1) + C$.

(c) $\displaystyle\int \sin^{-1} x\,dx$. *Ans.* $x\sin^{-1} x + \sqrt{1 - x^2} + C$.

(d) $\displaystyle\int x \cos 2x\,dx$. *Ans.* $\dfrac{x \sin 2x}{2} + \dfrac{\cos 2x}{4} + C$.

(e) $\int x \sin \frac{x}{2} dx.$ *Ans.* $4 \sin \frac{x}{2} - 2x \cos \frac{x}{2} + C.$

(f) $\int xe^{2x}\, dx.$ *Ans.* $\frac{e^{2x}}{4}(2x - 1) + C.$

(g) $\int x^2 e^{2x}\, dx.$ *Ans.* $\frac{e^{2x}}{4}(2x^2 - 2x + 1) + C.$

(h) $\int x^2 \sin 3x\, dx.$

(i) $\int \tan^{-1} 2x\, dx.$ *Ans.* $x \tan^{-1} 2x - \frac{\log (1 + 4x^2)}{4} + C.$

(j) $\int x \tan^{-1} \frac{x}{2} dx.$ *Ans.* $\frac{1}{2}(x^2 + 4) \tan^{-1} \frac{x}{2} - x + C.$

(k) $\int x^2 \cos \frac{x}{2} dx.$

(l) $\int x^3 \log 3x\, dx.$ *Ans.* $\frac{x^4}{16}(4 \log 3x - 1) + C.$

(m) $\int x^3\sqrt{4 - x^2}\, dx.$ HINT: Let $u = x^2$.

(n) $\int e^{2x} \cos 3x\, dx.$ *Ans.* $\frac{e^{2x}}{13}(2 \cos 3x + 3 \sin 3x) + C.$

(o) $\int \cos x \log \sin x\, dx.$

(p) $\int x \sin^2 3x\, dx.$ *Ans.* $\frac{x^2}{4} - \frac{x \sin 6x}{12} - \frac{\cos 6x}{72} + C.$

(q) $\int \sin ax \cdot \cos bx\, dx.$

(r) $\int \cos ax \cdot \cos bx\, dx.$

Ans. $\frac{a \sin ax \cos bx - b \cos ax \sin bx}{a^2 - b^2} + C.$

(s) $\int \frac{\sin^{-1} \sqrt{2x}}{\sqrt{1 - 2x}} dx.$ *Ans.* $\sqrt{2x} - \sqrt{1 - 2x} \sin^{-1} \sqrt{2x} + C.$

(t) $\int \tan^{-1} \frac{a}{x} dx.$

(u) $\int \log (1 + \sqrt{x}) dx.$

(v) $\int \frac{x\, dx}{\sin^2 2x}.$ *Ans.* $-\frac{x \operatorname{ctn} 2x}{2} + \frac{\log \sin 2x}{4} + C.$

(w) $\int \frac{\log (1 - x^2)}{x^2} dx.$

(x) $\int \log (3 - x^2) dx.$

(y) $\int e^x \sin x\, dx.$ *Ans.* $\frac{e^x}{2}(\sin x - \cos x) + C.$

2. Derive the formula:

$$\int \sin^n \theta\, d\theta = \frac{-\sin^{n-1} \theta \cos \theta}{n} + \frac{n - 1}{n}\int \sin^{n-2} \theta\, d\theta.$$

HINT: Set $u = \sin^{n-1} \theta$, $dv = \sin \theta\, d\theta$.

3. Use the formula of Prob. 2 to integrate

(a) $\int \sin^2 2x\, dx.$ (c) $\int \sin^4 x\, dx.$

(b) $\int \sin^3 x\, dx.$ (d) $\int \sin^5 \frac{x}{2} dx.$

4. In the formula of Prob. 2, replace n by $-m + 2$, and thus derive the new formula

$$\int \frac{d\theta}{\sin^m \theta} = \frac{-\cos \theta}{(m-1)\sin^{m-1}\theta} + \frac{m-2}{m-1}\int \frac{d\theta}{\sin^{m-2}\theta}.$$

5. Use the formula of Prob. 4 to integrate

(a) $\int \frac{dx}{\sin^3 x}.$ (c) $\int \csc^3 3x\, dx.$

(b) $\int \frac{dx}{\sin^4 2x}.$ (d) $\int \csc^5 x\, dx.$

6. Derive the formulas:

$$\int \cos^n \theta\, d\theta = \frac{\cos^{n-1}\theta \sin\theta}{n} + \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta,$$

and

$$\int \frac{d\theta}{\cos^m \theta} = \frac{\sin \theta}{(m-1)\cos^{m-1}\theta} + \frac{m-2}{m-1}\int \frac{d\theta}{\cos^{m-2}\theta}.$$

HINT: See Probs. 2 and 4.

7. By the formulas of Prob. 6, integrate:

(a) $\int \cos^4 \frac{x}{2} dx.$ (c) $\int \sec^4 2x\, dx.$

(b) $\int \cos^5 x\, dx.$ (d) $\int \sec^5 x\, dx.$

76. Integration of Rational Fractions. In treating integrals of the form $\int \frac{u(x)}{v(x)} dx$, where $u(x)$ and $v(x)$ are polynomials in x, we first reduce the integrand, in case it is an improper fraction, to the sum of a polynomial and a proper fraction. (A fraction is called *proper* if the degree of the numerator is less than that of the denominator; otherwise, it is *improper*.) Thus, $\frac{x^2 - x}{x^3 + 1}$ is a proper fraction, while $\frac{x^3 - 2x}{x^3 + 1}$ and $\frac{x^5 + 2x^4 - x^3 + 1}{x^3 + 1}$ are improper. The last two can be changed, by division, to $1 - \frac{2x + 1}{x^3 + 1}$ and $x^2 + 2x - 1 + \frac{-x^2 - 2x + 2}{x^3 + 1}$, respectively, *i.e.*, each to a polynomial plus a proper fraction. Since the integration of the polynomial part is immediate, we need concern ourselves only with the integration of proper fractions.

In turning now to the integration of a proper fraction, we make it our first task to write the integrand as a sum of (proper) *partial* fractions, in conformity with the following principles:

CASE I. *When the denominator of the fraction is a product of distinct real linear factors, as $x - a$, $x - b$, etc., each to the first power*, we set (calling the integrand $\frac{u(x)}{v(x)}dx$)

$$\frac{u(x)}{v(x)} = \frac{u(x)}{(x-a)(x-b)\cdots} = \frac{A}{x-a} + \frac{B}{x-b} + \cdots,$$

where the numerators are unknown constants, and proceed to determine A, B, etc. so that the equation shall be identically true.

For example, to integrate $\int \frac{x^2 + 2x - 1}{x^3 - x^2 - 6x}dx$, note that the denominator factors as $x(x + 2)(x - 3)$. The given proper fraction $\frac{x^2 + 2x - 1}{x(x+2)(x-3)}$ can, evidently, be the sum of proper fractions with only the linear factors x, $x + 2$, and $x - 3$ in their denominators. If each of these is taken as one denominator, the numerators can only be constants. Hence, we write

$$\frac{x^2 + 2x - 1}{x^3 - x^2 - 6x} = \frac{x^2 + 2x - 1}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}. \tag{109}$$

To determine A, B, and C, we clear of fractions and obtain

$$\begin{aligned} x^2 + 2x - 1 &= A(x+2)(x-3) + Bx(x-3) + Cx(x+2) \\ &= A(x^2 - x - 6) + B(x^2 - 3x) + C(x^2 + 2x), \end{aligned} \tag{110}$$

or

$$x^2 + 2x - 1 = (A + B + C)x^2 + (-A - 3B + 2C)x - 6A. \tag{111}$$

Recall that (109), and consequently (111), is intended to be an identity, *i.e.*, the numerators, A, B, and C are to be so determined that these equations are true for every value of x. Now two polynomials, in x, are identical, in that sense, if, and only if, the coefficients of like powers of x are the same in each member. (Reference, any good college algebra.) Hence, from (111)

$$\begin{aligned} A + B + C &= 1, \\ -A - 3B + 2C &= 2, \\ -6A &= -1. \end{aligned}$$

These three equations yield $A = +\frac{1}{6}$, $B = -\frac{1}{10}$ and $C = \frac{14}{15}$. Hence, we have

$$\frac{x^2 + 2x - 1}{x^3 - x^2 - 6x} = \frac{\frac{1}{6}}{x} - \frac{\frac{1}{10}}{x + 2} + \frac{\frac{14}{15}}{x - 3},$$

and

$$\int \frac{x^2 + 2x - 1}{x^3 - x^2 - 6x} dx = \frac{1}{6}\int \frac{1}{x} dx - \frac{1}{10}\int \frac{1}{x + 2} dx + \frac{14}{15}\int \frac{1}{x - 3} dx$$
$$= \tfrac{1}{6} \log x - \tfrac{1}{10} \log (x + 2) + \tfrac{14}{15} \log (x - 3) + C.$$

We may, if we wish to, write the result as

$$\log \sqrt[6]{x} - \log \sqrt[10]{x + 2} + \log \sqrt[15]{(x - 3)^{14}} + \log C,$$

or

$$\log \frac{C\sqrt[6]{x}\sqrt[15]{(x - 3)^{14}}}{\sqrt[10]{x + 2}}.$$

We might, as an alternative, have used the following method for determining A, B, and C. Using (110), which, we noted, must be true for every value of x, we substitute values that meet our convenience. Such values are $x = 0$, $x = -2$, and $x = 3$, since each makes two terms in the right-hand member of (110) vanish. In fact, we obtain,

$$\begin{array}{lrrl} \text{setting } x = & 0, & -1 = -6A, & \text{hence } A = \tfrac{1}{6}, \\ \text{setting } x = & -2, & -1 = 10B, & \text{hence } B = -\tfrac{1}{10}, \\ \text{setting } x = & 3, & 14 = 15C, & \text{hence } C = \tfrac{14}{15}. \end{array}$$

It should not be difficult for the student to satisfy himself that if the degree of $v(x)$ is n (and hence the number of its linear factors, $x - a$, $x - b$, etc., is n), then the number of equations from which we are to obtain the n proposed coefficients A, B, . . . will be precisely n if we follow the method first illustrated [the hypothesis being, all the while, that $u(x)$ is of lower degree than $v(x)$].

CASE II. *When the denominator of the fraction is a product of real linear factors, some (or all) of which occur to a power higher than the first,* we set up, corresponding to the factor, say, $(x - a)^p$, the partial fractions, $\frac{A}{(x - a)^p}$, $\frac{B}{(x - a)^{p-1}}$, . . . , $\frac{L}{x - a}$, and proceed as illustrated in the following example.

To integrate $\int \frac{x^2 - 3x + 1}{(x+2)^3(x-1)}dx$, we write the fraction in the integrand as

$$\frac{x^2 - 3x + 1}{(x+2)^3(x-1)} = \frac{A}{(x+2)^3} + \frac{B}{(x+2)^2} + \frac{C}{x+2} + \frac{D}{x-1}, \tag{112}$$

where A, B, C, and D are to be determined so that (112) is an identity. Clearing of fractions, we have

$$\begin{aligned} x^2 - 3x + 1 &= A(x-1) + B(x+2)(x-1) + \\ &\qquad C(x+2)^2(x-1) + D(x+2)^3, \qquad (113) \\ &= A(x-1) + B(x^2 + x - 2) + \\ &\qquad C(x^3 + 3x^2 - 4) + D(x^3 + 6x^2 + 12x + 8), \end{aligned}$$

or

$$\begin{aligned} x^2 - 3x + 1 = (C + D)x^3 + (B + 3C + 6D)x^2 + \\ (A + B + 12D)x + (-A - 2B - 4C + 8D), \end{aligned} \tag{114}$$

whence (equating coefficients of like powers of x on both sides) we obtain

$$\begin{aligned} C + D &= 0, \\ B + 3C + 6D &= 1, \\ A + B + 12D &= -3, \\ -A - 2B - 4C + 8D &= 1. \end{aligned}$$

These four equations yield $A = -11/3$, $B = 10/9$, $C = 1/27$, $D = -1/27$. We thus have

$$\frac{x^2 - 3x + 1}{(x+2)^3(x-1)} = \frac{-11/3}{(x+2)^3} + \frac{\frac{10}{9}}{(x+2)^2} + \frac{\frac{1}{27}}{x+2} - \frac{\frac{1}{27}}{x-1},$$

and

$$\begin{aligned} \int \frac{x^2 - 3x + 1}{(x+2)^3(x-1)}dx &= \frac{-11}{3}\int \frac{dx}{(x+2)^3} + \frac{10}{9}\int \frac{dx}{(x+2)^2} + \\ &\qquad \frac{1}{27}\int \frac{dx}{x+2} - \frac{1}{27}\int \frac{dx}{x-1} \\ &= \frac{11}{6(x+2)^2} - \frac{10}{9(x+2)} + \frac{1}{27}\log (x+2) \\ &\qquad - \frac{1}{27}\log (x-1) + C. \end{aligned}$$

Once again, we might have substituted values for x at pleasure

in (113), to determine A, B, C, and D. Very convenient values are, evidently, $x = -2$ and $x = 1$, since each makes all but one term of the right-hand member of (113) vanish. Two other values may be chosen, say as $x = 0$ and $x = -1$. By this method we obtain

$$\begin{array}{lll} \text{Setting } x = -2, & 11 = -3A, & \text{hence } A = -11/3 \\ \text{Setting } x = 1, & -1 = 27D, & \text{hence } D = -1/27 \\ \text{Setting } x = 0, & 1 = -A - 2B - 4C + 8D & \\ \text{Setting } x = -1, & 5 = -2A - 2B - 2C + D. & \end{array}$$

Making use of the known values of A and D, the last two equations reduce to

$$2B + 4C = 64/27,$$
$$2B + 2C = 62/27,$$

from which we obtain

$$B = 10/9, \qquad C = 1/27.$$

The student may, again in this case, easily satisfy himself that if the first method is followed, the number of equations obtained for the unknown numerators of the partial fractions will be precisely equal to the number of those unknowns, and equal to the degree of the denominator. HINT: If $v(x)$ is of degree n and $v(x) = (x - a)^p(x - b)^q(x - c)^r \cdots$, then $p + q + r + \cdots = n$.

CASE III. *When the denominator of the fraction contains real quadratic factors of the type $x^2 + px + q$, each such occurring to no higher power than the first,* then, corresponding to such a factor we assume partial fractions of the form $\dfrac{A(2x + p)}{x^2 + px + q}$ and $\dfrac{B}{x^2 + px + q}$, the coefficients, A and B, to be determined by the same plan as in the previous cases.

Example. To integrate $\displaystyle\int \frac{6x^2 + 4x - 3}{x^3 + 2x^2 - x}dx$, set

$$\frac{6x^2 + 4x - 3}{x^3 + 2x^2 - x} = \frac{6x^2 + 4x - 3}{x(x^2 + 2x - 1)} = \frac{A(2x + 2)}{x^2 + 2x - 1} + \frac{B}{x^2 + 2x - 1} + \frac{C}{x}.$$

From this

$$\begin{aligned} 6x^2 + 4x - 3 &= A(2x^2 + 2x) + Bx + C(x^2 + 2x - 1) \\ &= (2A + C)x^2 + (2A + B + 2C)x - C, \end{aligned}$$

whence

$$\begin{aligned} 2A \quad\quad + \ C &= 6, \\ 2A + B + 2C &= 4, \\ -C &= -3, \end{aligned}$$

yielding $A = \frac{3}{2}$, $B = -5$, $C = 3$. Hence

$$\frac{6x^2 + 4x - 3}{x^3 + 2x^2 - x} = \frac{\frac{3}{2}(2x + 2)}{x^2 + 2x - 1} + \frac{-5}{x^2 + 2x - 1} + \frac{3}{x},$$

and

$$\begin{aligned} \int \frac{6x^2 + 4x - 3}{x^3 + 2x^2 - x} dx &= \frac{3}{2} \int \frac{(2x + 2)dx}{x^2 + 2x - 1} - 5 \int \frac{dx}{x^2 + 2x - 1} + 3 \int \frac{dx}{x}. \\ &= \frac{3}{2} \log (x^2 + 2x - 1) - 5 \int \frac{dx}{(x + 1)^2 - 2} + \\ &\qquad 3 \log x + \log C. \\ &= \frac{3}{2} \log (x^2 + 2x - 1) - \frac{5}{2\sqrt{2}} \log \frac{x + 1 - \sqrt{2}}{x + 1 + \sqrt{2}} + \\ &\qquad \log Cx^3. \end{aligned}$$

CASE IV. *When the denominator of the fraction contains real quadratic factors of the type $x^2 + px + q$, some (or all) occurring to powers higher than the first,* we set up, corresponding to a factor, say $(x^2 + px + q)^r$, the partial fractions $\dfrac{A(2x + p) + B}{(x^2 + px + q)^r}$, $\dfrac{C(2x + p) + D}{(x^2 + px + q)^{r-1}}, \ldots, \dfrac{L(2x + p) + M}{x^2 + px + q}$, the unknown coefficients in the numerator to be determined in the same manner as in the previous cases.

For example, to integrate $\displaystyle\int \frac{2x^4 - 7x^3 + 13x^2 - 9x + 4}{x(x^2 - 2x + 2)^2} dx$, set

$$\frac{2x^4 - 7x^3 + 13x^2 - 9x + 4}{x(x^2 - 2x + 2)^2} = \frac{A(2x - 2) + B}{(x^2 - 2x + 2)^2} + \frac{C(2x - 2) + D}{x^2 - 2x + 2} + \frac{E}{x}.$$

Clear of fractions and obtain

$$\begin{aligned} 2x^4 - 7x^3 + 13x^2 - 9x + 4 &= A(2x^2 - 2x) + Bx + \\ &\quad C(2x^4 - 6x^3 + 8x^2 - 4x) + D(x^3 - 2x^2 + 2x) + \\ &\quad E(x^4 - 4x^3 + 8x^2 - 8x + 4). \\ &= (2C + E)x^4 + (-6C + D - 4E)x^3 \\ &\quad + (2A + 8C - 2D + 8E)x^2 + (-2A + B - 4C + 2D - 8E)x \\ &\quad + 4E. \end{aligned}$$

Hence, we have the equations

$$
\begin{aligned}
2C \qquad + \ E &= 2,\\
-6C + \ D - 4E &= -7,\\
2A \qquad + 8C - 2D + 8E &= 13,\\
-2A + B - 4C + 2D - 8E &= -9,\\
4E &= 4,
\end{aligned}
$$

yielding $A = \frac{1}{2}$, $B = 2$, $C = \frac{1}{2}$, $D = 0$, and $E = 1$. Hence

$$\frac{2x^4 - 7x^3 + 13x^2 - 9x + 4}{x(x^2 - 2x + 2)^2} = \frac{\frac{1}{2}(2x - 2) + 2}{(x^2 - 2x + 2)^2} + \frac{\frac{1}{2}(2x - 2)}{x^2 - 2x + 2} + \frac{1}{x},$$

and

$$\int \frac{2x^4 - 7x^3 + 13x^2 - 9x + 4}{x(x^2 - 2x + 2)^2}dx = \frac{1}{2}\int \frac{(2x - 2)dx}{(x^2 - 2x + 2)^2} + 2\int \frac{dx}{(x^2 - 2x + 2)^2} + \frac{1}{2}\int \frac{(2x - 2)dx}{x^2 - 2x + 2} + \int \frac{dx}{x}.$$

$$= -\tfrac{1}{2}(x^2 - 2x + 2)^{-1} + 2\textstyle\int \cos^2 \theta d\theta + \tfrac{1}{2} \log (x^2 - 2x + 2) + \log x + \log C,$$

where we have let $x - 1 = \tan \theta$ in order to integrate the second term. Continuing with the integration, we obtain, as the value of the original integral

$$\frac{-1}{2(x^2 - 2x + 2)} + \int (1 + \cos 2\theta)d\theta + \frac{1}{2} \log (x^2 - 2x + 2) + \log Cx$$

$$= \frac{-1}{2(x^2 - 2x + 2)} + \theta + \frac{1}{2} \sin 2\theta + \frac{1}{2} \log (x^2 - 2x + 2) + \log Cx$$

$$= \frac{-1}{2(x^2 - 2x + 2)} + \tan^{-1} (x - 1) + \frac{x - 1}{x^2 - 2x + 2} + \frac{1}{2} \log (x^2 - 2x + 2) + \log Cx$$

$$= \frac{2x - 3}{2(x^2 - 2x + 2)} + \tan^{-1} (x - 1) + \frac{1}{2} \log (x^2 - 2x + 2) + \log Cx.$$

Problems

1. Integrate each of the following:

(a) $\displaystyle\int \frac{(x+3)dx}{x^2+2x}$. *Ans.* $\frac{3}{2}\log x - \frac{1}{2}\log(x+2) + C$.

(b) $\displaystyle\int \frac{(7x+2)dx}{x^2-x-2}$. *Ans.* $\frac{5}{3}\log(x+1) + \frac{16}{3}\log(x-2) + C$.

(c) $\displaystyle\int \frac{(x^4-3x^3-4x-3)dx}{x^2-3x}$.

(d) $\displaystyle\int \frac{4x^2+9x-27}{x^3-9x}dx$.

Ans. $3\log x + 2\log(x-3) - \log(x+3) + C$.

(e) $\displaystyle\int \frac{(13x-2)dx}{x^3-x^2-2x}$.

(f) $\displaystyle\int \frac{(4\cos\theta + \sin\theta\cos\theta)d\theta}{\sin^2\theta - 2\sin\theta}$.

Ans. $3\log(2-\sin\theta) - 2\log\sin\theta + C$.

2. Integrate each of the following:

(a) $\displaystyle\int \frac{(3x^2+4x+3)dx}{x^3+2x^2+x}$. *Ans.* $3\log x + \frac{2}{x+1} + C$.

(b) $\displaystyle\int \frac{(x-2-3x^2)dx}{x^3-2x^2}$. *Ans.* $-\frac{1}{x} - 3\log(x-2) + C$.

(c) $\displaystyle\int \frac{2x^4-9x^3+12x^2-4x}{x^3-4x^2+4x}dx$. *Ans.* $x^2 - x + C$.

(d) $\displaystyle\int \frac{x^2-18x+27}{x^4-6x^3+9x^2}dx$. *Ans.* $\frac{2}{x-3} - \frac{3}{x} + C$.

(e) $\displaystyle\int \frac{(2x^3-6x^2+11x-2)dx}{x(x-1)^3}$. *Ans.* $2\log x - \frac{5}{2(x-1)^2} + C$.

(f) $\displaystyle\int \frac{(3x^3-4x^2-16x-16)dx}{x^3(x+2)^2}$. *Ans.* $\frac{2}{x^2} - \frac{3}{x+2} + C$.

(g) $\displaystyle\int \frac{4x^3+7x^2+15x+10}{x^4+3x^3+2x^2}dx$.

(h) $\displaystyle\int \frac{2e^{3x}+4e^{2x}-2e^x}{e^{3x}-e^x}dx$. *Ans.* $2\left(x + \log\frac{e^x-1}{e^x+1}\right) + C$.

3. Integrate each of the following:

(a) $\displaystyle\int \frac{x^5+7x^2-4x-2}{x^3+x}dx$.

(b) $\displaystyle\int \frac{(x^3+4x^2-2x+1)dx}{(x^2+x)(x^2-x+1)}$.

Ans. $\log\frac{Cx(x^2-x+1)}{(x+1)^2} + \frac{2}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$.

(c) $\displaystyle\int \frac{9x^3+13x^2+24x+36}{(x^2-3x)(x^2+4)}dx$.

Ans. $12\log(x-3) - 3\log x + 2\tan^{-1}\frac{x}{2} + C$.

(d) $\displaystyle\int \frac{3x^4+13x^3-x^2-6x+12}{(x^3+x^2)(x^2-3x+4)}dx$.

(e) $\displaystyle\int \frac{(4x^3 + 18x)dx}{(x^2 + 3)(x^2 - 2x + 3)}$.

Ans. $2 \log (x^2 - 2x + 3) - \sqrt{3} \tan^{-1} \dfrac{x}{\sqrt{3}} + \dfrac{7}{\sqrt{2}} \tan^{-1} \dfrac{x - 1}{\sqrt{2}} + C.$

(f) $\displaystyle\int \frac{(8x^3 - 17x^2 + 20x + 2)dx}{(x^2 - x + 2)(x^2 - 3x + 3)}$.

(g) $\displaystyle\int \frac{6x^3 + 16x^2 + 22x + 1}{(x + 2)^2(x^2 + 5)} dx$.

Ans. $2 \log [(x + 2)(x^2 + 5)] + \dfrac{3}{x + 2} - \dfrac{1}{\sqrt{5}} \tan^{-1} \dfrac{x}{\sqrt{5}} + C$

(h) $\displaystyle\int \frac{\theta + 1) \sec^2 \theta \, d\theta}{1 + \tan^3 \theta}$

Ans. $\log (\tan \theta + 1) + \dfrac{2}{\sqrt{3}} \tan^{-1} \dfrac{2 \tan \theta - 1}{\sqrt{3}} + C.$

4. Integrate each of the following:

(a) $\displaystyle\int \frac{3x^4 + 12x^2 - 4x + 12}{x^5 + 4x^3 + 4x} dx$.

Ans. $3 \log x - \dfrac{1}{\sqrt{2}} \left(\tan^{-1} \dfrac{x}{\sqrt{2}} + \dfrac{x\sqrt{2}}{2 + x^2}\right) + C.$

(b) $\displaystyle\int \left(\frac{x^2 - 1}{x^2 + 1}\right)^2 dx$. *Ans.* $x - 2 \tan^{-1} x + \dfrac{2x}{1 + x^2} + C.$

(c) $\displaystyle\int \frac{(3x - 2)dx}{(x^2 - 2x + 3)^2}$.

(d) $\displaystyle\int \frac{(x^2 - x)dx}{(x^2 + 1)^3}$.

(e) $\displaystyle\int \frac{x^3 + 2x - 1}{(x^3 - 1)^2} dx$.

(f) $\displaystyle\int \frac{3 \sin^3 \theta - 2 \sin^4 \theta - 16 + 8 \cos^2 \theta}{\sec \theta \sin^2 \theta (3 - \cos^2 \theta)} d\theta$.

77. Miscellaneous Problems. In his later work in the calculus, the student will find it desirable, and at times necessary, to employ a table of integrals, such as "A Short Table of Integrals" by B. O. Peirce, or others, found in the better editions of logarithmic and trigonometric tables. But even when using a table, he will be called upon to employ many of the principles and methods of integration dealt with in this chapter.

We conclude the work on integration by appending a list of problems that embody a review of the work that has been done in the preceding sections.

Miscellaneous Problems

Integrate the following:

1. $\displaystyle\int \frac{x^2 \, dx}{1 + x^6}$. *Ans.* $\frac{1}{3} \tan^{-1} x^3 + C.$

2. $\displaystyle\int \frac{x^3\,dx}{1 - x^8}$. *Ans.* $\dfrac{1}{8}\log\dfrac{x^4+1}{x^4-1} + C.$

3. $\displaystyle\int \frac{dx}{2x - 1}$.

4. $\displaystyle\int \frac{x\,dx}{2x - 1}$.

5. $\displaystyle\int \frac{dx}{2 + 3x - 2x^2}$.

6. $\displaystyle\int \frac{x\,dx}{2 + 3x - 2x^2}$.

7. $\displaystyle\int \frac{x^3 - 2x^2 + \sqrt[3]{x^2} - 1}{\sqrt[3]{x}}dx.$

8. $\displaystyle\int \frac{x^2\,dx}{\sqrt{(4 - 9x^2)^3}}$. *Ans.* $\dfrac{1}{27}\left(\dfrac{3x}{\sqrt{4 - 9x^2}} - \sin^{-1}\dfrac{3x}{2}\right) + C.$

9. $\displaystyle\int \frac{\sqrt{3x - 4}}{x + 2}dx.$

10. $\displaystyle\int \frac{dx}{\sqrt{x}(1 - 2\sqrt[3]{x})}$.

11. $\displaystyle\int \frac{\sqrt{x^2 - 9}}{x^2}dx.$

12. $\displaystyle\int \frac{x^2 + x}{(x - 3)^3}dx.$

13. $\displaystyle\int x^2 \log (x - 1)dx.$

14. $\displaystyle\int x^2\sqrt{4 + x^2}\,dx.$

15. $\displaystyle\int \frac{e^{\tan 2\theta}}{\cos^2 2\theta}d\theta.$

16. $\displaystyle\int \frac{x^3\,dx}{\sqrt{4 - 3x^2}}$.

17. $\displaystyle\int \frac{e^{2x}\,dx}{(e^x + 1)^m}$.

18. $\displaystyle\int \frac{dx}{2 \sin x + 3 \cos x}$.

19. $\displaystyle\int \frac{\log (\log 2x)}{x}dx.$

20. $\displaystyle\int \frac{x^3\,dx}{\sqrt{2x + 3}}$

21. $\displaystyle\int \frac{dx}{e^{2x} - 3e^x}$.

22. $\displaystyle\int \frac{dx}{(x^2 - 2x + 5)^3}$.

23. $\displaystyle\int \frac{dx}{x(1 - x - x^2 + x^3)}$.

24. $\displaystyle\int x \sin^4 (x^2)dx.$

25. $\int \cos\sqrt{x}\,dx.$ *Ans.* $2(\cos\sqrt{x} + \sqrt{x}\sin\sqrt{x}) + C.$

26. $\int \frac{\sqrt{1-\log 3x}}{x}dx.$

27. $\int \cos x \sin 3x\,dx.$

28. $\int \frac{x^2+10}{x^4-3x^2-4}dx.$

29. $\int \frac{\tan^{-1}\sqrt{x}}{x^{3/2}}dx.$

30. $\int \frac{dx}{x^4-x^3+4x^2}.$

31. $\int \frac{\sin^3 x\,dx}{1+2\cos x}.$

32. $\int \sin^3\frac{x}{2}\cos x\,dx.$

33. $\int \frac{dx}{\sin^4 x\cos^4 x}.$

34. $\int x^3(\log x)^2\,dx.$

35. $\int \frac{dx}{\sqrt{2+x-x^2}}.$ *Ans.* $\sin^{-1}\frac{2x-1}{3} + C.$

36. $\int \frac{dx}{1+\cos x}.$ *Ans.* $\csc x - \operatorname{ctn} x + C.$

37. $\int \frac{\sqrt{x^{2/3}+4}}{x^{1/3}}dx.$ *Ans.* $(x^{2/3}+4)^{3/2} + C.$

38. $\int \frac{dx}{2\csc x - \sin x}.$ *Ans.* $-\tan^{-1}(\cos x) + C.$

39. $\int \frac{\tan 2x\,dx}{\cos 2x\sqrt{1+\sec^2 2x}}.$

40. $\int x\tan(4x^2+9)dx.$

41. $\int \sqrt{\frac{1-\sqrt{x}}{x}}dx.$ *Ans.* $-\frac{4}{3}(1-\sqrt{x})^{3/2} + C.$

42. $\int \frac{(3x-1)dx}{\sqrt{2+x-x^2}}.$

43. $\int \frac{e^{\sqrt{3x+1}}}{\sqrt{3x+1}}dx.$

44. $\int x^3\sqrt{x^4+3}\,dx.$

45. $\int \frac{\cos 3x\,dx}{e^{4x}}.$

46. $\int x^5\sqrt{2-x^3}\,dx.$

47. $\int \sqrt{1+\cos^2 2x}\sin 2x\,dx.$

48. $\int \frac{dx}{1 - 4x^2}$.

49. $\int \frac{dx}{2 - \cos x}$.

50. $\int \frac{(1 + \cos x)^2}{\sin x} dx$. *Ans.* $\cos x + 2 \log \sin x + \log \frac{1 - \cos x}{1 + \cos x} + C$.

Integrate the following by the use of a table:

51. $\int \frac{x\, dx}{(3 - 2x)^3}$.

52. $\int \frac{x\, dx}{(3 - x)(2 + x)}$.

3. $\int \frac{dx}{x^2(4 - 3x^2)}$.

54. $\int \frac{x^2\, dx}{4 - x + 5x^2}$.

55. $\int \frac{dx}{x(3 - 2x + x^2)}$.

56. $\int \frac{dx}{x^2\sqrt{1 - x}}$.

57. $\int x^2\sqrt{4x^2 - 9}\, dx$.

58. $\int \frac{x^2\, dx}{(4 - x - x^2)^{3/2}}$.

59. $\int \sqrt{3x - x^2}\, dx$.

60. $\int \cos^3 \frac{x}{2} dx$.

61. $\int \frac{dx}{\sin^5 2x}$.

62. $\int \frac{\theta\, d\theta}{1 - \cos 2\theta}$.

63. $\int \sin 3x \cos^3 x\, dx$.

64. $\int x^3 \cos^{-1} x^2\, dx$.

65. $\int x^3 e^{2x}\, dx$.

66. $\int x^3 (\log x)^3 dx$

67. $\int x^3 \sinh 2x\, dx$.

68. $\int x^2 \sin 3x\, dx$.

69. $\int \frac{\sqrt{3x - x^2}}{x^2} dx$.

70. $\int \frac{dx}{x^2\sqrt{2 - x + x^2}}$.

71. Find the equation of a curve at any point of which the tangent has its slope equal to $2 + x$, where x is the abscissa of the point, and which passes through $(1, -2)$. HINT: $dy/dx = 2 + x$ and $y = -2$ when $x = 1$.

Ans. $y = \frac{x^2}{2} + 2x - \frac{9}{2}$

72. A curve passes through the point (1,0), and the slope of the tangent, at any point (x,y) of it, equals $3 - \frac{1}{x}$. Find its equation.

Ans. $y = 3x - \log x - 3$.

73. Find the equation of the curve which passes through the origin and has $x^2 - x + 1$ as the slope of the tangent to it at any point (x,y).

74. A curve passes through the points (0,2) and $(-1,1)$, and, at any point (x,y) of it, $d^2y/dx^2 = 2 - 4x$. Find its equation.

Ans. $y = -\frac{2}{3}x^3 + x^2 + \frac{8}{3}x + 2$.

75. A curve passes through the points (1,0) and $(e,2)$, and, at any point (x,y) of it, $\frac{d^2y}{dx^2} = 1 + \frac{2}{x}$. Find its equation.

76. A curve is tangent to the line $y = 3x - 4$ at the point (2,2) and, at any point (x,y) of the curve, $d^2y/dx^2 = 2x - x^2$. Find the equation of the curve.

Ans. $y = -\frac{x^4}{12} + \frac{x^3}{3} + \frac{5x}{3} - \frac{8}{3}$.

77. At any point (x,y) of a curve, $d^2y/dx^2 = 1 - x^2$, and the line $y = -x + 2$ touches the curve at $(1,1)$. Find the equation of the curve.

78. A particle moving on a straight line has its velocity, t sec. after the motion started, equal to $3 - t$. At the end of 2 sec. the particle is at the origin. Find the equation of its motion, *i.e.*, find s in terms of t, where s is the directed distance from the origin. *Ans.* $s = 3t - \dfrac{t^2}{2} - 4$.

79. The acceleration of a particle moving on a straight line, equals $t^2 + 2t$ at the time t. At the start of the motion the particle is 1 unit to the right of the origin; 2 sec. later it is 3 units to the left of the origin. Find the equation of its motion. *Ans.* $s = \dfrac{t^4}{12} + \dfrac{t^3}{3} - 4t + 1$.

80. The acceleration of a particle moving on a straight line, at the time t, equals $2t - 3$. At the end of the first second the particle is 2 units to the right of the origin and its velocity is 3 units per second. Find the equation of its motion.

81. The slope of a curve at any point (x,y) of it equals $2 + x^3$. By how much does y change when x changes from 2 to 6?

82. At the time t, the x-component of the velocity of a point equals $t + 2$, and the y-component equals $t - 2$. The particle starts from the point $(1,2)$. Find the equation of its path, with t as a parameter.

83. A particle is projected upward from the ground with an initial velocity of 20 ft./sec. If it is subjected to the acceleration of gravity only, which is 32 ft./sec.^{2} and directed downward, find how long the particle will be going upward, how far up it will go, and how soon it will strike the ground.

84. The rate at which a substance decomposes is proportional to the amount of the substance remaining. If the amount of the substance not decomposed changes from 8 lb. to 1 lb. in 3 hr. How much of the substance remains not decomposed after 1 more hour? *Ans.* $\frac{1}{2}$ lb.

85. Find the equation of the family of curves which are orthogonal, at every point (x,y) of the plane, to the curve of the family of parabolas, $x^2 = 4ay$, passing through that point. Find the particular curve of the new family passing through $(4,1)$. HINT: First establish that the slope, at any point (x,y), of a curve of the given family passing through that point, is $2y/x$. *Ans.* $x^2 + 2y^2 = 18$.

86. Find the equation of the family of curves which are orthogonal, at every point (x,y), to the curve of the family $x^3 + x = ay$ passing through that point. Find the particular curve of the new family passing through $(0, 1)$. *Ans.* $3x^2 + 9y^2 + 2 \log (3x^2 + 1) = 9$.

CHAPTER XI

DEFINITE INTEGRALS

78. Introduction and Definition. Let us perform the following summation. Starting with the function $f(x) = x^2 + 1$, and the interval $0 \leqq x \leqq 1$, on OX, we divide that interval say, into five equal parts, $\Delta_1 x$, $\Delta_2 x$, $\Delta_3 x$, $\Delta_4 x$, and $\Delta_5 x$, so that the points of division have the abscissas 0, 0.2, 0.4, 0.6, 0.8, and 1. Let us now compute the values of the function, $f(x)$, at some point in each partial interval, say, at its midpoint, *i.e.*, at $x_1 = 0.1$, $x_2 = 0.3$, $x_3 = 0.5$, $x_4 = 0.7$, and $x_5 = 0.9$. These values are readily found to be $f(x_1) = 1.01$, $f(x_2) = 1.09$, $f(x_3) = 1.25$, $f(x_4) = 1.49$, and $f(x_5) = 1.81$. We now form the sum

$$\begin{aligned} &f(x_1)\Delta_1 x + f(x_2)\Delta_2 x + f(x_3)\Delta_3 x + f(x_4)\Delta_4 x + f(x_5)\Delta_5 x \\ &\quad = (1.01)(0.2) + (1.09)(0.2) + (1.25)(0.2) + (1.49)(0.2) \\ &\qquad\qquad + (1.81)(0.2) = 1.330. \end{aligned}$$

If we next divide the original interval from 0 to 1 into 10 equal parts, $\Delta_1 x$, $\Delta_2 x$, $\Delta_3 x$, . . . , $\Delta_9 x$, and $\Delta_{10} x$, so that at the points of division the abscissas are 0, 0.1, 0.2, 0.3, . . . , 0.9, and 1, and again compute the value of $f(x)$ at some point in each partial interval, say, at the midpoints, where $x_1 = 0.05$, $x_2 = 0.15$, $x_3 = 0.25$, . . . , $x_9 = 0.85$, and $x_{10} = 0.95$, we obtain $f(x_1) = 1.0025$, $f(x_2) = 1.0225$, $f(x_3) = 1.0625$, $f(x_4) = 1.1225$, $f(x_5) = 1.2025$, $f(x_6) = 1.3025$, $f(x_7) = 1.4225$, $f(x_8) = 1.5625$, $f(x_9) = 1.7225$, and $f(x_{10}) = 1.9025$. Forming the same kind of sum as before, we obtain

$$\begin{aligned} &f(x_1)\Delta_1 x + f(x_2)\Delta_2 x + f(x_3)\Delta_3 x + \cdots + f(x_9)\Delta_9 x + \\ &\quad f(x_{10})\Delta_{10} x = (1.0025)(0.1) + (1.0225)(0.1) + (1.0625)(.01) + \\ &\qquad\qquad \cdots + (1.7225)(0.1) + (1.9025)(0.1) = 1.3325 \end{aligned}$$

Exercise 1. Subdivide the interval from 0 to 1 into 20 equal parts, $\Delta_1 x$, $\Delta_2 x$, $\Delta_3 x$, . . . , $\Delta_{19} x$, and $\Delta_{20} x$, compute the value of $f(x) = x^2 + 1$ at the midpoints x_1, x_2, x_3, . . . , x_{19}, x_{20} of the partial intervals, and evaluate

the sum

$$f(x_1)\Delta_1 x + f(x_2)\Delta_2 x + f(x_3)\Delta_3 x + \cdots + f(x_{19})\Delta_{19} x + f(x_{20})\Delta_{20} x.$$

NOTE: A table of squares will be a help in computing the values requested.

Ans. 1.333125.

The successive sums that we found in the above processes, *viz.*, 1.330, 1.3325, and 1.333125, will suggest the idea to the student that, as we divide the given interval into more and more parts and carry out the above summation process each time, the values of this sum approach a certain number as a limit. He will perhaps even guess that the limit is $\frac{4}{3}$, or 1.33333 . . . .

A glance at Fig. 126, where the graph of the function $x^2 + 1$ is shown and the ordinates $f(x_1)$, $f(x_2)$, $f(x_3)$, $f(x_4)$, and $f(x_5)$ are drawn (for the case of division of the interval into five parts), will help the student to interpret the first sum obtained, *viz.*, 1.330, as the sum of the areas of the five rectangles shown. Likewise, in the next two summations the results would represent the sum of the areas of 10 and 20 rectangles, respectively, situated between the lines $x = 0$ and $x = 1$, their bases lying in the x-axis and equal to the lengths of the partial intervals, and having as altitudes the values of $f(x_1)$, $f(x_2)$, etc.

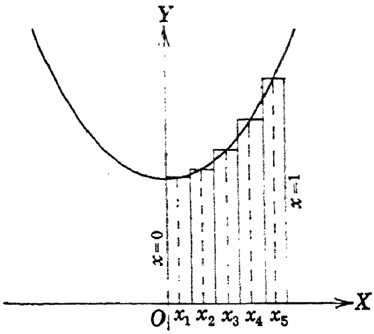

FIG. 126.

As the bases of these rectangles are made smaller and smaller and their number correspondingly larger and larger, it is tolerably clear that the sum of their areas approaches a limit, *viz.*, the area bounded by OX, the curve $y = x^2 + 1$, and the two lines $x = 0$ and $x = 1$.

One more point is worth noting in this example. The student will readily perceive that the successive division of the interval $0 \leqq x \leqq 1$ into *equal* parts was immaterial, as was also the choice of $x_1, x_2, x_3, \ldots$ at the *midpoints* of the partial intervals. In other words, should we divide the interval into a number of parts, $\Delta_1 x, \Delta_2 x, \Delta_3 x, \ldots$, *equal* or *unequal*, and choose x_1 *anywhere* in the partial interval $\Delta_1 x$, x_2 *anywhere* in the partial interval $\Delta_2 x$, x_3 *anywhere* in the partial interval $\Delta_3 x$, etc., and perform the

summation indicated above, then, as the number of subintervals is made larger and larger, while their lengths simultaneously become smaller and smaller, the sums obtained will approach a limit, *viz.*, the number measuring the area bounded by OX, the curve $y = x^2 + 1$, the line $x = 0$, and the line $x = 1$.

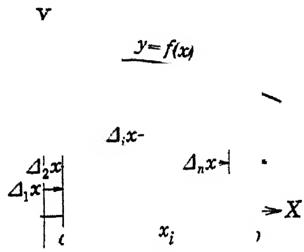

FIG. 127.

We now turn to any function $f(x)$, *continuous in the interval* $a \leqq x \leqq b$. Once more we divide the interval into a certain number, say n, parts of lengths $\Delta_1 x, \Delta_2 x, \ldots, \Delta_{n-1} x, \Delta_n x$ and choose an abscissa at random in each subinterval. Designate the abscissa chosen in the subinterval $\Delta_i x$ by x_i for $i = 1, 2, \ldots, n$, and form the sum

$$f(x_1)\Delta_1 x + f(x_2)\Delta_2 x + f(x_3)\Delta_3 x + \cdots + f(x_{n-1})\Delta_{n-1} x + f(x_n)\Delta_n x. \quad (115)$$

If, now, $F(x)$ is any value of $\int f(x)dx$, *i.e.*, any function such that $F'(x) = f(x)$, we have, by the theorem of the mean, page 220,

$$F(u_1) - F(a) = f(\xi_1)\Delta_1 x \qquad (a < \xi_1 < u_1) \quad (\text{where } u_1 = a + \Delta_1 x),$$
$$F(u_2) - F(u_1) = f(\xi_2)\Delta_2 x \qquad (u_1 < \xi_2 < u_2) \quad (\text{where } u_2 = u_1 + \Delta_2 x),$$
$$F(u_3) - F(u_2) = f(\xi_3)\Delta_3 x \qquad (u_2 < \xi_3 < u_3) \quad (\text{where } u_3 = u_2 + \Delta_3 x),$$

$$F(u_{n-1}) - F(u_{n-2}) = f(\xi_{n-1})\Delta_{n-1} x \qquad (u_{n-2} < \xi_{n-1} < u_{n-1}) \quad (\text{where } u_{n-1} = u_{n-2} + \Delta_{n-1} x),$$
$$F(b) - F(u_{n-1}) = f(\xi_n)\Delta_n x \qquad (u_{n-1} < \xi_n < b) \quad (\text{where } b = u_{n-1} + \Delta_n x).$$

Adding these n equalities, we obtain

$$F(b) - F(a) = f(\xi_1)\Delta_1 x + f(\xi_2)\Delta_2 x + f(\xi_3)\Delta_3 x + \cdots + f(\xi_{n-1})\Delta_{n-1} x + f(\xi_n)\Delta_n x. \quad (116)$$

Note that the left-hand member of (116), *viz.*, $F(b) - F(a)$, does not depend on the value of n, the number of subdivisions into which the interval from a to b is divided. Hence the sum on the right-hand member will continue to equal $F(b) - F(a)$ regard-

less of what values n is made to assume; in particular, if we continue increasing n beyond bound. In other words,

$$\lim_{\substack{n \to \infty \\ \Delta_i x \to 0^*}} [f(\xi_1)\Delta_1 x + f(\xi_2)\Delta_2 x + f(\xi_3)\Delta_3 x + \cdots + f(\xi_n)\Delta_n x] = F(b) - F(a)$$

In the integral calculus and its applications, a great deal of use is made of the limit of the sum (115), as n increases beyond bound while all the $\Delta_i x$ are made to approach zero, provided that limit exists and is independent of the choice of the x_i in each interval. The scope of this book precludes the possibility of arguing out under precisely what conditions the existence of that limit and its independence of the choice mentioned are actually assured. Here we must content ourselves with the assertion that if $f(x)$ is continuous in the interval $a \leqq x \leqq b$, the limit does indeed exist and its value is independent of the way the x_i are chosen in the respective intervals.

Now, if we choose, as the value of x_i in each interval, the ξ_i of the mean value theorem, employed above, the value of the limit, as displayed in (116), is $F(b) - F(a)$. This, then, is the value, in view of the assertion just made, of the limit in question, no matter how the x_i are chosen in the partial intervals.

The substance of the argument so far carried out may be summed up in the so-called

Fundamental Theorem of Integral Calculus. *Given a function $f(x)$ continuous in the interval $a \leqq x \leqq b$, if we divide that interval*

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* The symbol $\Delta_i x$ is intended to represent any one of $\Delta_1 x, \Delta_2 x, \Delta_3 x, \ldots$. We may, likewise, represent any one of the numbers $x_1, x_2, x_3, \ldots$ by x_i. With this agreement, we shall represent the sum (115) by the symbol

$$\sum_{i=1}^{n} f(x_i)\Delta_i x.$$

Likewise, the sum in the right-hand member of (116) can be represented by

$$\sum_{i=1}^{n} f(\xi_i)\Delta_i x.$$

The symbol Σ represents summation, and the notation put on it in the above cases is intended to indicate that i ranges from 1 to n.

into n parts, $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$, *and compute the value of* $f(x_i)$ *for some value* x_i *in each partial interval* $\Delta_i x$ *and form the sum*

$$\sum_{i=1}^{n} f(x_i)\Delta_i x = f(x_1)\Delta_1 x + f(x_2)\Delta_2 x + \cdots + f(x_{n-1})\Delta_{n-1} x + f(x_n)\Delta_n x,$$

then the value of that sum, as we increase n beyond bound and simultaneously cause all $\Delta_i x$ *to approach zero, approaches a limit.*

Furthermore, if $F(x)$ *is any function whose derivative is* $f(x)$, *then the limit of the above sum is* $F(b) - F(a)$.

The limit involved in the above theorem is called the *definite integral of* $f(x)$ *from* $x = a$ *to* $x = b$ and is represented by the symbol

$$\int_a^b f(x)dx.$$

We shall call a the *lower limit* of integration and b the *upper limit.* From the figures above it should be clear that in case $f(x)$ is positive or zero throughout the interval $a \leqq x \leqq b$, $\int_a^b f(x)dx$ represents the area bounded by the x-axis, the curve $y = f(x)$, the line $x = a$, and the line $x = b$.

The foregoing theorem is based on the assumption that the lower limit, a, is less than the upper limit, b. The cases of $a > b$ and $a = b$ are cared for by the following definitions.

Definition I. If $a > b$, then $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

Definition II. $\int_a^a f(x)dx = 0.$

If we also make use of the customary notation $F(x)\Big]_a^b$ for $F(b) - F(a)$, we may express the second part of the theorem by the equality

$$\lim_{\substack{n\to\infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} f(x_i)\Delta_i x = \int_a^b f(x)dx = F(x)\Big]_a^b = F(b) - F(a),$$

where

$$\sum_{i=1}^{n} \Delta_i x = b - a, \qquad \text{and} \qquad F(x) = \int f(x)dx.$$

Illustration. Let $f(x) = x$, $a = 1$, and $b = 2$. Then if

$$\Delta_1 x + \Delta_2 x + \cdots + \Delta_n x = 1,$$

$$\lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} f(x_i)\Delta_i x = \int_a^b f(x)dx = \int_1^2 x\,dx = \frac{x^2}{2}\Big]_1^2 = \frac{4}{2} - \frac{1}{2} = \frac{3}{2}.$$

We may, in this case, obtain directly the area bounded by the x-axis, the curve $y = f(x) = x$, and the ordinates $x = 1$ and $x = 2$, since it is a trapezoid with parallel sides equal to 1 and 2, and 1 unit apart. The area is thus obtained as $\frac{1}{2}(1)(1 + 2) = \frac{3}{2}$.

Exercise 2. Prove that $\{F(x) + C\}\Big]_a^b = F(x)\Big]_a^b$, and hence that $\int_a^b f(x)dx$ is independent of the choice of the constant of integration.

Problems

1. State the value of

$$\lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} [x_1^2\Delta_1 x + x_2^2\Delta_2 x + x_3^2\Delta_3 x + \cdots + x_n^2\Delta_n x] = \lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} x_i^2\Delta_i x,$$

where $\Delta_1 x, \Delta_2 x, \Delta_3 x, \ldots, \Delta_{n-1}x, \Delta_n x$ are subdivisions of the interval from $x = 1$ to $x = 3$ and $x_1, x_2, x_3, \ldots, x_{n-1}, x_n$ are abscissas chosen in the corresponding partial intervals. *Ans.* $\frac{26}{3}$.

2. State the value of

$$\lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} [\cos x_1\Delta_1 x + \cos x_2\Delta_2 x + \cdots + \cos x_n\Delta_n x] = \lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} \cos x_i\Delta_i x,$$

where $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$ are a set of subdivisions of the interval from $x = 0$ to $x = \pi/2$ and x_i, for each value of i, is an abscissa chosen in the subinterval $\Delta_1 x$. *Ans.* 1.

3. State the value of

$$\lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} [(\sqrt{x_1} + 1)\Delta_1 x + (\sqrt{x_2} + 1)\Delta_2 x + \cdots + (\sqrt{x_n} + 1)\Delta_n x] =$$

$$\lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} (\sqrt{x_i} + 1)\Delta_i x,$$

where the $\Delta_i x$ are subintervals of the interval from $x = 1$ to $x = 4$ and x^i is an abscissa in the subinterval $\Delta_i x$ for each value of i from 1 to n.

Ans. $\frac{23}{3}$.

4. Compute the definite integrals:

(a) $\int_1^e \frac{1}{x}dx$. *Ans.* 1. (c) $\int_{\frac{\pi}{2}}^{\pi} \sin x\,dx$. *Ans.* 1.

(b) $\int_0^2 (x^3 - x + 1)dx$. (d) $\int_0^1 e^{3x}\,dx$.

(e) $\displaystyle\int_1^3 \frac{x\,dx}{\sqrt{3+x^2}}$.

(f) $\displaystyle\int_1^2 \frac{x\,dx}{2+x^2}$. *Ans.* $\frac{1}{2}\log 2$.

(g) $\displaystyle\int_0^4 \sqrt{1+2x}\,dx$.

(h) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sin^2 x\,dx$.

(i) $\displaystyle\int_0^{\frac{\pi}{4}} \cos^3 2x\,dx$.

(j) $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{dx}{\sin x \cos x}$.

(k) $\displaystyle\int_0^a \sqrt{a^2-x^2}\,dx$.

(l) $\displaystyle\int_0^{\frac{\pi}{8}} \tan^2 2x\,dx$. *Ans.* $\frac{4-\pi}{8}$.

(m) $\displaystyle\int_1^2 \frac{dx}{\sqrt{2x-x^2}}$. *Ans.* $\pi/2$

(n) $\displaystyle\int_{\frac{1}{2}}^1 \sqrt{\frac{1-x}{x}}\,dx$. *Ans.* $\frac{\pi-2}{4}$.

(o) $\displaystyle\int_0^{\frac{\pi}{4}} x \sin 2x\,dx$.

(p) $\displaystyle\int_0^{\frac{\pi}{12}} \frac{dx}{\cos^4 3x}$.

5. Take $f(x) = 1$ and verify that $\int_a^b dx = b - a$, by considering

$$\lim_{\substack{n\to\infty\\ \Delta_i x\to 0}} [1\cdot\Delta_1 x + 1\cdot\Delta_2 x + 1\cdot\Delta_3 x + \cdots + 1\cdot\Delta_{n-1}x + 1\cdot\Delta_n x],$$

where $\Delta_1 x, \Delta_2 x, \Delta_3 x, \ldots, \Delta_n x$ are a subdivision of the interval from $x = a$ to $x = b$.

6. Take $f(x) = x$ and verify that $\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}$ by considering

$$\lim_{\substack{n\to\infty\\ \Delta_i x\to 0}} [x_1\Delta_1 x + x_2\Delta_2 x + x_3\Delta_3 x + \cdots + x_n\Delta_n x],$$

where $\Delta_1 x, \Delta_2 x, \Delta_3 x, \ldots, \Delta_n x$ are subdivisions of the interval from $x = a$ to $x = b$.

HINT: Take $\Delta_1 x = \Delta_2 x = \Delta_3 x = \cdots \Delta_n x = \frac{b-a}{n} = \Delta x$, and also $x_1 = a$, $x_2 = a + \Delta x$, $x_3 = a + 2\Delta x$, $\cdots$, $x_n = a + (n-1)\Delta x$. Recall that $1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2}$.

7. If $f(-x) = -f(x)$ (a function having this property is called an *odd* function; *e.g.*, $\sin x$ is such a function, as is also x^3), prove $\int_{-a}^a f(x)dx = 0$.

79. Properties of Definite Integrals. We now quote some important properties of definite integrals as exercises for the student to prove.

Exercise 1. Prove that $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx$.

Exercise 2. Prove that if $f(x)$ is positive in the interval $a \leqq x \leqq b$, then $\int_a^b f(x)dx$ is a positive number, and if $f(x)$ is negative in the interval, then $\int_a^b f(x)dx$ is a negative number.

Exercise 3. Prove that $\int_a^b f(x)dx = (b - a)f(\xi)$, where ξ is some number satisfying $a \leqq \xi \leqq b$. (Theorem of the mean for integration.)

HINT: Divide the interval from a to b into n equal parts and choose some abscissa x_i in each. Then

$$\frac{f(x_1) + f(x_2) + \cdot \cdot \cdot + f(x_{n-1}) + f(x_n)}{n} = f(z),$$

where $a \leqq z \leqq b$. This z will vary with the choice of n and the choice of the x_i but will always satisfy the above inequality as the student may see by employing Property 1, page 161, for continuous functions. Now multiply the numerator and denominator of the fraction on the left-hand side of the equation by Δx and let $n \to \infty$.

NOTE: The value $f(\xi)$ in this exercise is called the *mean value* of the function in the interval $a \leqq x \leqq b$.

Problems

1. Find the mean value of the function, and the corresponding value of the independent variable, in each of the following:

(*a*) $f(x) = x^2$, in the interval $1 \leqq x \leqq 3$. *Ans.* $^{13}\!/_3$;

(*b*) $f(x) = \sin 2x$, in the interval $0 \leqq x \leqq \pi/6$. *Ans.* $3/2\pi$.

(*c*) $f(x) = \log x$, in the interval $1 \leqq x \leqq e^2$.

(*d*) $f(x) = \sin^2 x$, in the interval $0 \leqq x \leqq \pi$.

2. Find the mean value of the ordinate y of the part of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

lying in the first quadrant,

(*a*) When y is expressed as a function of the abscissa, x.

(*b*) When y is expressed as a function of the eccentric angle θ, as $y = b \sin \theta$. *Ans.* (*a*) $\pi b/4$; (*b*) $2b/\pi$.

3. Find the mean value of the ordinate of the parabola $x = 2t - 1$, $y = t^2 + 3$ in the interval between the vertex and the y-axis,

(*a*) When the ordinate is expressed as a function of the abscissa. *Ans.* $^{37}\!/_{12}$.

(*b*) When the ordinate is expressed as a function of the parameter. *Ans.* $^{37}\!/_{12}$.

4. Find the mean value of the abscissa of the parabola of Prob. 3 in the interval given there,

(*a*) When it is expressed as a function of the ordinate.

(*b*) When it is expressed as a function of the parameter.

5. For the part of the curve $r = 2\cos\theta$ in the first quadrant,

(a) Find the mean value of r (expressed as a function of θ).

(b) Find the mean value of θ (expressed as a function of r).

6. Find the mean density of a rod of length 2 units if the density at a point of distance x from one end is $1 + x^2$. At what point of the rod does the density attain its mean value? *Ans.* $7/3$; $2/\sqrt{3}$ units from end.

7. Find the mean value of y for the upper part of the circle $x^2 + y^2 = a^2$ [y expressed as a function of s, the length of arc from $(a,0)$ to (x,y)].

80. Improper Integrals. In our definition of the definite integral $\int_a^b f(x)dx$, a and b were understood to be finite numbers and the function $f(x)$ was understood to be continuous, and hence finite, in the interval $a \leqq x \leqq b$. If either of these conditions is lacking, *i.e.*, if the limits a and b are not both finite or if the integrand function $f(x)$ becomes infinite anywhere in the interval, the definite integral $\int_a^b f(x)dx$ is said to be *improper*. The question that arises in connection with an improper integral is whether it has a numerical value at all, in other words, whether the limit, in terms of which the definite integral was defined, exists. To decide that question and to find the value of the integral when it does exist, we define an improper integral as the limit of a proper integral, as follows:

1. $\int_a^\infty f(x)dx$ is defined as $\lim\limits_{u\to\infty} \int_a^u f(x)dx$.

2. $\int_{-\infty}^b f(x)dx$ is defined as $\lim\limits_{u\to-\infty} \int_u^b f(x)dx$.

3. If the integrand function becomes infinite at the lower limit a, we define

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0} \int_{a+\epsilon}^b f(x)dx.$$

4. If the integrand function becomes infinite at the upper limit b, we define

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0} \int_a^{b-\epsilon} f(x)dx.$$

5. If the integrand function becomes infinite for a value c intermediate to a and b, we define

$$\int_a^b f(x)dx = \lim_{\epsilon\to 0} \int_a^{c-\epsilon} f(x)dx + \lim_{\epsilon\to 0} \int_{c+\epsilon}^b f(x)dx.$$

If the limit, in any of these cases, exists, that limit is said to be the value of the definite integral.

Illustration 1. $\int_0^\infty \frac{dx}{(x+1)^2}$. By 1, above, we define this integral as

$$\lim_{u\to\infty}\int_0^u \frac{dx}{(x+1)^2} \quad \lim_{u\to\infty}\left[\frac{-1}{x+1}\right]_0^u = \lim_{u\to\infty}\left(1 - \frac{1}{u+1}\right) = 1.$$

Illustration 2. $\int_{-1}^{1}\frac{dx}{\sqrt[3]{x^4}}$. Here the integrand function becomes infinite at $x = 0$. By 5, above, we define it as

$$\lim_{\epsilon\to 0}\int_{-1}^{0-\epsilon}\frac{dx}{\sqrt[3]{x^4}} + \lim_{\epsilon\to 0}\int_{0+\epsilon}^{1}\frac{dx}{\sqrt[3]{x^4}} = \lim_{\epsilon\to 0}\left[\frac{-3}{\sqrt[3]{x}}\right]_{-1}^{-\epsilon} + \lim_{\epsilon\to 0}\left[\frac{-3}{\sqrt[3]{x}}\right]_{\epsilon}^{1}$$
$$= \lim_{\epsilon\to 0}\left[\frac{3}{\sqrt[3]{\epsilon}} - 3\right] + \lim_{\epsilon\to 0}\left[-3 + \frac{3}{\sqrt[3]{\epsilon}}\right],$$

and both limits are infinite. This integral is, hence, undefined.

Note that if we took no account of the discontinuity of the integrand and performed formal steps uncritically, we would get the false result

$$\int_{-1}^{1}\frac{dx}{\sqrt[3]{x^4}} = \frac{-3}{\sqrt[3]{x}}\Bigg]_{-1}^{1} = \frac{3}{\sqrt[3]{x}}\Bigg]_{1}^{-1} = -3 - 3 = -6.$$

Problems

1. In each of the following cases find whether the given integral is defined, and if so, find its value.

(a) $\int_2^\infty \frac{dx}{x^4}$. *Ans.* $\frac{1}{24}$.

(b) $\int_{-\infty}^{1} \frac{x\,dx}{(4+x^2)^{3/2}}$.

(c) $\int_0^2 \frac{dx}{\sqrt{2-x}}$. *Ans.* $2\sqrt{2}$.

(d) $\int_2^4 \frac{x\,dx}{\sqrt{x-2}}$. *Ans.* $16\sqrt{2}/3$.

(e) $\int_1^\infty \frac{dx}{1+x^2}$. *Ans.* $\pi/4$.

(f) $\int_0^2 \frac{dx}{\sqrt{4-x^2}}$.

(g) $\int_{-\infty}^{-1} \frac{dx}{x^2}$. *Ans.* 1.

(h) $\int_1^5 \frac{x\,dx}{\sqrt{x^2-1}}$. *Ans.* $\sqrt{24}$.

(i) $\int_0^e \frac{dx}{x}$.

(j) $\int_0^9 \frac{dx}{\sqrt[3]{x-1}}$. *Ans.* $\frac{9}{2}$.

(k) $\int_\infty^0 e^x\,dx$.

(l) $\int_0^2 \frac{x\,dx}{1-x}$.

(m) $\int_{-\infty}^0 \frac{dx}{x^4}$.

(n) $\int_{-\infty}^\infty \frac{dx}{x^2+4x+5}$.

(o) $\int_0^\infty \sin 2x\,dx$.

(p) $\int_2^\infty \frac{dx}{x^2(x-1)}$.

2. Show that $\int_a^b \frac{dx}{(b-x)^n}$ exists if, and only if, $n < 1$. Find also the conditions for the existence of $\int_a^b \frac{dx}{(x-a)^n}$.

81a. Areas in Rectangular Coordinates. Let it be required to find the area bounded by two curves $y = f(x)$ and $y = g(x)$ where $f(x) \geqq g(x)$, $a \leqq x \leqq b$ (Fig. 128), the area extending from the line $x = a$ to the line $x = b$ and so situated that any line, between $x = a$ and $x = b$ and parallel to the y-axis, meets each of the bounding curves once.

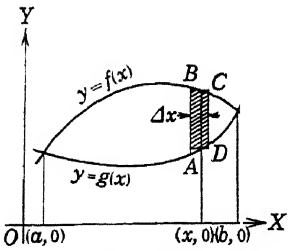

Fig. 128.

If we divide the interval $a \leqq x \leqq b$ into any number, say n, parts and erect ordinates at the points of division, those ordinates will divide the area in question into an equal number of parts. One such part, whose boundary is AB, arc BC, CD, and arc DA, is shown in Fig. 128. If we choose an abscissa x, intermediate to the abscissa of A and D (or, for that matter, equal to either of them) and draw the corresponding ordinate, a rectangle is formed, of area

$$F(x)\Delta x,$$

where $F(x) = f(x) - g(x)$ and Δx is the length of the particular subinterval. The area of this rectangle is called an *element* of area. It is not identical with the area $ABCD$, and hence the sum of all n of such rectangles, *i.e.*,

$$\sum_{i=1}^{n} F(x_i)\Delta_i x,$$

is not identical with the area sought. But it is clear that as we increase the number n of subdivisions indefinitely, while the values of Δx simultaneously approach zero, the sum of the n elements of area approaches, as a limit, the area in question. In view of the fundamental theorem of integral calculus (Sec. 78), we have, then, as a measure of this area,

$$\lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} F(x_i)\Delta_i x = \int_a^b F(x)dx,$$

where $F(x) = f(x) - g(x)$.

Exercise 1. Show that the area (Fig. 129) bounded by the two curves $x = f(y)$ and $x = g(y)$, extending from $y = a$ to $y = b$ and so situated that

every line through it and parallel to OX meets each of the bounding curves once, is measured by

$$\lim_{\substack{n\to\infty \\ \Delta_i y\to 0}} \sum_{i=1}^{n} F(y_i)\Delta_i y = \int_a^b F(y)dy,$$

where $F(y) = f(y) - g(y)$.

Illustration 1. Find the area bounded by the two curves $y = x^2 + 13$ and $y = 2x^2 + 4$.

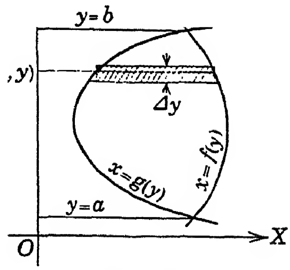

FIG. 129.

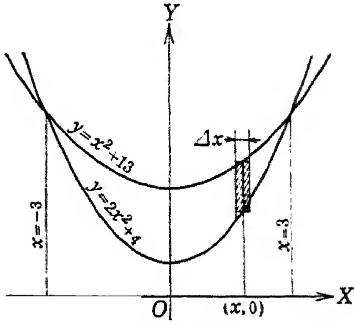

FIG. 130.

Solution. To find the points of intersection, we solve the two equations simultaneously, obtaining $x = -3$ and $x = 3$. These values are the limits of integration. The element of area is a rectangle of dimensions Δx and $F(x) = f(x) - g(x) = (x^2 + 13) - (2x^2 + 4) = 9 - x^2$. The area requested is, then, given by

$$\lim_{\substack{n\to\infty \\ \Delta_i x\to 0}} \sum_{i=1}^{n} (9 - x_i^2)\Delta_i x = \int_{-3}^{3} (9 - x^2)dx = \left[9x - \frac{x^3}{3}\right]_{-3}^{3} = 18 - (-18) = 36.$$

Note that, owing to the symmetry of this area with respect to OY, we might have computed the right-hand half of the area, by integrating from $x = 0$ to $x = 3$, and doubled the result. In other words we might have represented the area by

$$2\int_0^3 (9 - x^2)dx = 2\left[9x - \frac{x^3}{3}\right]_0^3 = 2(18 - 0) = 36.$$

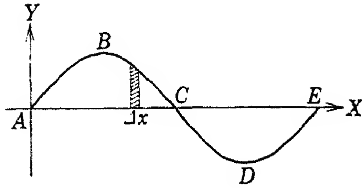

FIG. 131.

Illustration 2. Find the area bounded by the curve $y = \sin x$ and the x-axis in the interval from $x = 0$ to $x = 2\pi$.

Solution. The element of area is evidently, $(\sin x - 0)\,\Delta x = \sin x\Delta x$. Inasmuch as the area extends from $x = 0$ to $x = 2\pi$, we might be inclined to compute the area as

$$\int_0^{2\pi} \sin x\,dx = -\cos x\Big]_0^{2\pi} = \cos x\Big]_{2\pi}^{0} = 1 - 1 = 0.$$

The result is evidently absurd, the area in question certainly having a value different from zero. The fault lies in overlooking the fact that the element of area, as used, *viz.*, $\sin x\,\Delta x$, is positive in the interval from $x = 0$ to $x = \pi$ and negative in the interval from $x = \pi$ to $x = 2\pi$. The integral $\int_0^{\pi} \sin x\,dx$ has, then, a positive value and $\int_{\pi}^{2\pi} \sin x\,dx$ has a negative value (Exercise 2, Sec. 79). The first represents the area ABC, and the second represents the negative of the area CDE.
Now,

$$\int_0^{\pi} \sin x\,dx = -\cos x\Big]_0^{\pi} = \cos x\Big]_{\pi}^{0} = 1 - (-1) = 2,$$

and

$$\int_{\pi}^{2\pi} \sin x\,dx = -\cos x\Big]_{\pi}^{2\pi} = \cos x\Big]_{2\pi}^{\pi} = -1 - 1 = -2.$$

Each area is, therefore, equal to 2 and the total area is 4 units. We might have noted that the area of CDE is the same as that of ABC, and computed the entire area as

$$2\int_0^{\pi} \sin x\,dx = -2\cos x\Big]_0^{\pi} = 2\cos x\Big]_{\pi}^{0} = 2 - (-2) = 4.$$

Illustration 3. Find the area bounded by the parabola $y^2 + 20x = 96$ and the two lines $y = x$ and $y = 2x$, and lying in the first quadrant.

Solution. Solving the pairs of equations

$$\begin{cases} y^2 + 20x = 96, \\ y = x, \end{cases}$$

and

$$\begin{cases} y^2 + 20x = 96, \\ y = 2x, \end{cases}$$

simultaneously, we find (4,4) and (3,6) as the points in the first quadrant in which the two lines intersect the parabola. We pass a line through the point C (4,4) parallel to OX, to divide the required area into two portions. Portion OAC is bounded by the lines $y = x$, $y = 2x$ and $y = 4$. Portion ABC is bounded by the lines $y = 2x$, $y = 4$ and the parabola $y^2 + 20x = 96$. In the latter portion the element of area is a rectangle of dimensions Δy and $f(y) - g(y) = \dfrac{96 - y^2}{20} - \dfrac{y}{2}$, and hence

$$\begin{aligned}\text{Area } ABC &= \int_4^6 \left(\frac{96 - y^2}{20} - \frac{y}{2}\right) dy = \int_4^6 \frac{96 - y^2 - 10y}{20}\,dy \\ &= \frac{1}{20}\left(96y - \frac{y^3}{3} - 5y^2\right)\Big]_4^6 = \frac{1}{20}\left(324 - \frac{848}{3}\right) \\ &= {}^{124}\!/_{60} = 2^1\!/_{15}.\end{aligned}$$

The element of area in OAC is a rectangle of dimensions Δy and

$$f(y) - g(y) = y - \frac{y}{2} = \frac{y}{2},$$

and, hence

$$\text{Area } OAC = \int_0^4 \frac{y}{2}dy = \frac{y^2}{4}\Big]_0^4 = 4.$$

The total area OBC is, then, equal to $6\frac{1}{15}$ units.

Exercise 2. Draw a line through the point (3,6) of Fig. 132 parallel to OY and compute the area of Illustration 3 in two parts, using the formula

$$\int_a^b F(x)dx.$$

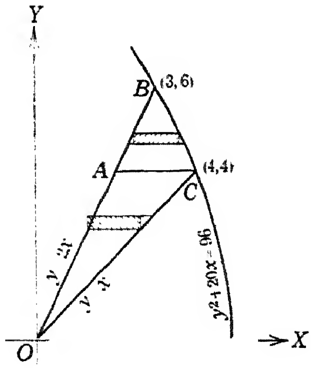

FIG. 132.

Problems

1. Compute the areas bounded as follows:

(a) By the curve $x^2 + 4y = 36$ and the x-axis. *Ans.* 72.

(b) By the curve $x^2 + y + 5 = 6x$ and the x-axis. *Ans.* $32/3$.

(c) By the curve $y^2 + 4x = 16$ and the y-axis. *Ans.* $64/3$.

(d) By the curve $y^2 - 6x = 36$ and the y-axis. *Ans.* 48.

(e) By the curve $x^2 - 5y = 25$ and the x-axis.

(f) By the curve $x^2 + 2x - y = 3$ and the x-axis. *Ans.* $32/3$

(g) By the curve $x^2 - 3x + y - 2 = 0$ and the line $y = 2$

(h) By the curve $y = \cos x$, and the lines, $y = 0$, $x = \pi$.

(i) By the curves $y^2 = 4x - x^2$ and $4y = 4x - x^2$ (smaller area).

(j) By the curves $x^2 = 9 - y$ and $5x^2 = 4y$. *Ans.* 24.

2. Compute the areas bounded by the following curves:

(a) $y^2 + 8x = 16$, $3y = 4x$, and $x = 0$ (two such areas).
Ans. $19/6$, $40/3$.

(b) $x^2 + 8y = 16$ and $4y = 3x$. *Ans.* $125/6$.

(c) $x^2 + y^2 = 9$ and $x = 1$ (smaller area).

(d) $y = x^2 - 6x$ and $x + y = 6$.

(e) $x^2 = 4y$ and $y(4 + x^2) = 8$. *Ans.* $2/3(3\pi - 2)$.

(f) $x^2 - 4x - y = 0$, $y = -x$, and $y = -3x$. *Ans.* $13/3$.

(g) $x^2 + 4y^2 = 16$ and $2y^2 + 3x = 0$ (larger area).

(h) $y = \log 2x + 1$, $y = 3 \log x$, and $y = 0$.

(i) $y = e^{2x} \cos 3x$, $y = e^{2x} \sin 3x$, $x = 0$ and lying the first quadrant and to the left of $12x = \pi$.

3. Compute the following:

(a) The area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2}$ 1. *Ans.* πab.

(b) The area of the loop of the curve $4y^2 = x^4(4 + x)$

(c) The area bounded by the $3)^2 (y - 2)$ and the line $y = 4$.

(d) The area bounded by the curve $x^{1/2} + y^{1/2} = a^{1/2}$ and the line

$$x + y = a.$$

(e) The area enclosed by the curve $9x^2 + 5y^4 = 20y^2$.

(f) The area enclosed by the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$.

Ans. $\frac{3}{8}\pi ab$.

(g) The area enclosed by the curve $y^2 + 2x^3 + x^4 = 0$.

4. Compute the following areas:

(a) Bounded by the curve $xy^2 = 8 - 4x$ and its asymptote, the y-axis. *Ans.* 4π.

(b) Bounded by the curve $y^2(6 - x) = x^3$ and its asymptote, the line $x = 6$. *Ans.* 27π.

(c) Bounded by the curve $y^2(3 - x) = x^2(3 + x)$ and its asymptote, the line $x = 3$. *Ans.* $18 + 9\pi/2$.

5. Find the area under one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. HINT: Express the element of area, $y\,dx$, in terms of θ. The limits on θ are $\theta = 0$ and $\theta = 2\pi$. *Ans.* $3\pi a^2$.

6. (a) Find the area enclosed by the four-cusped hypocycloid $x = a\sin^3\theta$, $y = a\cos^3\theta$.

(b) Find the area bounded by the hyperbola $x = a\sec\theta$, $y = b\tan\theta$, and the chord $x = 2a$.

(c) Find the area bounded by one arch of the curve $x = a(1 - \cos\theta)$, $y = a\theta$, and the y-axis. *Ans.* $2\pi a^2$.

7. Show that the area bounded by a parabola and a chord perpendicular to its axis is two-thirds of the area of the circumscribing rectangle.

81b. Areas in Polar Coordinates. Let it be required to find the area OAB included by the lines OA and OB and the arc AB of a curve whose equation, in polar coordinates, is $r = f(\theta)$.

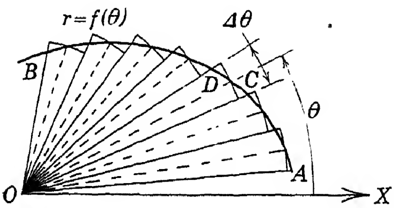

FIG. 133.

Divide the arc AB into n parts, as in Fig. 133, and join the pole O to the points of division. In each interval of arc, choose any value of θ at random, and with the corresponding value of r as radius, and O as center, draw a circle intersecting the two radii drawn to the adjacent points of division. We obtain, thus, n circular sectors, or *elements of area*, none of them necessarily identical with the actual subdivisions of the area AOB. But it is clear that as n is permitted to increase indefinitely, while simultaneously the values of $\Delta\theta$ approach zero, the sum of the circular sectors actually approaches the area desired as a limit.

Now, the area of a typical circular sector is $\frac{r}{2}(r\ \Delta\theta)$, or half the radius, multiplied by the arc, where $r = f(\theta)$. The sum of n sectors is

$$\sum_{i=1}^{n} \frac{r_i^2}{2}\,\Delta_i\theta,$$

and the area sought is

$$\lim_{\substack{n\to\infty\\ \Delta_i\theta\to 0}} \sum_{i=1}^{n} \frac{r_i^2}{2}\Delta\theta = \int_\alpha^\beta \frac{r^2}{2} d\theta = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\,d\theta,$$

where α is the amplitude of OA and β is the amplitude of OB, and

$$\Delta_1\theta + \Delta_2\theta + \cdots + \Delta_n\theta = \beta - \alpha.$$

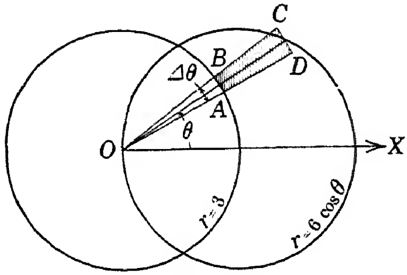

FIG. 134.

Illustration. Find the area outside the circle $r = 3$ and inside the curve $r = 6\cos\theta$.

Solution. The *element of area* $ABCD$ equals

$$\tfrac{1}{2}r_2^2\,\Delta\theta - \tfrac{1}{2}r_1^2\,\Delta\theta,$$

where $r_2 = 6\cos\theta$ and $r_1 = 3$, *i.e.*, $\frac{1}{2}(36\cos^2\theta - 9)\Delta\theta$. The limits of integration are found by solving the equations of the two curves simultaneously, giving $\theta = -\pi/3$ and $\theta = \pi/3$.

They are, thus, $\alpha = -\pi/3$ and $\beta = \pi/3$, and the area is given by

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{1}{2}(36\cos^2\theta - 9)d\theta = \frac{9}{2}\Big[2\theta + \sin 2\theta - \theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{9}{2}\Big[\theta + \sin 2\theta\Big]_{-\frac{\pi}{3}}^{\frac{\pi}{3}}$$

$$= \frac{9}{2}\left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right] = \frac{9}{2}\left(\frac{2\pi}{3} + \sqrt{3}\right)$$

Note that on account of the symmetry of the area with respect to the polar axis we might have taken the limits of integration as 0 and $\pi/3$ and doubled the result, *i.e.*, we might have written for the area

$$2\int_0^{\frac{\pi}{3}} \tfrac{1}{2}(36\cos^2\theta - 9)d\theta = 9\Big[\theta + \sin 2\theta\Big]_0^{\frac{\pi}{3}}$$

$$= 9\left[\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) - 0\right] = 9\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right).$$

Problems

1. Compute the area inside

(a) The curve $r = a(1 - \cos\theta)$. *Ans.* $3\pi a^2/2$.

(b) The curve $r = a \sin 3\theta$. *Ans.* $\pi a^2/4$.

(c) The curve $r = 3 \cos 2\theta$. *Ans.* $9\pi/2$.

(d) The curve $r^2 = a^3 \sin 2\theta$.

(e) The curve $r = 4 \sec^2 \dfrac{\theta}{2}$ and to the right of the line $\theta = \pi/2$.

(f) The first spire of the spiral of Archimedes, $r = a\theta$.

(g) One loop of the curve $r = 3 \sin 4\theta$.

(h) The curve $r = \sin 2\theta + \cos 2\theta$. *Ans.* π.

(i) The inner loop of the curve $r = 1 + 2 \cos\theta$.

2. Compute the area bounded by the curves

(a) $r = 2$ and $r = 4 \sin\theta$ (each of the three areas).

(b) $r = a(1 + \sin\theta)$ and $\theta = \pi/6$ (each of the three areas).

(c) $r = \dfrac{6}{2 - \sin\theta}$ and $r = \dfrac{3}{1 + \sin\theta}$ (each of the two areas).

(d) $r = 1 - \sin\theta$ and $r = 1$ (each of the three areas).

(e) $r = \dfrac{2}{1 - \cos\theta}$ and $r = \dfrac{2}{1 + \cos\theta}$. *Ans.* $16/3$.

(f) $r^2 = a^2 \cos 2\theta$ and $r = a$. *Ans.* $(\pi - 1)a^2$.

(g) $r^2 = 2 \sin 2\theta$ and $r = 2 \sin\theta$ (each of three areas).

3. For the hyperbolic spiral $r\theta = a$, show that the area bounded by any two radii vectors and the curve is proportional to the difference in length of the two radii vectors.

4. For the spiral of Archimedes, $r = a\theta$, show that the area added by the nth spire is proportional to $n - 1$.

5. (a) Find the area of the loop of the folium of Descartes,

$$x^3 + y^3 = 3axy,$$

by transforming to polar coordinates. *Ans.* $3a^2/2$.

(b) Find the area between the folium of Descartes and its asymptote $x + y = -a$ $\left(\text{or, in polar coordinates, } r = \dfrac{-a}{\sin\theta + \cos\theta}\right)$.

6. Find the area enclosed by the curve $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$ by changing to polar coordinates.

82. Lengths of Curves. We define the length of an arc AB of a curve as *the limit of the sum of the lengths of the inscribed chords, as the number of these chords is increased indefinitely while simultaneously their lengths all approach zero.*

If, now, we divide the arc AB (Fig. 135) into n parts and draw the corresponding chords, we obtain as their lengths

$$\sqrt{\Delta_1{}^2x + \Delta_1{}^2y},\ \sqrt{\Delta_2{}^2x + \Delta_2{}^2y},\ \cdots,\ \sqrt{\Delta_n{}^2x + \Delta_n{}^2y}.$$

Let us write the length of any one of these chords, say the ith, as

$$\sqrt{\Delta_i^2 x + \Delta_i^2 y} = \sqrt{1 + \left(\frac{\Delta_i y}{\Delta_i x}\right)^2}\, \Delta_i x = \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x,$$

where $y = f(x)$ is the equation of the curve and ξ_i is the value of x in the ith interval, satisfying the theorem of the mean (Sec. 62)—the continuity of $f(x)$ and $f'(x)$ in the interval from A to B, which justifies the theorem of the mean, being, of course, a part of our hypothesis. The length, then, of arc AB, by our definition, and in view of the fundamental theorem of integral calculus, is

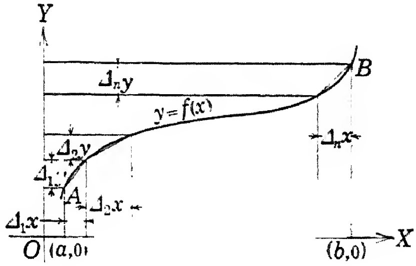

FIG. 135.

$$\lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} \sqrt{1 + [f'(\xi_i)]^2}\, \Delta_i x = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (117)$$

If, along the arc AB, x is a function of y, say $x = \varphi(y)$, we may write the length of the ith chord as

$$\Delta_i l = \sqrt{\left(\frac{\Delta_i x}{\Delta_i y}\right)^2 + 1}\, \Delta_i y = \sqrt{[\varphi'(\eta_i)]^2 + 1}\, \Delta_i y,$$

where η_i is a value of y in the ith interval satisfying the theorem of the mean. An alternative expression for the length of arc AB is, then,

$$\lim_{\substack{n \to \infty \\ \Delta_i y \to 0}} \sum_{i=1}^{n} \sqrt{1 + [\varphi'(\eta_i)]^2}\, \Delta_i y = \int_c^d \sqrt{1 + [\varphi'(y)]^2}\, dy. \qquad (118)$$

If the curve of Fig. 135 is defined parametrically in rectangular coordinates as

$$x = g(t),$$
$$y = h(t),$$

where $\begin{cases} a = g(t_1), \\ c = h(t_1), \end{cases}$ and $\begin{cases} b = g(t_2), \\ d = h(t_2), \end{cases}$ then $f'(x) = \dfrac{dy}{dx} = \dfrac{h'(t)}{g'(t)}$, and the element of integration in (117) assumes the form

$$\sqrt{1 + \left(\frac{h'(t)}{g'(t)}\right)^2}\, g'(t)dt = \sqrt{[g'(t)]^2 + [h'(t)]^2}\, dt.$$

The formula for the length of arc therefore becomes

$$\int_{t_1}^{t_2} \sqrt{[g'(t)]^2 + [h'(t)]^2}\, dt. \tag{119}$$

The same integral would have been obtained from (118) by use of the equation $\varphi'(y) = dx/dy = g'(t)/h'(t)$.

Note that the element of integration in (119) equals

$$\sqrt{[g'(t)dt]^2 + [h'(t)dt]^2} = \sqrt{dx^2 + dy^2} = ds. \qquad [cf.\ (102).] \tag{120}$$

If, lastly, the equation of arc AB is in polar coordinates, in the form $r = f(\theta)$, we transform the element of integration in (120) by the equations $x = r \cos \theta$, $y = r \sin \theta$ to

$$\sqrt{(-r \sin \theta\, d\theta + \cos \theta\, dr)^2 + (r \cos \theta\, d\theta + \sin \theta\, dr)^2}$$
$$= \sqrt{r^2\overline{d\theta^2} + \overline{dr^2}}, \qquad (cf.\ \text{Sec. } 70)$$

and if at A, $\theta = \alpha$, $r = r_1$, while at B, $\theta = \beta$, $r = r_2$, we obtain for the length of AB

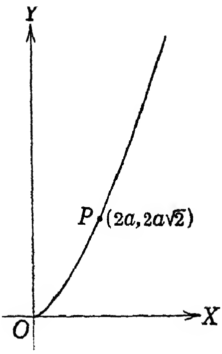

FIG. 136.

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \tag{121}$$

or

$$\int_{1}^{r_2} \sqrt{r^2 \qquad + 1}\, dr. \tag{122}$$

Illustration 1. Find the length of the semicubical parabola $ay^2 = x^3$ from the origin to the point $P\ (2a, 2a\sqrt{2})$.

Solution. Using (117) with $2ay\frac{dy}{dx} = 3x^2$ and $\frac{dy}{dx} = \frac{3x^2}{2ay}$, we get for the length of OP (see Fig. 136),

$$OP = \int_0^{2a} \sqrt{1 + \frac{9x^4}{4a^2y^2}}dx$$
$$= \int_0^{2a} \sqrt{1 + \frac{9x^4}{4ax^3}}dx$$
$$= \int_0^{2a} \sqrt{\frac{4a + 9x}{4a}}dx$$
$$\frac{1}{2\sqrt{a}} \cdot (4a + 9x)^{3/2} \cdot \frac{2}{27}\Big]_0^{2a}$$
$$\frac{}{27\sqrt{a}}[(22a)^{3/2} - (4a)^{3/2}]$$
$$\frac{22\sqrt{22} - 8}{27}a$$

Illustration 2. Find the length of the circumference of the circle

$$x = a\cos\theta,$$
$$y = a\sin\theta.$$

Solution. The expression (119)—with t replaced by θ—gives the desired length as

$$\int_0^{2\pi}\sqrt{(-a\sin\theta)^2 + (a\cos\theta)^2}\,d\theta = \int_0^{2\pi}\sqrt{a^2}\,d\theta = a\theta\Big]_0^{2\pi} = 2\pi a.$$

Note that α and $\alpha + 2\pi$, with α arbitrary, could be employed for the limits of integration, just as well.

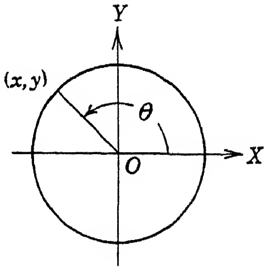

FIG. 137.

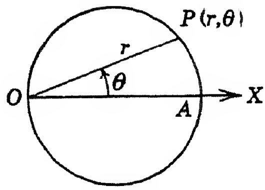

FIG. 138.

Illustration 3. Find the length of the curve $r = a\cos\theta$.

Solution. The expression (121) gives, as the length,

$$2\int_0^{\pi/2}\sqrt{r^2 + (-a\sin\theta)^2}\,d\theta = 2\int_0^{\pi/2}\sqrt{a^2\cos^2\theta + a^2\sin^2\theta}\,d\theta = 2\int_0^{\pi/2} a\,d\theta$$
$$= 2a\theta\Big]_0^{\pi/2} = \pi a,$$

where, on account of symmetry with respect to the polar axis, we compute the length APO and double the result. Obviously $\theta = 0$ at A and $\theta = \pi/2$ at O.

Problems

1. Compute the length of

(*a*) The parabola $y^2 = 4x$ from the origin to $(a^2, -2a)$.

Ans. $\sqrt{a^4 + a^2} + \log(a + \sqrt{1 + a^2})$.

(*b*) The catenary $y = a\cosh\dfrac{x}{a}$ from the point of minimum y to the point where $x = a$.

(*c*) The entire curve $x^2 = 6y - y^2$. *Ans.* 6π.

(*d*) The circle $x^2 = y(4 - y)$ from the origin to $(2,2)$.

(*e*) The curve $y = \log\sin x$ from $(\pi/2, 0)$ to $(5\pi/6, -\log 2)$.

(*f*) $x^2 = (y - 1)^3$ from $(0,1)$ to $(5\sqrt{5}/27, 14/9)$.

(*g*) The hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$. *Ans.* $6a$.

(*h*) The tractrix $x = a\log\dfrac{a + \sqrt{a^2 - y^2}}{y} - \sqrt{a^2 - y^2}$ from $(0,a)$

to the point where $y = v$. *Ans.* $a \log \frac{v}{a}$.

(*i*) The curve $e^y = \dfrac{e^x + 1}{e^x - 1}$ from $x = 1$ to $x = 2$.

(*j*) The curve $y = \log (1 - x^2)$ from $x = 0$ to $x = \frac{1}{3}$.

2. Compute the length of

(*a*) One arch of the cycloid $x = a(\theta - \sin \theta)$, $y = a(1 - \cos \theta)$. *Ans.* $8a$.

(*b*) The curve $x = a(\cos t + t \sin t)$, $y = a(\sin t - t \cos t)$ from $t = \alpha$ to $t = \beta$. *Ans.* $\frac{a}{2}(\beta^2 - \alpha^2)$.

(*c*) The curve $x = e^t \cos t$, $y = -e^t \sin t$ from $t = 0$ to $t = \alpha$.

3. Compute the length of

(*a*) The entire curve $r = a \sin \theta$. *Ans.* πa.

(*b*) The entire curve $r = a(1 - \cos \theta)$. *Ans.* $8a$.

(*c*) The first spire of the spiral of Archimedes $r = a\theta$.

(*d*) The entire curve $r = a \sin^3 \frac{\theta}{3}$. *Ans.* $3\pi a/2$.

(*e*) The curve $r = \dfrac{4}{1 + \sin \theta}$, above the polar axis.

4. Show that the length of the ellipse $x = a \sin \varphi$, $y = b \cos \varphi$ is given by $4a \int_0^{\pi/2} \sqrt{1 - e^2 \sin^2 \varphi}\, d\varphi$, where e is the eccentricity of the ellipse.

This is an example of a so-called *elliptic* integral. It cannot be evaluated in terms of the familiar (algebraic, trigonometric, etc.) functions. By methods given later in this book, its value can be approximated. Its values are also tabulated in most good tables. Look up its value for the case $e = \sqrt{3}/2$ in a table.

5. Given that P is a point on the curve $r = a(1 + \cos \theta)$, such that the length of its arc from $(2a, 0°)$ to the point P equals $2a$, find the area bounded by that arc, the polar axis and the line OP. (O is the pole.)

Ans. $\frac{a^2}{4}\left(\pi + \frac{9\sqrt{3}}{4}\right)$.

6. Find the length of the *cylindrical helix* $x = a \cos \theta$, $y = a \sin \theta$, $z = b\theta$, from $\theta = 0$ to $\theta = 2\pi$. HINT: Show that $\overline{ds}^2 = \overline{dx}^2 + \overline{dy}^2 + \overline{dz}^2$ in the case of a space curve. *Ans.* $2\pi\sqrt{a^2 + b^2}$.

7. Find the length of the curve $x = e^t$, $y = e^{-t}$, $z = t\sqrt{2}$ from $t = 0$ to $t = 1$. *Ans.* $e - \frac{1}{e}$.

83. Volumes of Solids. We begin with an illustration. Let us find the volume of a sphere of radius a. Figure 139 shows the sphere with one-eighth of it bounded by three perpendicular planes through its center. Let us divide the radius OA into n parts and pass, through the points of division, planes perpendicular to OA. These planes will divide the hemisphere to the right of the plane

COB into n parts. (Since the figure shows only one-eighth of the sphere, it shows only one quarter of one of these parts, as $PQRS$.)

If OA is chosen as an axis, say the x-axis, and any value of x is chosen in the interval TU, say $x = OV$, let us pass a plane through V perpendicular to OA, which will intersect the sphere in a circle parallel to the planes of the circles PQ and RS. If now we project that circle upon those planes we obtain, as *the element of volume*,

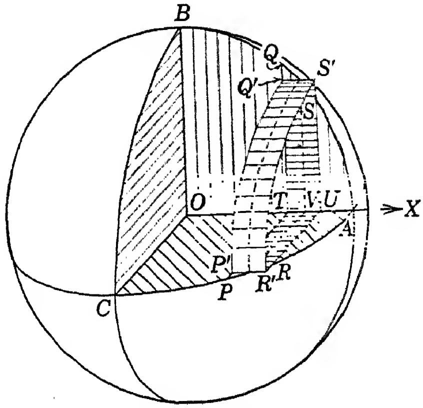

FIG. 139.

the circular cylinder $P'Q'R'S'$, of altitude Δx and whose base has a radius equal to $\sqrt{a^2 - x^2}$. Its volume is, therefore, equal to $\pi(a^2 - x^2)\Delta x$.

It is evident that while this cylinder is not identical in volume with the portion $PQRS$, the sum of such cylinders actually approaches the volume of the hemisphere as the number of divisions is increased indefinitely and the intervals Δx all approach zero. Hence, by the fundamental theorem of integral calculus, the volume of the hemisphere is represented by

$$\lim_{\Delta_i x \to 0} \sum_i^n \pi(a^2 - x_i^2)\Delta_i x = \int_0^a \pi(a^2 - x^2)dx,$$

where $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$ are a set of subdivisions of the interval from $x = 0$ to $x = a$. The volume of the sphere, then, is

$$2\pi\int_0^a (a^2 - x^2)dx = 2\pi\Big[a^2x - \tfrac{1}{3}x^3\Big]_0^a$$
$$= 2\pi(a^3 - \tfrac{1}{3}a^3) = \tfrac{4}{3}\pi a^3,$$

a well-known result.

Suppose, now, we deal with any volume such that the area of a section cut from its boundary by any plane perpendicular to some line, say OX, is a known function of x, say $f(x)$, and that the portion of OX intercepted by the boundary of the surface is AB, where $OA = a$ and $OB = b$. By dividing AB into n parts and passing, through the points of division, planes perpendicular to AB, we divide the volume into n parts. (TU in Fig. 140 is the width of one such part.) We pick any value of x in the interval TU and pass a plane at right angles to OX through the corresponding point. This plane cuts from the given figure a section whose area, by hypothesis, is $f(x)$. If we project this section upon the planes through T and U, a cylinder is formed whose altitude is Δx and whose bases have the area $f(x)$. The sum of the volumes of all such cylinders is an approximation to the volume sought, and the limit of that sum, as the number of cylinders increases indefinitely and their altitudes all approach zero, is actually that volume. In other words, the volume of the solid is

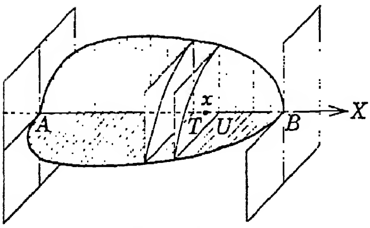

FIG. 140.

$$V = \lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} f(x_i)\Delta_i x = \int_a^b f(x)dx.$$

Illustration 1. A solid is such that every section of it, at right angles to OX, is an equilateral triangle whose base is the double ordinate of the parabola $y^2 = 4x$. Find its volume if it extends from $x = 0$ to $x = 4$.

Solution. The area of a section, at a distance x to the right of the origin, is that of an equilateral triangle of side $2y$, and is equal to $y^2\sqrt{3}$, or $4x\sqrt{3}$. The element of volume is a triangular prism whose base has the area $4x\sqrt{3}$ and whose altitude is Δx. By the above treatment, the volume sought is

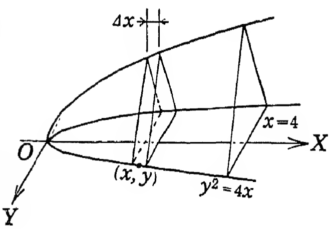

FIG. 141.

$$= \int_0^4 4x\sqrt{3}\, dx \quad \frac{4\sqrt{3}x^2}{2}\Bigg]_0^4 \quad 32\sqrt{3} \text{ cubic units.}$$

Illustration 2. The area under the portion of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ is revolved about the x-axis Find the volume of the solid generated.

Solution. The area of a section at a distance x to the right of the origin is that of a circle of radius y and equals πy^2, or $\pi \sin^2 x$. As the element of

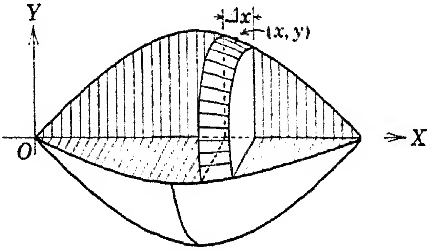

FIG. 142.

volume we take a cylindrical plate whose width is Δx and whose base has the area $\pi \sin^2 x$. The volume we are finding is, then,

$$V = \int_0^\pi \pi \sin^2 x \, dx = \frac{\pi}{2}\left[x - \frac{1}{2} \sin 2x\right]_0^\pi = \frac{\pi}{2}(\pi - 0) = \frac{\pi^2}{2} \text{ cubic units.}$$

Illustration 3. The same area as in Illustration 2 is revolved about the y-axis. Find the volume of the solid generated.

Solution. It is convenient to take here, as the element of volume, the volume between two concentric cylinders (cylindrical shell). The volume of this shell is $2\pi xy\Delta x$ (where y is its altitude, Δx the thickness of its walls, and

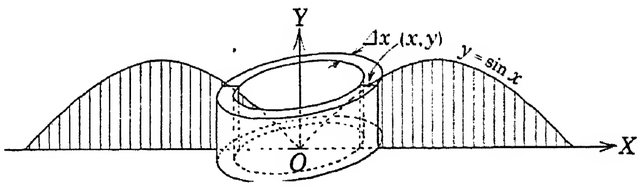

FIG. 143.

x the mean of its inner and outer radii) or $2\pi x \sin x\Delta x$, since $y = \sin x$. Hence,

$$V = \int_0^\pi 2\pi x \sin x \, dx = 2\pi\left[\sin x - x \cos x\right]_0^\pi = 2\pi[\pi - 0] = 2\pi^2 \text{ cubic units.}$$

NOTE: This problem could be solved by taking the element of integration as that bounded by two planes perpendicular to the axis of revolution, just as was done in the preceding problem. The area of a section at a distance y above the x-axis is, then, $\pi x_2^2 - \pi x_1^2$, and the volume of the element is $(\pi x_2^2 - \pi x_1^2)\Delta y$. Now, from the equation $y = \sin x$, we obtain $x = \sin^{-1} y$. The *principal* value of $\sin^{-1} y$ is x_1 and $x_2 = \pi - x_1 = \pi - \sin^{-1} y$. Hence the volume of the element is equal to

$$\pi[(\pi - \sin^{-1} y)^2 - (\sin^{-1} y)^2]\Delta y = \pi(\pi^2 - 2\pi \sin^{-1} y)\Delta y.$$

The limits on y, in the solid formed, are $y = 0$ and $y = 1$, and the volume is

$$V = \pi\int_0^1 (\pi^2 - 2\pi \sin^{-1} y)dy = \pi\Big[\pi^2 y - 2\pi y \sin^{-1} y - 2\pi\sqrt{1 - y^2}\Big]_0^1$$
$$= \pi\left[\left(\pi^2 - 2\pi \cdot \frac{\pi}{2} - 0\right) - (-2\pi)\right] = 2\pi^2, \text{ as above.}$$

The student will, probably, prefer the cylindrical-shell element of integration as more convenient in the case of this problem.

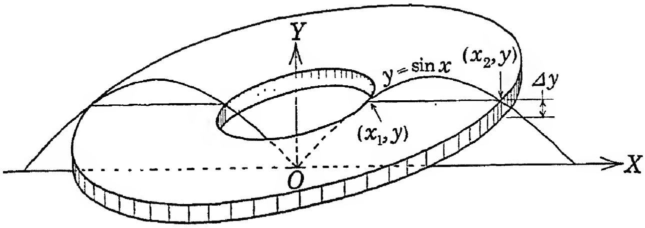

FIG. 144.

Exercise 1. Show that the volume of the solid generated by revolving about OX the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$, is represented by $\pi\int_a^b [f(x)]^2\, dx$.

Exercise 2. Show that the volume of the solid generated by revolving about OY the area bounded by the curve $x = F(y)$, the y-axis, and the abscissas $y = c$ and $y = d$, is represented by $\pi\int_c^d [F(y)]^2\, dy$.

Problems

1. Find the volume of the solid generated by revolving the given area about OX in each case.

(a) The circle $x^2 + y^2 = a^2$. *Ans.* $\frac{4\pi}{3}a^3$.

(b) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. *Ans.* $\frac{4\pi}{3}ab^2$.

(c) Either area bounded by the curve $y = \cos x$ and the coordinate

(d) The area bounded by the curve $y^2 = 3x + 12$ and the y-axis.

(e) The area bounded by one arch of the cycloid

$$\begin{cases} a(\theta - \sin \\ y = a(1 - \cos\theta) \end{cases}$$ and the x-axis. *Ans.*

(f) The area bounded by the curve $4y - x^2 = 12$ and the line $y = 6$. *Ans.* $384\pi\sqrt{3}/5$.

(g) The area bounded by the curve $x^2 = 4y^3$ and the line $y = 4$.

(h) The area of the loop of the curve $\begin{cases} x = t^2 \\ y = t^3 - 9t \end{cases}$. *Ans.* $2187\pi/4$.

2. Find the volume of the solid obtained by revolving the given area about OY in each case.

(*a*) The area bounded by the parabola $y^2 + 2x = 6$ and the y-axis. *Ans.* $48\pi\sqrt{6}/5$.

(*b*) The area bounded by the lines $y = 3x$, $x = 0$, and $y = 3$.

(*c*) The area bounded by the curve $x^2 = 4y$ and the line $y = 2x$. *Ans.* $512\pi/3$.

(*d*) The area below the curve $y = x^3 - 4x$ and above the x-axis.

(*e*) The area bounded by $y = \cos 2x$, $\pi y = 3x$, and $x = 0$.

(*f*) The area of Prob. 1(*e*). *Ans.* $6\pi^3a^3$.

(*g*) The area inside the curve $x^{2/3} + y^{2/3} = a^{2/3}$. *Ans.* $32\pi a^3/105$.

(*h*) The area of Prob. 1(*h*).

3. The area bounded by the curve $y^2 = 4x + 8$ and the line $x = 2$ is revolved about that line. Find the volume generated. *Ans.* $1024\pi/15$.

4. The area bounded by the curve $(x - 4)^2 = 2y$ and the coordinate axes is revolved about the line $y = 8$. Find the volume generated.

5. The area of Prob. 4 is revolved about the line $x = 4$. Find the volume generated.

6. The area bounded by the parabola $x^2 = 4y$ and the line $x + 2y = 4$, is revolved about the line $x = 3$. Find the volume. *Ans.* 72π.

7. The area between the cissoid $y^2(6 - x) = x^3$ and its asymptote is revolved about the asymptote. Find the volume generated. *Ans.* $54\pi^2$.

8. The area bounded by the parabola $(x - 4)^2 = 2y$ and the line $y = 8$ is revolved about the line $y = 8$. Find the volume generated.

9. Find the volume obtained by revolving a circle of radius a about a line in its plane whose distance from the center is b, where $b > a$. This figure is called a *torus*. *Ans.* $2\pi^2a^2b$.

10. The area of Prob. 2(*e*) is revolved about the x-axis. Find the volume generated.

11. If the area bounded by the curve $y = ax^2 - 2x$ $(a > 0)$, the x-axis, and the line $ax = 1$ equals $1/6$, find the volume obtained by revolving that area about the x-axis. Also find the volume generated when that area is revolved about the line $ax = 1$.

12. The area bounded by the curve $y = e^x$, the line $y = 1$, and the line $x = 2$, is revolved about the x-axis. Find the mean value of the area of a section of this solid at right angles to the x-axis, the area being considered as a function of its distance from the y-axis. *Ans.* $\frac{\pi}{4}(e^4 - 5)$.

13. A solid is such that each section of it at right angles to the x-axis is an equilateral triangle whose base extends from the x-axis to the line $y = 2x$. Find its volume if it extends from $x = 0$ to $x = 3$. *Ans.* $9\sqrt{3}$.

14. Find the volume of a solid such that each section of it at right angles to the y-axis is a square having one diagonal extending from the line $y = -x$ to the parabola $x = 2\sqrt{y}$, given that the solid extends from $y = 0$ to $y = 9$.

15. The sections of a solid at right angles to a line are rectangles whose dimensions are each proportional to the distance of the section from a fixed point A on that line, one dimension being twice the other. The solid inter-

cepts a distance AB on the line, equal to h. If the area of the section at a distance $h/2$ from A equals $2h^2$, find the entire volume. *Ans.* $\frac{8}{3}h^3$.

16. Every section, at right angles to OX, of a solid is an ellipse whose major axis extends from the line $y = x$ to the line $y = 3x$ and whose eccentricity is $3/5$. The solid extends to the right from the origin, and the area of its greatest section is 20π square units. Find its volume. Hint: The area of an ellipse whose semiaxes are u and v, is πuv.

17. Find the entire volume under the surface $4x^2 + y^2 = 4(9 - z)$ and above the xy-plane. Hint: Sections at right angles to OZ and at a distance z above the xy-plane are ellipses of semiaxes $\sqrt{9 - z}$ and $2\sqrt{9 - z}$.

Ans. 81π.

18. Find the volume in the first octant, bounded by the cylinder $y^2 + z^2 = 9$, the plane $x + 3y = 9$, and the coordinate planes. Hint: Sections at right angles to the y-axis are rectangles of dimensions x and z, where $x = 9 - 3y$ and $z = \sqrt{9 - y^2}$.

19. Find the volume in the first octant bounded by the cylinder $x^2 + z^2 = 16$, the plane $y + 2z = 8$, and the coordinate planes.

Ans. $32\pi - 128/3$.

20. Find the volume bounded by the yz-plane and the surface

$y^2 + 3z^2 = 12 - x$. *Ans.* $24\pi\sqrt{3}$.

21. Find the volume of the pyramid formed by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the coordinate planes.

22. Find the volume in the first octant, bounded by the coordinate planes, the cylinder $x^2 + y^2 = 16$, and the plane $2x + y + 3z = 12$.

23. Find the volume common to the cylinders $x^2 + y^2 = 9$ and $y^2 + z^2 = 9$.

24. A football is 10 in. long and a plane section containing a seam is an ellipse 6 in. broad. Find the volume of the ball, assuming that the leather is so stiff that every plane section perpendicular to the long axis is a square.

Ans. 120.

25. A wedge is cut from a cylinder whose base is a circle of radius 2 ft., by a plane passing through a diameter of the base and inclined at an angle of 60° to the base. Find the volume of the wedge. *Ans.* $16\sqrt{3}/3$.

26. From each point of the ellipse $\frac{x^2}{9} + \frac{z^2}{16} = 1$, lying in xz-plane, two lines are drawn parallel to the yz-plane and intersecting the circle

$$x^2 + y^2 = 9,$$

lying in the xy-plane. Find the volume formed. *Ans.* 96.

84. Areas of Surfaces of Revolution. Let the arc AB be revolved about the x-axis. Desired, to compute the area of the surface generated.

To solve this problem, let us divide the arc AB into n parts of lengths, $\Delta_1 s$, $\Delta_2 s$, . . . , $\Delta_n s$ and draw the chords through the

successive points of division. Let us call the lengths of the corresponding chords $\Delta_1 c$, $\Delta_2 c$, . . . , $\Delta_n c$. The area swept out by any chord $\Delta_i c$ is that of a frustum of a cone whose dimensions are indicated in Fig. 145, and is thus equal to $\pi(y_i + y_i + \Delta_i y)\Delta_i c$. The total area swept out by all the chords, *i.e.*, by the broken line AB, is, then

$$\pi \sum_{i=1}^{n} (2y_i + \Delta_i y)\Delta_i c.$$

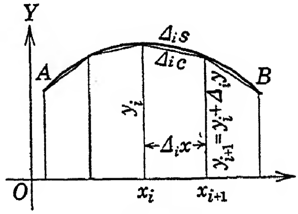

FIG. 145.

Now, the surface generated by the arc AB, in its revolution about the x-axis, is defined as the limit of the sum of the surfaces generated by the n chords, as n increases beyond limit and the length of each chord tends to zero. In other words, the surface in question is defined as

$$\lim_{\substack{n \to \infty \\ \Delta_i c \to 0}} \pi \sum_{i=1}^{n} (2y_i + \Delta_i y)\Delta_i c = \pi \lim_{\substack{n \to \infty \\ \Delta_i c \to 0}} \sum_{i=1}^{n} (2y_i + \Delta_i y)\Delta_i c.$$

But

$$\lim_{\Delta_i c \to 0} \left(\frac{2y_i \Delta_i s}{(2y_i + \Delta_i y)\Delta_i c} \right) = \lim_{\substack{\Delta_i c \to 0 \\ \Delta_i y \to 0}} \left(\frac{2y_i}{2y_i + \Delta_i y} \right) \lim_{\Delta_i c \to 0} \left(\frac{\Delta_i s}{\Delta_i c} \right) =$$

for each of the values $i = 1, 2, 3, \ldots, n$, since each limit in the product equals 1. It can be shown, furthermore, that if y and y' are continuous functions of x for $a_1 \leqq x \leqq a_2$, then, if we set $(2y_i + \Delta_i y)\Delta_i c = u_i$ and $2y_i \Delta_i s = v_i$, the infinitesimals u_i, v_i satisfy the hypothesis of Duhamel's theorem. (Art. 65.)

By that theorem, the area of the surface that we are finding is

$$S = \pi \lim_{\substack{n \to \infty \\ \Delta_i c \to 0}} \sum_{i=1}^{n} (2y_i + \Delta_i y)\Delta_i c = \pi \lim_{\substack{n \to \infty \\ \Delta_i c \to 0}} \sum_{i=1}^{n} 2y_i \Delta_i s = 2\pi \int y\,ds.$$

The differential ds, under the integral sign, will be put in the form $\sqrt{1 + (dy/dx)^2}\,dx$ or $\sqrt{1 + (dx/dy)^2}\,dy$, whichever is found more convenient. The limits of integration, if A is (a_1,b_1) and B is (a_2,b_2), will be a_1 and a_2 in the first case and b_1 and b_2 in the second.

Exercise 1. Show that the area of the surface of revolution obtained by revolving an arc AB about OY is represented by the definite integral $2\pi \int x\,ds$.

Illustration. (*a*) The upper half of the circle $x^2 - 4x + y^2 = 0$ revolves about the x-axis. Find the area of the surface formed.

Solution.—By the formula above, the area is represented by

$$2\pi \int y ds = 2\pi \int_0^4 \sqrt{4x - x^2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\pi \int_0^4 \sqrt{4x - x^2} \sqrt{1 + \frac{(2-x)^2}{4x - x^2}}\, dx$$

$$= 2\pi \int_0^4 \sqrt{(4x - x^2) + (2 - x)^2}\, dx = 2\pi \int_0^4 \sqrt{4}\, dx$$

$$= 4\pi x \Big]_0^4 = 16\pi \text{ square units.}$$

(*b*) Let us also revolve this semicircle about the y-axis and compute the area of the surface formed.

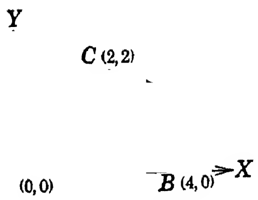

FIG. 146.

Solution. By Exercise 1, this area is represented by

$$2\pi \int x\, ds = 2\pi [\int x_1\, ds + \int x_2\, ds]$$

where

$$x_1 = 2 - \sqrt{4 - y^2} \quad \text{and} \quad x_2 = 2 + \sqrt{4 - y^2},$$

the first equation expressing the abscissa of a point on the arc AC as a function of y and the second expressing the abscissa of a point on the arc CB, also as a function of y. When $x = x_1$

$$\frac{dx}{dy} = \frac{d(2 - \sqrt{4 - ;}}{dy}$$

and when $x = x_2$

$$\frac{dx}{dy} = \frac{d(2 + \sqrt{4 - y^2})}{dy} \qquad -y$$

We find it convenient to use $\sqrt{1 + (dx/dy)^2}\, dy$ for ds, and hence, the surface is given by

$$S = 2\pi \left[\int_0^2 (2 - \sqrt{4 - y^2}) \sqrt{1 + \frac{y^2}{4 - y^2}} dy + \right.$$

$$\int_0^2 (2 + \qquad /1 + \frac{}{4 - y^2} dy$$

$$= 2\pi \left[\int_0^2 (2 - \sqrt{4 - y^2}) \cdot \frac{2}{\sqrt{4 - y^2}} dy + \right.$$

$$\int_0^2 (2 + \sqrt{4 - y^2}) \cdot \frac{2}{\sqrt{4 - y^2}} dy$$

$$= 2\pi \left[\int_0^2 \left(\frac{4}{\sqrt{4 - y^2}} - 2 \right) dy + \int_0^2 \left(\frac{4}{\sqrt{4 - y^2}} + 2 \right) dy \right]$$

$$= 2\pi \int_0^2 \frac{8}{\sqrt{4 - y^2}} dy = 16\pi \sin^{-1} \frac{y}{2} \Big]_0^2 = 16\pi \left(\frac{\pi}{2} - 0 \right) = 8\pi^2 \text{ square units.}$$

Problems

1. Obtain the area of the surface of a sphere of radius a by revolving the semicircle $x^2 + y^2 = a^2$ about one of the coordinate axes. *Ans.* $4\pi a^2$.

2. Revolve each curve below about the axis indicated and compute the area of the surface formed.

(*a*) The portion of the line $y = 3x$ from (0,0) to (3,9) about OX; also about OY; also about the line $x = -1$.

Ans. $27\pi\sqrt{10}$, $9\pi\sqrt{10}$, $15\pi\sqrt{10}$.

(*b*) The portion of the parabola $y^2 = x$ from (0,0) to $(12,\sqrt{12})$ about OX; also about OY.

$$Ans.\quad \frac{171\pi}{3},\ \pi\left[\frac{679\sqrt{12}}{16} - \frac{1}{32}\log\,(2\sqrt{12} + 7)\right].$$

(*c*) The portion of the line $2x + y + 6 = 0$ from $(-2,-2)$ to $(-1,-4)$ about OX; also about OY. *Ans.* $6\pi\sqrt{5}$, $3\pi\sqrt{5}$.

(*d*) The circle $x^2 + y^2 - 4y + 3 = 0$ about OY; also about OX.

(*e*) A loop of the curve $8y^2 = x^2 - x^4$ about OX. *Ans.* $\pi/4$.

(*f*) The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about OX; also about OY.

(*g*) The portion of the catenary $y = a\cosh\dfrac{x}{a}$ from $(0,a)$ to

$$(a,\ a\cosh 1)$$

about OX; also about OY; also about the tangent drawn at $(0,a)$.

(*h*) The hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ about OX; also about OY.

3. Revolve each curve below about the axis indicated and compute the area of the surface formed.

(*a*) The arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about OX; also about the tangent at the point of maximum y; also about the normal at the point of maximum y.

(*b*) The portion of the curve $x = e^\theta \sin\theta$, $y = e^\theta\cos\theta$ from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$, about OX; also about OY.

(*c*) The cardioid $r = a(1 - \cos\theta)$ about the polar axis. HINT: $y = r\sin\theta$, $ds = \sqrt{r^2\overline{d\theta}^2 + \overline{dr}^2}$. *Ans.* $32\pi a^2/5$.

(*d*) The right-hand half of the lemniscate $r^2 = 4\cos 2\theta$ about the polar axis.

4. Compute the lateral area of the finite portion of the paraboloid of revolution $y^2 + z^2 = 8x$ cut off by the plane $x = 8$.

5. If a closed curve having an axis of symmetry revolves about a line parallel to that axis, not intersecting the curve, and at a distance a from the axis, prove (*a*) that the area of the surface formed is equal to $2\pi aL$, where L is the length of the curve; (*b*) that the volume generated equals $2\pi aA$, where A is the area enclosed by the curve.

6. Use the statements made in Prob. 5 to obtain the surface and volume of the torus formed by revolving a circle of radius a about a line in its plane at a distance of b units from the center ($b > a$).

7. The segment of the line $x + y = 5$, included between the coordinate axes, revolves about the line $3x - 4y = 12$. Find the area of the surface formed.

85. Work. If a force f of constant magnitude acts in a straight line over a distance s, the work done by the force is defined in mechanics as equal to fs (units of work).

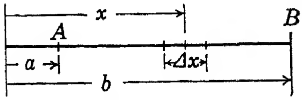

FIG. 147.

Let, now, a variable force act over the distance AB (Fig. 147) in such a manner that at each point it is a function of the distance x from a fixed point O on the line AB, say, the function $f(x)$. Required, the work done by the force.

The student has, probably, by this time caught the spirit of the integral calculus to the end that his first move will be to divide the line AB into a number of parts, select any value of x in each interval, and consider the product $f(x)\Delta x$. Such a product measures the amount of work done by the force in the interval Δx, on the assumption that its value throughout that interval is constant and equals $f(x)$. This amount of work is not, in general, identical with the actual work performed by the force in the interval. It is clear, however, that as the number of intervals is permitted to increase beyond bound, while their lengths all tend to zero, the sum of these partial amounts of work approaches the exact total work as a limit. In other words, the amount of work done by the force over the distance AB is

$$W = \lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} f(x_i)\Delta_i x = \int_a^b f(x)dx.$$

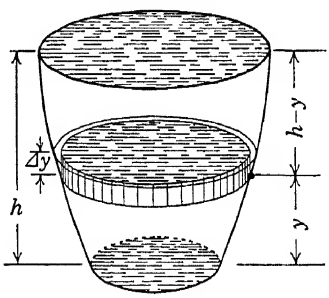

FIG. 148.

Again, consider the amount of work done in pumping a liquid out of a tank such that a horizontal section of it, at a distance y above the bottom has an area equal to $f(y)$. Now, the work done in lifting a mass of weight w a distance s equals ws units of work. Since all of the liquid in the tank is not carried up the same distance, we consider an element of volume, and the element of work done in raising it, choosing the element of volume so that we may assume the depth to be uniform throughout it. Such an element is $f(y)\Delta y$, and its weight is $\delta \cdot f(y) \cdot \Delta y$, where δ is

the density of the liquid. If the height of the tank is h, the distance the element of volume is raised is $h - y$, and the element of work is $\delta \cdot f(y) \cdot (h - y) \cdot \Delta y$.

Without further details, the student will immediately identify the total amount of work as

$$\int_0^h \delta \cdot f(y) \cdot (h - y)dy.$$

Illustration. Find the work done in pumping the water from a hemispherical tank of radius 5 ft. and containing water to a depth of 3 ft. at the start, if the outflow pipe is 2 ft. above the top of the tank and water weighs $\frac{1}{32}$ ton/cu. ft.

Solution. It will be convenient to take the origin at the center of the sphere (this will simplify the relation between x and y) and consider the y-axis as positive downward (else all the y's in our volume would be negative). An element of volume is $\pi x^2 \cdot \Delta y$, and its weight is

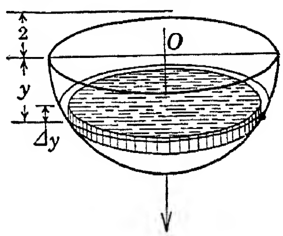

FIG. 149.

$$\frac{\pi x^2 \cdot \Delta y}{32} = \frac{\pi(25 - y^2) \cdot \Delta y}{32}$$

in tons, the right-hand expression coming from $x^2 + y^2 = 25$, the equation of the circular section through the xy-plane. The element is raised a distance $y + 2$, and the element of work is

$$\frac{\pi}{32}(25 - y^2)(y + 2) \quad \Delta y.$$

The total amount of work, then, in foot-tons, done in emptying the tank, is

$$\int_2^5 \frac{\pi}{32}(25 - y^2)(y + 2)dy,$$

the least y for any element of volume being 2, at the top surface of the water, and the greatest y being 5, at the bottom of the tank.

Problems

1. A cylindrical tank of base radius 3 ft. and height 12 ft. is filled with oil weighing $\frac{1}{40}$ ton/cu. ft., to within 2 ft. of the top. The oil is pumped out of the tank, the outflow pipe being 5 ft. above the top of the tank. Compute the work expended. *Ans.* 27π ft.-tons.

2. A conical tank 6 ft. in diameter across the top and 12 ft. deep is filled with water to half its depth. Find the work done in pumping the tank empty, the outflow pipe being 3 ft. above the top of the tank.

3. A tank is in the shape of a paraboloid of revolution, with the vertex at the bottom. It is 6 ft. deep and 8 ft. wide across the top. If it is full of

water to start with and the outflow pipe is at the top of the tank, find the work done in pumping it empty. *Ans.* 3π ft.-tons.

4. A horizontal cylindrical tank 12 ft. from end to end and 6 ft. in diameter is full of oil weighing $\frac{1}{40}$ ton/cu. ft. Find the work done in pumping it empty if the outflow pipe is at the top.

5. A tank, in the form of a hemisphere of radius 5 ft. surmounted by a cylinder of the same base radius and 12 ft. high, is filled with water to a point 3 ft. from the top. The outflow pipe is at the top of the cylinder. Find the work done in pumping the tank empty.

6. A bag containing originally 60 lb. of flour is lifted through a vertical distance of 9 ft. While it is being lifted, flour is leaking from the bag at such a rate that the number of pounds lost is proportional to the square root of the distance traversed. If the total loss of flour is 12 lb., find the amount of work done in lifting the bag. *Ans.* 468 ft.-lb.

7. The force required to stretch a spring is proportional to the amount of stretch, and a force of 1 lb. stretches a particular piece of wire of normal length 40 in. to a length of 40.3 in. Find the work done in stretching this wire from its original length of 40 in. to a length of 41 in. *Ans.* $\frac{5}{3}$ in.-lb.

8. The force required to compress a spring is proportional to the amount of compression. If the normal length of the spring is 8 in. and a force of 40 lb. is required to compress it $\frac{1}{2}$ in., find the work done in compressing it from its length of 8 in. to a length of 7 in.; also the work done in compressing it from a length of 7 in. to a length of 6 in. *Ans.* 40 in.-lb., 120 in.-lb.

9. A meteorite, a miles from the center of the earth to start with, falls to the earth's surface. If the force of gravity is inversely proportional to the square of the distance from the center of the earth, find the work done by gravity if the weight of the meteorite is w lb. at the surface of the earth. Call the radius of the earth R. *Ans.* $R^2w\left(\frac{1}{R} - \frac{1}{a}\right)$.

10. A quantity of air with an initial volume of 100 cu. ft. and pressure of 40 lb./sq. in. is confined in a cylinder. It is compressed to a volume of 80 cu. ft. Find the work done

(a) on the basis that $pv = c$; *Ans.* $576{,}000 \log \frac{5}{4}$ ft.-lb.

(b) on the basis that $pv^{1.4} = c$,

where p is the pressure per square unit of area, v is the volume of air, and c is a constant.

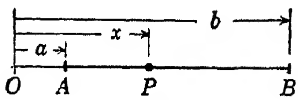

FIG. 150.

11. In mechanics, the *kinetic energy* of a particle of mass m and possessing a velocity v is defined as $\frac{1}{2}mv^2$ (units of energy). Prove that if a force whose value at any point P is $f(x)$ moves a body from A to B, then the work done is equal to the kinetic energy at the point B minus the kinetic energy at the point A, *i.e.*, prove that

$$\int_a^b f(x)dx = \tfrac{1}{2}mv_2^2 - \tfrac{1}{2}mv_1^2$$

where v_1 is the velocity of the body at A and v_2 is the velocity of the body at

B. HINT: Force = mass · acceleration. Hence

$$f = m\frac{dv}{dt} = m\frac{dv}{dx}\cdot\frac{dx}{dt} = mv\frac{dv}{dx}.$$

12. Assuming that below the surface of the earth the force of gravity varies directly as the distance from the center, find what velocity a particle whose weight at the surface of the earth equals w, must possess at a distance a below the surface if it is traveling against gravity and its velocity becomes zero when it reaches the surface. HINT: The mass m of a particle, whose weight at the surface of the earth is w, equals w/g (g is approximately 32). Call the radius of the earth R; also see Prob. 11.

Ans. $\sqrt{\frac{g}{R}(2aR - a^2)}$.

86. Attraction. Two material particles, according to Newton's law, attract each other with a force directed along the line joining them, proportional to the product of their masses and inversely proportional to the square of the distance between them, the force being equal, in other words, to km_1m_2/R^2, where m_1 and m_2 are the two masses and R is the distance between them. The value of k, a constant, depends upon the choice of units of mass, distance and force. In fact, we may, and shall, assume the units to have been so chosen that $k = 1$.

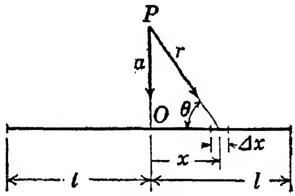

FIG. 151.

Let us now find the total attraction of a homogeneous rod of length $2l$ and density δ upon a particle of mass m situated on a perpendicular bisector of the rod, at a distance a from it.

Solution. Consider an *element* of attraction, *viz.*, the attraction of an element of length Δx, of the rod upon the particle situated at P. By Newton's law, it equals $\dfrac{\delta\cdot m\cdot\Delta x}{r^2}$, and is directed as indicated in Fig. 151. This element of attraction has a component at right angles to the rod, directed along PO, and a component parallel to the rod. Owing to the symmetry of the rod with respect to the line OP, the components parallel to the rod will add to zero for pairs of equal elements Δx equidistant from O and on opposite sides of it. Hence their resultant is zero. We need, then, to sum only the components of the attraction along PO. Now the magnitude of such a component is

$$\frac{m\delta\cdot\Delta x\cdot\sin\theta}{r^2}$$

and the total attraction, the student will perceive immediately, is

$$2\int_0^l \frac{m\delta \sin\theta\, dx}{r^2}.$$

To evaluate this integral we must express the integrand in terms of x. Now, $r = \sqrt{a^2 + x^2}$ and $\sin\theta = a/\sqrt{a^2 + x^2}$. Hence, the total attraction is

$$2m\delta a \int_0^l \frac{dx}{(a^2 + x^2)^{3/2}} = 2m\delta a \cdot \frac{x}{a^2\sqrt{a^2 + x^2}}\Big]_0^l = \frac{2m\delta l}{a\sqrt{a^2 + l^2}}$$

units of force.

Problems

1. Find the attraction for the example worked out in the text, on the basis that the density at any point of the rod, at a distance x from its center, is $x/2$. *Ans.* $m\left(1 - \dfrac{a}{\sqrt{l^2 + a^2}}\right)$.

2. A particle of mass m is placed in the line of a rod of length l and density δ, and at a distance a from the nearer end. Find the attraction of the rod upon a particle:

(*a*) Assuming δ to be constant, *i.e.*, the rod homogeneous;

Ans. $\dfrac{m\delta l}{a(l + a)}$.

(*b*) Assuming δ at any point of the rod to equal $2x$, where x is the distance of that point from the end near the particle.

Ans. $2m\left[\log\dfrac{a + l}{a} - \dfrac{l}{a + l}\right]$.

3. A particle of mass 1 is placed on the line through the center of a circular loop of wire, perpendicular to the plane of the wire, and at a distance a from the center. The wire is of negligible thickness and uniform density δ. Find the attraction of the wire upon the particle, the radius of the circle being equal to r. *Ans.* $2\pi ar\delta/(a^2 + r^2)^{3/2}$.

4. A particle of mass 1 is placed on the line through the center of a circular disk, of radius r and perpendicular to the plane of the disk, the particle being at a distance a from the center of the disk. Find the attraction of the disk upon the particle assuming the density of the disk

(*a*) to be δ, a constant; *Ans.* $2\pi\delta\left(1 - \dfrac{a}{\sqrt{a^2 + r^2}}\right)$.

(*b*) at a distance x from its center to be $x/3$.

5. A material cylinder has a radius of base equal to r, height equal to h and density δ. A particle of mass 1 is placed on the axis of the cylinder produced, at a distance a from the nearer base. Find the attraction of the cylinder upon the particle. *Ans.* $2\pi\delta[h + \sqrt{a^2 + r^2} - \sqrt{(a + h)^2 + r^2}]$.

6. (*a*) If the particle, in the example worked out in the text, is pulled away, against the attraction of the rod, from its initial position at P to a point Q on the line OP such that $OQ = b$, find the amount of the work done.

$$Ans.\quad 2m\delta \log \left(\frac{b}{a} \cdot \frac{l + \sqrt{l^2 + a^2}}{l + \sqrt{l^2 + b^2}}\right).$$

(*b*) Find the limit of the quantity found in (*a*) as $b \to \infty$. (This limit, with m set equal to 1, is called the *potential* at P due to the rod.)

$$Ans.\quad 2\delta \log \frac{l + \sqrt{l^2 + a^2}}{a}.$$

7. A particle of mass m_1 is placed at O. Another particle, of mass m_2, moves along some path from P to Q, against the attraction of the first particle. Show that the amount of work done upon it is equal to

$$m_1 m_2\left(\frac{1}{b} - \frac{1}{a}\right),$$

where $a = OQ$ and $b = OP$, regardless of the path. HINT: Use polar coordinates with pole at O and recall that $\tan \alpha = r\frac{d\theta}{dr}$, where α is the angle between the radius from O to a point on the arc PQ and the tangent drawn at that point. The element of work is the product of the component along the tangent, by Δs. Also recall that $ds = \sqrt{\overline{dr}^2 + r^2 d\theta^2}$.

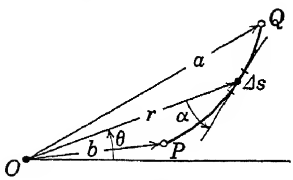

FIG. 152.

NOTE: The limit of the amount of work as $a \to \infty$ is called the *potential* at P due to m_1 being located at O (see Prob. 6) if $m_2 = 1$. It reduces, thus, to m_1/b.

87. Center of Gravity. It is tolerably well known to the student that two bodies of masses m_1 and m_2 will balance when placed on opposite sides of the fulcrum of a lever if the distances d_1 and d_2 are such that $m_1 d_1 = m_2 d_2$. Such a product, md, of a mass m, by a distance d, is called the *moment* of the mass about a point, line or plane which is at the distance d from it. In this language, we say that the two masses m_1 and m_2 will balance on the lever if they lie on opposite sides of the fulcrum and have equal moments.

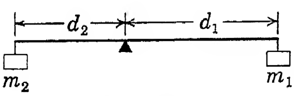

FIG. 153.

If, instead of two masses there are p of them, $m_1, m_2, \ldots, m_p$ on one side of the fulcrum O, at the distances $d_1, d_2, \ldots, d_p$ and q of them $\mu_1, \mu_2, \ldots, \mu_q$ on the other side at the distances $\delta_1, \delta_2, \ldots, \delta_q$, the system will still balance at O, provided the distances and masses are such that

$$m_1 d_1 + m_2 d_2 + \cdots + m_p d_p = \mu_1\delta_1 + \mu_2\delta_2 + \cdots + \mu_q\delta_q.$$

i.e., provided the sum of the moments of the masses on one side of O is equal to the sum of the moments of the masses on the other side of O.

If we now employ the notion of directed distances and use the abscissa of a point to represent its directed distance, we have

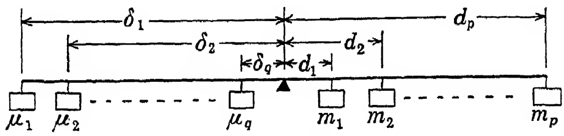

FIG. 154.

$x_1 = d_1, x_2 = d_2, \cdots, x_p = d_p, x_{p+1} = -\delta_1, x_{p+2} = -\delta_2, \cdots, x_{p+q} = -\delta_q$ and write $m_{p+1} = \mu_1, m_{p+2} = \mu_2, \cdots, m_{p+q} = \mu_q,$ the above equation can be written as

$$m_1x_1 + m_2x_2 + \cdots + m_px_p = -m_{p+1}x_{p+1} - m_{p+2}x_{p+2} - \cdots - m_{p+q}x_{p+q}$$

or

$$m_1x_1 + m_2x_2 + \cdots + m_nx_n = 0,$$

where $n = p + q$. We thus see that, by defining the moment of a given mass about a given point as *the mass* times *the directed distance* from the point, the masses will balance about a point so located that the sum of the moments of the masses about it is equal to zero.

We propose, now, the problem, given a set of point masses $m_1, m_2, \ldots, m_n,$ located on the x-axis with abscissas $x_1, x_2, \ldots, x_n,$ respectively, to find the abscissa $\bar{x}$ of the point about which they will balance. To solve this we first note that the directed distances of the given point masses from the point whose abscissa is $\bar{x}$ are the quantities $x_1 - \bar{x}, x_2 - \bar{x}, \cdots, x_n - \bar{x}$. Since the sum of the moments about this point must be zero, we have

$$0 = (x_1 - \bar{x})m_1 + (x_2 - \bar{x})m_2 + \cdots + (x_n - \bar{x})m_n$$

or, finally

$$\bar{x} = \frac{x_1m_1 + x_2m_2 + \cdots + x_nm_n}{m_1 + m_2 + \cdots + m_n}, \tag{123}$$

locating the *center of gravity* of the system.

Exercise 1. Prove that the sum of the moments of the masses $m_1, m_2, \ldots, m_n,$ located on the x-axis at points whose abscissas are, respectively,

$x_1, x_2, \ldots, x_n$, about any fixed point P on the x-axis, is equal to the moment about the point P of a mass equal to $(m_1 + m_2 + \cdots + m_n)$ and located at $\bar{x}$ as given in Eq. (123).

Exercise 2. Derive an equation analogous to (123) modified for the case in which the masses are on the y-axis and obtain

$$\bar{y} = \frac{y_1 m_1 + y_2 m_2 + \cdots + y_n m_n}{m_1 + m_2 + \cdots + m_n}.$$

It should be clear to the student that, if a system of point masses, m_1 at (x_1,y_1), m_2 at (x_2,y_2), $\ldots$, m_n at (x_n,y_n), is to balance at a point $(\bar{x},\bar{y})$, that the system should still balance after all points are projected upon each coordinate axis. Hence we obtain the equations

$$\begin{aligned} \bar{x} &= \frac{x_1 m_1 + x_2 m_2 + \cdots + x_n m_n}{m_1 + m_2 + \cdots + m_n}, \\ \bar{y} &= \frac{y_1 m_1 + y_2 m_2 + \cdots + y_n m_n}{m_1 + m_2 + \cdots + m_n}, \end{aligned} \tag{124}$$

locating the center of gravity of the system.

Similarly, in three dimensions, the coordinates, $(\bar{x},\bar{y},\bar{z})$, of the center of gravity of a system of point masses, $m_1, m_2, \ldots, m_n$, located, respectively, at the points (x_1,y_1,z_1), (x_2,y_2,z_2), $\ldots$, (x_n,y_n,z_n) are given by the equations

$$\begin{aligned} \bar{x} &= \frac{x_1 m_1 + x_2 m_2 + \cdots + x_n m_n}{m_1 + m_2 + \cdots + m_n}, \\ \bar{y} &= \frac{y_1 m_1 + y_2 m_2 + \cdots + y_n m_n}{m_1 + m_2 + \cdots + m_n}, \\ \bar{z} &= \frac{z_1 m_1 + z_2 m_2 + \cdots + z_n m_n}{m_1 + m_2 + \cdots + m_n} \end{aligned} \tag{125}$$

Since these sets of equations, locating the center of gravity of a system of point masses, are based on an assumed system of coordinates, it might at first seem that the result would depend, in some way, upon the choice of that system. That this is not the case, and that the position of the center of gravity depends only on the masses themselves and their positions relative to each other, we establish by means of the following exercises.

Exercise 3. If the axes are translated, with a new origin at (h,k), show that Eqs. (124), stated in the new coordinates, will define the same point as they do in the old coordinates.

Exercise 4. If the axes are rotated through an angle θ, show that Eqs. (124), stated in the new coordinates, will define the same point as they do in the old coordinates.

Exercise 5. If the axes are translated, with new axes at (h,k,l), show that Eqs. (125), stated in the new coordinates, will define the same point as they do in the old coordinates.

Exercise 6. If the axes are rotated, so that the new axes have as their direction cosines with respect to the old axes, l_1,m_1,n_1, l_2,m_2,n_2, l_3,m_3,n_3, show that Eqs. (125), stated in the new coordinates, will define the same point as they do in the old coordinates.

So far, we have dealt with the center of gravity of a system of point masses merely as the point about which the system will be balanced in the field of gravity. The point, however, has another interesting property, *viz.*, that the sum of the moments of the masses about any fixed line or plane is the same as the moment, about that line or plane, of a single mass located at the center of gravity and equal in magnitude to the sum of the masses of the system. The proof is left to the student in the following exercises.

Exercise 7. Prove that the sum of the moments of a set of point masses, located in a plane, about an arbitrary line in that plane, is equal to the moment about that line of a single mass equal to the sum of the masses of the set and located at their center of gravity.

Exercise 8. Prove that the sum of the moments of a set of point masses in space about an arbitrary plane is equal to the moment about that plane of a single mass equal in magnitude to the sum of the masses of the set and located at their center of gravity.

If now instead of dealing with a finite number of point masses we deal with an infinite number of particles, forming an arc of a curve, an area of a surface, or filling a portion of space, and desire to find their center of gravity we resort to the consideration of an *element* of mass whose shape or size is such that we may consider it as located at a point. Assuming first the entire mass divided into a finite number, say n, of such elements we may apply formulas (123), (124), or (125) to them and then extend the result in the usual fashion of the integral calculus, as n is allowed to become infinite. Thus, if the element of mass, $\Delta_i m$, has x_i as the abscissa of its center of gravity, we have, for the case of n elements of mass

$$\bar{x} = \frac{x_1\,\Delta_1 m + x_2\,\Delta_2 m + \cdots + x_n\,\Delta_n m}{\Delta_1 m + \Delta_2 m + \cdots + \Delta_n m} = \frac{\sum_{i=1}^{n} x_i\,\Delta_i m}{M},$$

where M is the total mass. Now, if we define the abscissa of the center of gravity of our configuration as the limiting value of the above as n becomes infinite and as each element of mass tends to

zero as a limit, we obtain

$$\bar{x} = \frac{\lim\limits_{\substack{n\to\infty \\ \Delta_i m\to 0}} \sum_{i=1}^{n} x_i \, \Delta_i m}{M} = \frac{\int x \, dm}{M}.$$

Also, in like manner, we derive

$$\bar{y} = \frac{\int y \, dm}{M},$$

$$\bar{z} = \frac{\int z \, dm}{M}$$

the limits in the integration being determined by the figure under consideration.

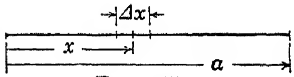

FIG. 155.

Illustration 1. Find the center of gravity of a homogeneous straight line of length a.

Solution. Assume an axis, say the x-axis, to coincide with the line, with the origin at one end of it. If the density of the line is δ, we have $\Delta m = \delta \cdot \Delta x$ and

$$\bar{x} = \frac{\lim\limits_{\substack{n\to\infty \\ \Delta_i x\to 0}} \sum_{i=1}^{n} \delta \cdot x_i \cdot \Delta_i x}{\delta a} = \frac{\int_0^a \delta \cdot x \cdot dx}{\delta a} = \frac{\left.\frac{\delta x^2}{2}\right]_0^a}{\delta a} = \frac{a}{2};$$

hence, the center of gravity of a homogeneous straight line is its midpoint.

Illustration 2. Find the center of gravity of a homogeneous circular disk of radius a.

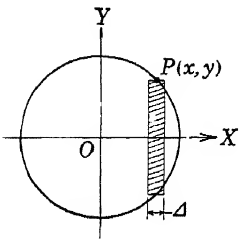

FIG. 156.

Solution. Assume as the coordinate axes two perpendicular diameters. The element of mass is now an element of area, $2y \cdot \Delta x$, multiplied by the density δ, and

$$\bar{x} = \frac{\int_{-a}^{a} \delta \cdot 2yx \, dx}{\delta \cdot a^2\pi} = \frac{\int_{-a}^{a} 2x\sqrt{a^2 - x^2} \, dx}{\pi a^2} = \frac{\left.-\frac{2}{3}\sqrt{(a^2 - x^2)^3}\,\right]_{-a}^{a}}{\pi a^2} = 0.$$

To find $\bar{y}$, we employ another element of area than the one just used, since y does not have a constant value throughout it. The element we choose now is parallel to OX, so that $\Delta m = \delta \cdot 2x \cdot \Delta y$ and

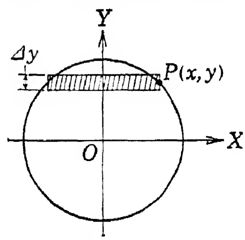

FIG. 157.

$$\bar{y} = \frac{\int_{-}^{a} \delta \cdot 2xy \, dy}{\delta \cdot \pi a^2} = \frac{\int_{-}^{a} 2y\sqrt{a^2 - y^2} \, dy}{\pi a^2} = 0.$$

Hence, the center of gravity of a homogeneous circular disk is its center.

Exercise 9. Prove that the center of gravity of a rectangle of uniform density is its center.

Exercise 10. Find the ordinate, $\bar{y}$, for the circle in Illustration 2 by using an element as shown in Fig. 156 and, since the element is a rectangle, using the center of the element as its center of gravity.

Problems

1. Find the center of gravity of the following:

(*a*) Masses 2, 1, 3, 5 placed, respectively, at $(0,1),(2,-3),(-4,1),(3,2)$. *Ans.* $(\frac{5}{11},\frac{12}{11})$.

(*b*) Five equal masses placed at $(2,0),(3,-1),(2,-4),(-4,3),(5,1)$. *Ans.* $(\frac{8}{5},-\frac{1}{5})$.

(*c*) Masses 3, 2, 4 placed, respectively, at $(0,2,1),(3,-1,2),(-2,3,-1)$.

(*d*) Three equal masses placed at $(0,0,3),(-2,-4,3),(4,1,-2)$.

2. (*a*) Given $p + q$ point masses, $m_1, m_2, \ldots, m_p, M_1, M_2, \ldots, M_q$ show that their center of gravity can be found by assuming one mass equal to $(m_1 + m_2 + \cdots + m_p)$, located at the center of gravity of the first p particles, and another mass equal to $(M_1 + M_2 + \cdots + M_q)$, situated at the center of gravity of the other q particles.

(*b*) Show that the center of gravity of an area consisting of two parts can be found by assuming each part to be concentrated at its center of gravity.

3. Show that the center of gravity (*a*) of a triangular area is the intersection of the medians of the triangle; (*b*) of a set of three equal point masses is the intersection of the medians of the triangle formed by joining the points.

4. Show that the center of gravity of a semicircular area is on the bisecting radius and at a distance from the center equal to $4/3\pi$ times the radius.

5. Show that if an area or volume has an axis of symmetry, that axis passes through the center of gravity.

6. A homogeneous area consists of a rectangle of dimensions $2a$ and $2b$, to which is adjoined a semicircle that has one of the sides of length $2a$ as a diameter. Find the center of gravity of the combined figure.

7. Find the center of gravity of

(*a*) The arc of the circle $x^2 + y^2 = a^2$ lying in the first quadrant.

HINT: $dm = \delta \cdot ds$ and $M = \int \delta\, ds = \dfrac{\pi a}{2}\delta$. *Ans.* $\bar{x} = \bar{y} = 2a/\pi$.

(*b*) The arc of the hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the first quadrant. *Ans.* $\bar{x} = \bar{y} = 2a/5$.

(*c*) The arc of the first arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$. *Ans.* $(\pi a, 4a/3)$.

(*d*) The arc of the catenary $y = a\cosh\dfrac{x}{a}$ between $x = 0$ and $x = 1$.

(*e*) The area bounded by the lines $x = 0$, $y = 3$, $y = 2x$. *Ans.* $(\frac{1}{2},2)$.

(*f*) The area bounded by $y^2 = 4ax$ and the line $x = a$. *Ans.* $(3a/5,0)$.

(*g*) The area bounded by $y^2 = 4ax$ and $x^2 = 4ay$.
Ans. $(9a/5, 9a/5)$.

(*h*) The area bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and to the right of OY.
Ans. $(4a/3\pi, 0)$.

(*i*) The area bounded by $x^2 = 16y$ and the line $4y = x + 8$ (find $\bar{x}$ only).

(*j*) The area bounded by $y = x^2 + 13$ and $y = 2x^2 + 4$.

(*k*) The area of the loop of the curve $x^2 = 5y^2 - y^3$.
Ans. $(0, {}^{20}\!/_7)$.

(*l*) The area bounded by the parabola $x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}$ and the coordinate axes. *Ans.* $(a/5, a/5)$.

(*m*) The area bounded by the cissoid $y^2(6 - x) = x^3$ and its asymptote, $x = 6$. *Ans.* $(5,0)$.

(*n*) The area bounded by the first arch of the cycloid of (*c*) and OX.
Ans. $(\pi a, 5a/6)$.

(*o*) The area in the second quadrant bounded by the axes and the hypocycloid of (*b*).

8. The density of a rod, of length a, at any point is proportional to the distance of that point from one end. Find the distance from that end to the center of gravity. *Ans.* $2a/3$.

9. The density of a semicircular area, of radius a, is proportional, at any point, to the distance from that point to the bounding diameter. Find the distance from that diameter to the center of gravity. *Ans.* $3\pi a/16$.

10. Find the center of gravity of

(*a*) A homogeneous right circular cylinder.

(*b*) A homogeneous hemisphere of radius a. *Ans.* $3a/8$ from base.

(*c*) That part of the volume inclosed by $x^2 + y^2 + z^2 = a^2$ and lying in the first octant.

(*d*) The volume above the xy-plane and inside the surface $x^2 + y^2 = 6 - z$. *Ans.* $(0,0,2)$.

(*e*) A hemisphere of radius a in which the density, at any point, is proportional to the distance from the bounding plane.
Ans. $8a/15$ from base.

(*f*) The volume obtained by revolving that portion of the line $\frac{x}{a} + \frac{y}{b} = 1$ included between the axes about the y-axis.

(*g*) The volume obtained by revolving about the line $x = 4$ the area bounded by that line and the curve $y = 2\sqrt{x}$.

11. Prove the two theorems of Pappus:

(*a*) *The surface generated by revolving an arc of a curve about a line in its plane that does not intersect the arc, equals the product of the length of the arc by the circumference described by its center of gravity.*

HINT: Assume the axis of revolution as the x-axis and recall that the area of the surface formed is, then $2\pi \int y \, ds$.

(*b*) *The volume generated by revolving a plane area about a line in its plane, that does not cross the area, equals the product of the area by the circumference described by its center of gravity.*

HINT: Assume the axis of revolution as the x-axis and recall that if a line parallel to OX crosses the area in two points (x_1,y) and (x_2,y) the volume of revolution formed is $\int 2\pi y(x_2 - x_1)dy$. (If the area is crossed by lines parallel to OX in more than two points, it can be broken up into separate parts which are crossed by such lines in not more than two points, and the argument may be applied to the several such parts.)

12. Using the theorems of Pappus, find

(*a*) The surface and the volume of the torus generated by revolving a circle of radius a about a line in its plane which is b units from its center $(b > a)$.

(*b*) The surface and volume generated by revolving a rectangle of dimensions a and b about a line in its plane and c units from its center, the line lying outside the rectangle.

(*c*) The volume generated by revolving the right-hand half of the area enclosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, about the line $x = c$ $(c > a)$.

Ans. $\pi^2 ab\,(c - 4a/3\pi)$.

13. An area consists of a rectangle of dimensions $2a$ and $2b$ $(b > a)$ from which a semicircle with one of the sides $2a$ as diameter has been removed. Find its center of gravity.

14. From a right circular cone of radius a and altitude h the volume of the inscribed sphere is removed. Find the center of gravity of the remaining volume.

15. Find the center of gravity of the larger area bounded by the parabola $y^2 = 3x$ and the circle $x^2 - 6x + y^2 = 0$. From that, by the theorem of Pappus, find the volume generated by revolving that area about the y-axis.

88. Moment of Inertia. The *moment of inertia* of a particle of mass m about a line is defined as the product of its mass by the square of its distance from the line. The moment of inertia of a system of particles about a line is defined as the sum of the moments of inertia of the several particles about the line. Thus, if the particles are of masses $m_1, m_2, \cdots, m_n$, situated at distances $d_1, d_2, \cdots, d_n$ from a line l, the moment of inertia of the system of particles about l is

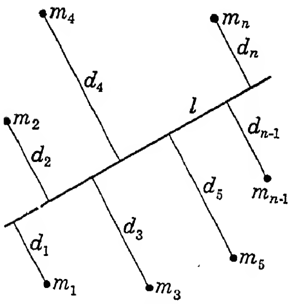

FIG. 158.

$$I_l = m_1 d_1^2 + m_2 d_2^2 + \cdots + m_n d_n^2 = \sum_{i=1}^{n} m_i d_i^2.$$

When dealing with a continuous physical body, whether shaped as an arc of a curve, an area, or a portion of volume, we consider,

in the customary manner of the integral calculus, *elements* of mass and their moments about the line specified, and define the *moment of inertia* of the body as the limit of the sum of the moments of the elements, as their number is made to become infinite, while their masses tend to zero. Thus, for a physical body,

$$I_l = \lim_{\substack{n \to \infty \\ \Delta_i m \to 0}} \sum_{i=1}^{n} d_i^2 \, \Delta_i m = \int d^2 \, dm,$$

the integrand to be expressed in terms of a suitable variable and the limits of integration suggested by the contour of the configuration.

Illustration. To find the moment of inertia of a homogeneous rod of density δ and length a about a perpendicular axis through one of its ends.

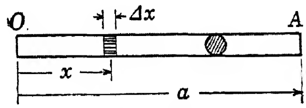

FIG. 159.

Solution. The element of mass is $\delta \cdot \Delta x$, the distance from the axis through O to a point at random in the interval Δx is x, and the moment of inertia, consequently is

$$I = \lim_{\substack{n \to \infty \\ \Delta_i x \to 0}} \sum_{i=1}^{n} \delta \cdot x_i^2 \, \Delta_i x = \delta \int_0^a x^2 \, dx = \frac{\delta a^3}{3}.$$

Let us now inquire at what point of the rod its entire mass would need to be concentrated, in order that its moment of inertia about the same axis be the same as it is with the mass spread uniformly from O to A. If the distance from O to that point is called r, we evidently have, representing the total mass of the rod by $M = \delta a$,

$$Mr^2 = \frac{\delta a^3}{3},$$

or

$$r^2 = \frac{\delta a^3}{3\delta a} = \frac{a^2}{3},$$

and

$$r = \frac{a}{\sqrt{3}}.$$

This length r is called the *radius of gyration* of the body, for the given axis, and, in any case, is defined by the equation

$$Mr^2 = I,$$

where I is the moment of inertia of the body about the line specified and M is its total mass.

NOTE: As a matter of notation, it is customary to designate the moment of inertia about the x-axis by I_x, the moment of inertia about the y-axis by I_y, the moment of inertia about a line l by I_l, etc.

Problems

1. Calculate the moment of inertia of the following systems:

(*a*) A system of three particles: one of mass 3, placed at (1,3); one of mass 2, placed at (5,0); and one of mass 1, placed at (3, −2); (I) about the x-axis; (II) about the y-axis; (III) about the line $5x + 2y = 11$.

Ans. 31, 62, $392/29$.

(*b*) A system of four particles, each of mass m, placed at the vertices of a square of side a: (I) about an axis through its center perpendicular to the plane of the square; (II) about such an axis through one of the vertices of the square; (III) about a diagonal of the square; (IV) about a side of the square. *Ans.* $2ma^2$, $4ma^2$, ma^2, $2ma^2$.

2. Calculate the moment of inertia and the radius of gyration of the following:

(*a*) A homogeneous rod of length a about a perpendicular axis through its center. *Ans.* $\delta a^3/12$, $a/\sqrt{12}$.

(*b*) An arc of a semicircle of radius a, (I) about the bounding diameter; (II) about the tangent parallel to the bounding diameter.

$$Ans.\quad \text{(I) } \pi a^3\delta/2,\ a/\sqrt{2};\ \text{(II) } a^3\delta\left(\frac{3\pi}{2} - 4\right),\ a\sqrt{\frac{3\pi - 8}{2\pi}}.$$

(*c*) A homogeneous wire bent in the form of an isosceles triangle, with base $2b$ and altitude h: (I) about the base; (II) about an axis at the vertex, perpendicular to the plane of the triangle.

$$Ans.\quad \text{(II) } \frac{2\delta}{3}(b^2 + h^2)^{3/2} + 2b\delta\left(\frac{b^2}{3} + h^2\right).$$

(*d*) A homogeneous wire bent in the form of a rectangle of sides $2a$ and $2b$: (I) about one of the sides of length $2b$; (II) about an axis through the center, perpendicular to the plane of the rectangle.

(*e*) A rod of length a whose density, at any point, is proportional to the square of the distance of that point from one end, about a perpendicular axis through that end. *Ans.* $r = a\sqrt{3/5}$.

3. Calculate the moment of inertia and the radius of gyration of the following:

(*a*) A homogeneous circular disk of radius a about a diameter.

Ans. $r = a/2$.

(*b*) A homogeneous rectangular area of dimensions $2a$ and $2b$: (I) about one of the sides of length $2b$; (II) about a line through its center, parallel to one of the sides of length $2b$.

Ans. (I) $r = 2a/\sqrt{3}$; (II) $r = a/\sqrt{3}$.

(*c*) A rectangular area, of dimensions $2a$ and $2b$, whose density at any point is half the distance of that point from one of the sides of length $2b$: (I) about that side; (II) about a line through the center, parallel to that side.

(*d*) An area in the form of an isosceles triangle, of base $2b$ and altitude h: (I) about the base; (II) about the altitude.

4. Prove the following theorem: *The moment of inertia of a mass about any line equals its moment of inertia about a parallel line through its center of gravity, increased by the product of the mass and the square of the distance between the two lines.*

HINT: If the distance from an element of mass Δm to the line specified, is x and the distance to the parallel line through the center of gravity is x', then $x^2 = x'^2 + a^2 \pm 2bx'$, the sign depending on whether the angle opposite x is acute or obtuse. Hence

$$\int x^2\,dm = \int x'^2\,dm + \int a^2\,dm \pm \int 2bx'\,dm.$$

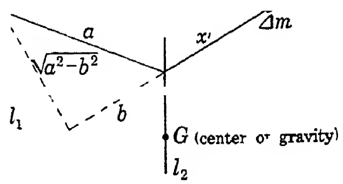

FIG. 160.

5. Apply the theorem of Prob. 4 to find the moment of inertia of

(*a*) A circular disk of radius a about a tangent [using the result of Prob. 3(*a*)]. *Ans.* $\dfrac{5\pi}{4}a^4\delta$.

(*b*) A homogeneous rod of length a about a perpendicular axis, a distance of $a/4$ from one end [using the result of Prob. 2(*a*)]. *Ans.* $\frac{7}{48}a^3\delta$.

(*c*) A homogeneous wire bent into the form of an isosceles triangle of base $2b$ and altitude h, about a line through its center of gravity parallel to the base [using the result of Prob. 2(*c*)].

6. Calculate the moment of inertia of the area

(*a*) Bounded by the parabola $x^2 = 4y$ and the line $y = 4$; about the x-axis. *Ans.* $\frac{1024}{7}\delta$.

(*b*) Bounded by the circle $x^2 - 4x + y^2 = 0$ and the line $x = y\sqrt{3}$; about the y-axis (upper area).

(*c*) Between the ellipse $x^2 + 4y^2 = 4$ and the circle $x^2 + y^2 = 1$: (I) about the x-axis; (II) about the y-axis; (III) about a common tangent to the two curves.

(*d*) Bounded by the x-axis and the arc of the cycloid

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta)$$

from $\theta = 0$ to $\theta = 2\pi$: (I) about the y-axis; (II) about the line $x = \pi a$.

(*e*) Bounded by the curves $y = \cos x$, $y = \sin x$, the y-axis, and lying in the first quadrant: (I) about the y-axis; (II) about the x-axis.

7. Calculate the moment of inertia of the following:

(*a*) A homogeneous sphere of radius a: (I) about a diameter; (II) about a tangent line. HINT: In (I) use as element of mass the mass of a cylindrical shell concentric with a diameter. Obtain (II) from (I) by the theorem of Exercise 4. *Ans.* (I) $r = a\sqrt{2/5}$; (II) $r = a\sqrt{7/5}$.

(*b*) A homogeneous right cylinder of radius a and altitude h: (I) about its axis; (II) about an element of the cylinder.

Ans. (I) $r = a/\sqrt{2}$; (II) $r = a\sqrt{3/2}$.

(*c*) A homogeneous right circular cone of radius of base a and altitude h: (I) about its axis; (II) about a line through the circumference of the base and parallel to the axis. *Ans.* (I) $r = a\sqrt{3/10}$; (II) $r = a\sqrt{13/10}$.

(*d*) That part of the paraboloid $x^2 + y^2 = 9 - z$ lying above the xy-plane and intercepted by the cylinder $x^2 + y^2 = 4$; about the z-axis.

Ans. $152\pi\delta/3$.

(*e*) The volume bounded by the surface $y^2 + z^2 = 3x$ and the plane $x = 3$; about the x-axis. *Ans.* $81\pi\delta/2$.

(*f*) The volume generated by revolving about the y-axis the area bounded by the curve $y = \log x$, the line $x = 3$, and the x-axis; about the y-axis.

(*g*) The volume generated by revolving about the x-axis the area between the parabola $y^2 = 4x$ and the line $y = x$; about the x-axis.

(*h*) The volume generated by revolving the right triangle of sides 3, $\sqrt{27}$, and 6 about the line in its plane passing through the vertex of the 60° angle parallel to the side opposite, the moment of inertia being also taken about that line.

89. Liquid Pressure. Let a plate be immersed vertically in a liquid of density ω. Desired, to compute the pressure exerted by the liquid upon it. We consider the pressure upon an element of area $ABCD$ of the plate, where AD and BC are parallel to the upper surface of the liquid. Now, if the depth of this element were uniform (*i.e.*, if it had a horizontal position), the pressure against it, by a principle of hydrostatics, would be equal to the weight of a column of liquid with $ABCD$ as base and its depth as altitude. It is also a principle of hydrostatics that the pressure exerted by a liquid at any point is the same in all directions.

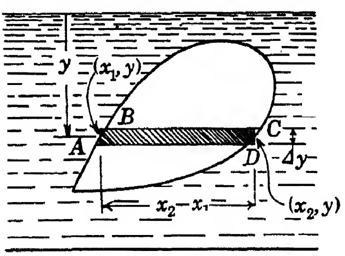

FIG. 161.

Hence, we may take the pressure on the element $ABCD$ as equal to $\omega \cdot$ (area $ABCD$) $\cdot\, y$, where y is some depth intermediate to the depth of BC and AD. The *element* of pressure, then, is $\omega \cdot (x_2 - x_1) \cdot y \cdot \Delta y$. The sum of all such elements of pressure is an approximation to the total pressure upon the plate, which is actually equal to the limit of that sum as the number of elements is increased indefinitely while the thickness of each element tends to zero. In other words, the pressure upon the plate is equal to

$$\lim_{\substack{n \to \infty \\ \Delta_i y \to 0}} \sum_{i=1}^{n} \omega \cdot (x_{2_i} - x_{1_i}) \cdot y_i \cdot \Delta_i y = \omega \int (x_2 - x_1) y \, dy.$$

The limits of integration, as well as the expressions for x_2 and x_1 in terms of y, are, of course, determined by the contour of the area immersed.

Illustration. To compute the pressure upon a circular disk of radius 5 ft. immersed vertically in water, with its center 7 ft. below the water level.

Solution. Assume axes as in Fig. 162, so that the y-axis passes through the center of the circle. Since the least and greatest values of y over the area are clearly 2 and 12, respectively, the pressure is represented by

$$P = \omega \int_2^{12} [x - (-x)]y\,dy = 2\omega \int^{12} xy\,dy.$$

From the equation, $x^2 + (y - 7)^2 = 25$, we have

$$x = \sqrt{25 - (y - 7)^2}$$
$$= \sqrt{-y^2 + 14y - 24},$$

whence, the pressure is

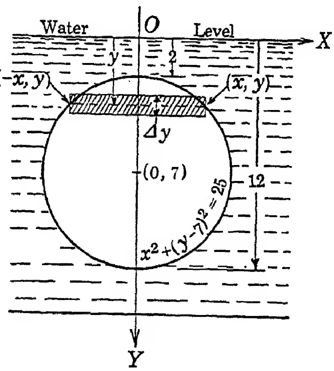

FIG. 162.

$$P = 2\omega \int_2^{12} y\sqrt{-y^2 + 14y - 24}\,dy$$
$$= 2\omega\left[\frac{(-y^2 + 14y - 24)^{3/2}}{-3} + 7\left(\frac{-2y + 14}{-4}\sqrt{-y^2 + 14y - 24} - \frac{25}{2}\sin^{-1}\frac{-2y + 14}{10}\right)\right]_2^{12}$$
$$175\pi\omega.$$

The dimensions here being in feet, and the weight of water approximately $\frac{1}{32}$ ton/cu. ft., we have $\omega = \frac{1}{32}$, and $P = (175\pi)/(32)$ tons.

Note that since the choice of axes is at our disposal we might have taken the origin at the center, obtaining a simpler equation for the circle. The student may do well to solve the problem over again, with that choice of axes.

Observe that the result in the above illustration equals $\frac{1}{32}(25\pi)(7)$, *i.e.*, the product of the density of the liquid, the area of the submerged surface, and the depth of the center of gravity under the liquid level. That this is in keeping with a general proposition, we ask the student to establish by

Exercise 1. Show that the pressure on any vertical plane surface submerged in a liquid is measured by the product of the area of the surface, the density of the liquid, and the depth under the liquid level of the center of gravity of the surface.

Problems

1. Compute the total pressure on the following:

(*a*) The upper half of the circular disk in the illustration of the text.

Ans. $\omega\left(\frac{175\pi}{2} - \frac{250}{3}\right)$.

(*b*) A rectangular plate 10 ft. high and 6 ft. wide, submerged in water, with the top of the plate at the water level. *Ans.* $\frac{75}{8}$ tons.

(*c*) A plate in the form of a right triangle, with a vertical leg of 8 ft. and a horizontal leg of 6 ft., submerged in water, with the top of the plate at the water level: (I) when submerged with the 6-ft. side up; (II) when submerged with the 6-ft. side down. *Ans.* (I) 2 tons; (II) 4 tons.

(*d*) The same plate as in (*c*), with the hypotenuse at the water level.

(e) A trapezoidal plate, with bases of 6 ft. and 12 ft., and altitude 4 ft. submerged vertically in water, with the 6-ft. base uppermost and 1 ft. below the water level. *Ans.* $2\frac{9}{8}$ tons.

(f) A plate in the form of a vertical isosceles triangle of base 12 ft. and altitude 8 ft.: (I) when submerged with the base uppermost and 2 ft. below the water level; (II) when submerged in water with the base lowermost and 11 ft. below the water level.

(g) The end of a water trough, 6 ft. wide at the top and 6 ft. deep, whose cross section parallel to the end is a parabola, if the water is 4 ft. deep at the deepest point. *Ans.* $4\sqrt{6}/15$ tons.

(h) The end of a semi-elliptical water trough, 6 ft. wide at the top and 4 ft. high, the trough containing 3 ft. of water at the deepest point.

2. Compute the total pressure on the following:

(a) The upper area bounded by the curve $y^2 = 8x - x^2$ and the x-axis, the y-axis assumed to be positive upward, if the level of the liquid is at $y = 6$. The dimensions are in feet and the weight of the liquid is 50 lb./cu. ft. *Ans.* $800(3\pi - \frac{8}{3})$ lb.

(b) The area bounded by the curve $2x = 5y - y^2$ and the line $x + y = 0$, the y-axis assumed positive upward and the level of the liquid at $y = 8$. The dimensions are in feet and the weight of the liquid is 50 lb./cu. ft. *Ans.* $6431\frac{1}{4}$ lb.

(c) The area bounded by the curves $3x + 2y^2 = 18$, $3x + 3y - y^2 = 0$, the y-axis assumed positive upward and the level of the liquid at $y = 5$, if the dimensions are in feet and the liquid is water.

3. An area in the form of a rectangle surmounted by a semicircle is immersed vertically in water with the top of the area 3 ft. below the water level. The radius of the semicircle is 2 ft. Find the height of the rectangle if the pressure upon it is the same as the pressure upon the semicircle.

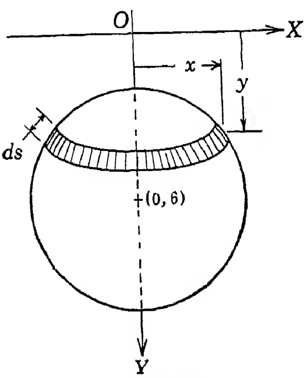

FIG. 163.

4. Find the pressure on a sphere 8 ft. in diameter, immersed in water, with center 6 ft. below the water level.

HINT: Take as element of pressure, the pressure on a zone of the sphere. Its area is $2\pi x \cdot ds$, and ds is found from the equation, $x^2 + (y - 6)^2 = 16$, of a great circle cut from the sphere by a vertical plane through the center. *Ans.* 12π tons.

5. (a) Find the total pressure on a cylinder of radius 4 ft. and altitude 8 ft., immersed in water, with its bases horizontal and the upper base 3 ft. below the water level. *Ans.* 21π tons.

(b) A cylinder is of the same dimensions and in the same position as in Prob. 5(a), but its upper half is immersed in liquid weighing 40 lb./cu. ft., while the liquid below is water. Find the total pressure upon it.

6. Find the pressure upon a right circular cone of radius of base 3 ft. and altitude 6 ft., immersed in water, with its base horizontal: (I) when the base

is at the water level; (II) when the vertex of the cone is at the water level.

Ans. (I) $\dfrac{9\pi\sqrt{5}}{16}$ tons.

7. A vessel is in the form of a paraboloid of revolution. Its width at the top is 8 in., and its height is 12 in. It is filled with water to within 2 in. of the top. Find the pressure on its surface.

90. Approximate Integration. The definite integrals that we have so far encountered have all been computed by first performing the quadrature, in conformity with the definition of $\int_a^b f(x)dx$ as $F(b) - F(a)$, where $F(x)$ is one of the values of $\int f(x)dx$. It may happen, however, that the quadrature cannot be carried out, as is the case with $\int\sqrt{(1 - x^2)(1 - 4x^2)}dx$, *i.e.*, no function $F(x)$ of the type familiar to the student—algebraic, trigonometric, exponential, etc.—can be found such that $F'(x) = \sqrt{(1 - x^2)(1 - 4x^2)}$. Should the same integrand confront us, however, in a *definite integral*, *e.g.*, $\int_0^{1/2}\sqrt{(1 - x^2)(1 - 4x^2)}dx$, we shall find means for finding its value to any desired degree of accuracy by approximation,* by methods developed below.

We show that an approximation to the value of $\int_a^b f(x)dx$ [the continuity of $f(x)$ in the interval of integration being understood] can be found in any one of the expressions:

I. $$\frac{\Delta x}{2}\Big[f(a) + 2f(a + \Delta x) + 2f(a + 2\,\Delta x) + \cdots + 2f(a + \overline{n - 1}\,\Delta x) + f(b)\Big]. \qquad \left(\Delta x = \frac{b - a}{n}.\right)$$

II. $$\frac{b - a}{6}\left[f(a) + 4f\left(\frac{a + b}{2}\right) + f(b)\right].$$

III. $$\frac{\Delta x}{3}\Big[f(a) + 4f(a + \Delta x) + 2f(a + 2\,\Delta x) + 4f(a + 3\,\Delta x) + 2f(a + 4\,\Delta x) + \cdots + 2f(a + \overline{n - 2}\,\Delta x) + 4f(a + \overline{n - 1}\,\Delta x) + f(b)\Big]. \qquad \left(\Delta x = \frac{b - a}{n}, \text{ and } n \text{ even.}\right)$$

* It may be remarked that some definite integrals can even be computed precisely, though the corresponding indefinite integral cannot be found, by methods that we have no space for in this book. For example, $\int_0^\infty \cos x^2\,dx$ is found to be equal to $\frac{1}{2}\sqrt{\frac{\pi}{2}}$, though the quadrature $\int \cos x^2\,dx$ cannot be performed.

The rules giving these expressions as approximations to $\int_a^b f(x)dx$ are called, respectively, the *trapezoidal rule*, the *prismoidal formula*, and *Simpson's rule*. Many other rules for the approximation of definite integrals have been developed, of which we shall presently give two, *Euler's formula* and *Gauss's formula*.

The first of the above rules, *i.e.*, the trapezoidal rule, amounts to assuming that the area bounded by the curve $y = f(x)$, the x-axis, and the ordinates $x = a$ and $x = b$, [which area, of course, is measured by the precise value of $\int_a^b f(x)dx$] is replaced by a finite number of trapezoids of altitude Δx and bases $f(a), f(a + \Delta x)$, $f(a + 2\Delta x), \cdots, f(a + \overline{n-1}\,\Delta x)$, and $f(b)$, where all bases but $f(a)$ and $f(b)$ are common to two adjacent trapezoids, as shown in Fig. 164. The total area of the several trapezoids is, then,

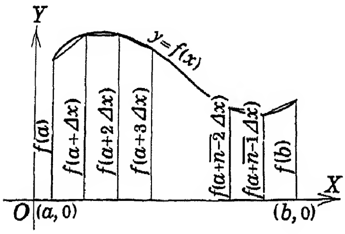

FIG. 164.

$$\frac{\Delta x}{2}\Big[\overline{f(a) + f(a + \Delta x)} + \overline{f(a + \Delta x) + f(a + 2\,\Delta x)} + \cdots$$

$$+ \overline{f(a + \overline{n-2}\,\Delta x) + f(a + \overline{n-1}\,\Delta x)} + \overline{f(a + \overline{n-1}\,\Delta x) + f(b)}\Big],$$

whence the rule.

The second of the above approximations, *i.e.*, the prismoidal formula, may be arrived at by passing through the three points $[a,f(a)]$, $\left[\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right]$, $[b,f(b)]$, of the curve $y = f(x)$, a curve whose equation has the form $y = R + Sx + Tx^2 = g(x)$. That this is always possible may be readily shown by the student, which he is requested to do in Exercise 1 below. The curve $y = g(x)$ intersects the curve $y = f(x)$ at $x = a$, $x = \frac{a+b}{2}$, and $x = b$, and to obtain the prismoidal formula, we simply employ $g(x)$ as an approximation for $f(x)$. By direct integration it can be shown that $\int_a^b g(x)\,dx$ is equal to

$$b - a\Big[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)$$

and the student is requested to make the demonstration in Exercise 2 below. Hence, replacing $\int_a^b f(x)dx$ by $\int_a^b g(x)dx$ and noting the equalities $g(a) = f(a)$, $g\left(\frac{a+b}{2}\right) = f\left(\frac{a+b}{2}\right)$, and $g(b) = f(b)$, we have the desired result.

Exercise 1. Prove that constants R, S, and T can always be found so that the curve $y = R + Sx + Tx^2$ shall pass through the three points $[a,f(a)]$, $\left[\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right]$, $[b,f(b)]$.

Exercise 2. Prove that $\int_a^b g(x)dx = \frac{b-a}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right]$, if $g(x)$ is a function of the form $R + Sx + Tx^2$, where R, S, and T are constants.

The approximation netted by the prismoidal formula will naturally be better the smaller the value of $b - a$. When $b - a$ is large, it is, therefore, well to divide it into smaller parts and apply the rule to each part by noting the relation

$$\int_a^c f(x)dx + \int_c^d f(x)dx + \cdots + \int_l^b f(x)dx = \int_a^b f(x)dx,$$

where (in this case) $a < c < d < \cdots < l < b$.

In particular, if we divide the interval $b - a$ into an *even* number of *equal* parts of length Δx and apply the prismoidal formula to each successive double interval (*i.e.*, of length $2\,\Delta x$), we obtain, as an approximation to $\int_a^b f(x)dx$

$$\begin{aligned}
&\frac{2\,\Delta x}{6}\Big[f(a) + 4f(a + \Delta x) + f(a + 2\,\Delta x)\Big] + \frac{2\,\Delta x}{6}\Big[f(a + 2\,\Delta x) + \\
&\qquad 4f(a + 3\,\Delta x) + f(a + 4\,\Delta x)\Big] + \cdots + \\
&\qquad 2\,\Delta x\Big[f(a + \overline{n-2}\,\Delta x) + 4f(a + \overline{n-1}\,\Delta x) + f(b)\Big] = \\
&\qquad \frac{\Delta x}{3}\Big[f(a) + 4f(a + \Delta x) + 2f(a + 2\,\Delta x) \\
&\qquad + 4f(a + 3\,\Delta x) + \cdots + 2f(a + \overline{n-2}\,\Delta x) \\
&\qquad + 4f(a + \overline{n-1}\,\Delta x) + f(b)
\end{aligned}$$

which is Simpson's rule.

Exercise 3. From the mode of their derivation the prismoidal formula and Simpson's rule will give precise values for $\int_a^b f(x)dx$ in case

$$f(x) = R + Sx + Tx^2.$$

Show that the prismoidal formula, and hence, Simpson's rule, will also give the precise value in case $f(x) = R + Sx + Tx^2 + Ux^3$, where R, S, T, and U are constants. HINT: It is sufficient to make the proof for the case $f(x) = x^3$.

As an illustration of the various methods of approximation, we consider an integral which we can evaluate directly, for the purpose of appraising the accuracy of each. The integral is

$$\int_1^2 \log x\,dx = x\log x - x\Big]_1^2 = 2\log 2 - 1 = \log 4 - 1 = 0.38629,$$

correct to 5 places.

By the trapezoidal rule with $\Delta x = 0.2$, we have

x	$f(x)$	$f(x)$
1.0	0	
1.2		0.18232
1.4		0.33647
1.6		0.47000
1.8		0.58779
2.0	0.69315	
Sum	0.69315	1.57658

Whence $T = \frac{0.2}{2}[0.69315 + 2(1.57658)]$
$= 0.38463$ to 5 places.

By Simpson's rule with $\Delta x = 0.25$, we have

x	$f(x)$	$f(x)$	$f(x)$
1.0	0		
1.25		0.22314	
1.5			0.40547
1.75		0.55962	
2.0	0.69315		
Sum	0.69315	0.78276	0.40547

Whence $S = \frac{0.25}{3}[0.69315 + 4(.78276) + 2(0.40547)]$
$= 0.38626$ to 5 places.

The prismoidal formula gives

$$\frac{2-1}{6}[\log 1 + 4\log(1.5) + \log 2] = \frac{1}{6}[0 + 4(0.40547) + (0.69315)] = 0.38584$$ as a value for the integral.

By the trapezoidal rule with $\Delta x = 0.1$, we have

x	$f(x)$	$f(x)$
1.0	0	
1.1		0.09531
1.2		0.18232
1.3		0.26236
1.4		0.33647
1.5		0.40547
1.6		0.47000
1.7		0.53063
1.8		0.58779
1.9		0.64185
2.0	0.69315	
Sum	0.69315	3.51220

Whence $T = \frac{0.1}{2}[0.69315 + 2(3.51220)] = 0.38588$ to 5 places.

By Simpson's rule with $\Delta x = 0.1$, we have

x	$f(x)$	$f(x)$	$f(x)$
1.0	0		
1.1		0.09531	
1.2			0.18232
1.3		0.26236	
1.4			0.33647
1.5		0.40547	
1.6			0.47000
1.7		0.53063	
1.8			0.58779
1.9		0.64185	
2.0	0.69315		
Sum	0.69315	1.93562	1.57658

Whence $S = \frac{0.1}{3}[0.69315 + 4(1.93562) + 2(1.57658)] = 0.38629$ to 5 places.

It is seen, as was to be expected, that both the trapezoidal rule and Simpson's rule improve in the quality of approximation as Δx is made smaller, and that Simpson's rule, for the same Δx, gives a closer result than does the trapezoidal rule. Such will generally be the case.

From the mean value theorem for integrals (page 291), we have

$$\int_a^b f(x)dx = (b - a)f(\xi),$$

where ξ is some value for x between a and b. It appears, thus, that if we could find, or approximate to, the value of ξ, we could evaluate $\int_a^b f(x)dx$ by computing $f(x)$ for a single value of x

between a and b. Should we take as an approximation to ξ the number midway between a and b, we should have, as an approximation for $\int_a^b f(x)dx$

$$(b - a)f\left(\frac{a + b}{2}\right). \tag{126}$$

Exercise 4. Show that the expression (126) represents the precise value of $\int_a^b f(x)dx$ if $f(x)$ is a linear function of x.

Since, now, we can obtain the precise value for the integral of a linear function by the use of just one value of the function, it is plausible to expect that we should be able to obtain the precise value of the integral of higher degree polynomials by the use of more values of that polynomial. In fact, *Gauss's formula* for approximation to a definite integral is an answer to the following question: For a given number, n, of partitions of the interval from a to b into subintervals, $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$ in each of which is to be selected a value of x, say $x_1, x_2, \ldots, x_n$, how shall these selections be made in order that

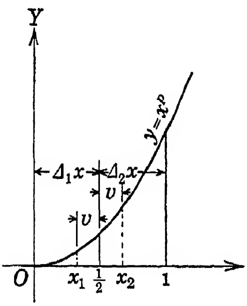

FIG. 165.

$$f(x_1)\Delta_1 x + f(x_2)\Delta_2 x + \cdots + f(x_n)\Delta_n x \tag{127}$$

shall represent $\int_a^b x^p\, dx$ precisely for $p = 0, 1, 2, \ldots, m$, and m to be as large as possible?

For uniformity of treatment, recognize that the integral $\int_a^b f(x)dx$ may be transformed by the substitution $x = a + (b - a)u$ into the integral $\int_0^1 g(u)du = \int_0^1 g(x)dx$. Let us assume that the subintervals $\Delta_i x$ and the x_i selected are symmetrically situated with respect to the midpoint, $x = \frac{1}{2}$, of the interval from 0 to 1.

Taking $n = 2$, then, we have $\Delta_1 x = \Delta_2 x = \frac{1}{2}$, $x_1 = \frac{1}{2} - v$, $x_2 = \frac{1}{2} + v$ (see Fig. 165), and we wish to satisfy as many as possible of the equations

$$x_1^0\,\Delta_1 x + x_2^0\,\Delta_2 x = (\tfrac{1}{2} - v)^0 \cdot (\tfrac{1}{2}) + (\tfrac{1}{2} + v)^0 \cdot (\tfrac{1}{2}) = \int_0^1 x^0\, dx = 1,$$

$$x_1\,\Delta_1 x + x_2\,\Delta_2 x = (\tfrac{1}{2} - v) \cdot (\tfrac{1}{2}) + (\tfrac{1}{2} + v) \cdot (\tfrac{1}{2}) = \int_0^1 x\, dx = \tfrac{1}{2},$$

$$x_1^2\,\Delta_1 x + x_2^2\,\Delta_2 x = (\tfrac{1}{2} - v)^2 \cdot (\tfrac{1}{2}) + (\tfrac{1}{2} + v)^2 \cdot (\tfrac{1}{2}) = \int_0^1 x^2\,dx = \tfrac{1}{3},$$

$$x_1^p\,\Delta_1 x + x_2^p\,\Delta_2 x = \left(\frac{1}{2} - v\right)^p \cdot \left(\frac{1}{2}\right) + \left(\frac{1}{2} + v\right)^p \cdot \left(\frac{1}{2}\right) = \int_0^1 x^p\,dx = \frac{1}{p+1},$$

The first two of these equations are identities, the third and fourth reduce to $12v^2 = 1$, while the fifth reduces to $80v^4 + 120v^2 - 11 = 0$, which is not satisfied by $v = \pm \dfrac{1}{2\sqrt{3}}$. Thus, four of these equations are satisfied by $x_1 = \dfrac{1}{2} - \dfrac{1}{2\sqrt{3}}$, $x_2 = \dfrac{1}{2} + \dfrac{1}{2\sqrt{3}}$, and $\int_0^1 g(x)dx$ is precisely evaluated by (127) if $g(x)$ is a polynomial of degree 3 or less.

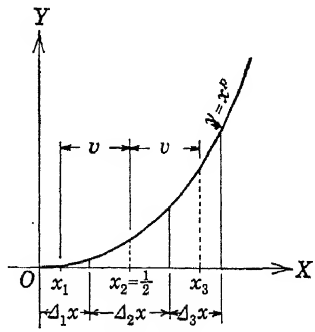

FIG. 166.

Taking, next, $n = 3$, we have (see Fig. 166)

$$x_1 = \tfrac{1}{2} - v,\ x_2 = \tfrac{1}{2},\ x_3 = \tfrac{1}{2} + v,\ \Delta_1 x = \Delta_3 x,$$

and we wish to satisfy the equation

$$x_1^p\,\Delta_1 x + x_2^p\,\Delta_2 x + x_3^p\,\Delta_3 x \quad \frac{1}{2} - v\,)\,\Delta_1 x + \left(\frac{1}{2}\right)^p \Delta_2 x + (\,\frac{1}{2} + v\,)\,\Delta_1 x = \int_0^1 x^p\,dx = \frac{1}{p+1},$$

for p equal to as many as possible of the consecutive values, $0, 1, 2, 3, \ldots$. This can be done as far as $p = 5$, the solutions being $\Delta_1 x = \Delta_3 x = \tfrac{5}{18}$, $\Delta_2 x = \tfrac{4}{9}$, $v = \tfrac{1}{2}\sqrt{\tfrac{3}{5}}$, whence $x_1 = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{\tfrac{3}{5}}$ and $x_3 = \tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{3}{5}}$.

Continuing in this way, setting $n = 4$, $n = 5$, etc., we find in general that

$$x_1^p\,\Delta_1 x + x_2^p\,\Delta_2 x + \cdots + x_n^p\,\Delta_n x = \int_0^1 x^p\,dx = \frac{1}{p+1},$$

the equation holding precisely for $n = 0, 1, 2, \ldots, 2n - 1$, and the values of the x_i and the $\Delta_i x$ being given in the table

Number of intervals	Values of x_i	Values of $\Delta_i x$
$n = 1$	$x_1 = 0.5$	$\Delta_1 x = 1$
$n = 2$	$x_1 = 0.21132487$ $x_2 = 0.78867513$	$\Delta_1 x = 0.5$ $\Delta_2 x = 0.5$
$n = 3$	$x_1 = 0.11270167$ $x_2 = 0.5$ $x_3 = 0.88729833$	$\Delta_1 x = 0.27777778$ $\Delta_2 x = 0.44444444$ $\Delta_3 x = 0.27777778$
$n = 4$	$x_1 = 0.06943184$ $x_2 = 0.33000948$ $x_3 = 0.66999052$ $x_4 = 0.93056816$	$\Delta_1 x = 0.17392742$ $\Delta_2 x = 0.32607258$ $\Delta_3 x = 0.32607258$ $\Delta_4 x = 0.17392742$
$n = 5$	$x_1 = 0.04691008$ $x_2 = 0.23076534$ $x_3 = 0.5$ $x_4 = 0.76923466$ $x_5 = 0.95308992$	$\Delta_1 x = 0.11846344$ $\Delta_2 x = 0.23931434$ $\Delta_3 x = 0.28444444$ $\Delta_4 x = 0.23931434$ $\Delta_5 x = 0.11846344$
$n = 6$	$x_1 = 0.03376524$ $x_2 = 0.16939531$ $x_3 = 0.38069041$ $x_4 = 0.61930959$ $x_5 = 0.83060469$ $x_6 = 0.96623476$	$\Delta_1 x = 0.08566225$ $\Delta_2 x = 0.18038079$ $\Delta_3 x = 0.23395697$ $\Delta_4 x = 0.23395697$ $\Delta_5 x = 0.18038079$ $\Delta_6 x = 0.08566225$

Since Gauss's formula gives the precise* value of $\int_a^b f(x)dx$ when $f(x)$ is a polynomial of degree $2n - 1$ or less, it serves excellently as an approximation to the integral when $f(x)$ is continuous and has continuous derivatives, an approximation, moreover, which is highly accurate for the number of values of $f(x)$ which it employs.

To approximate by Gauss's formula the integral, $\int_1^2 \log x\, dx$, previously evaluated by other methods, we set $x = 1 + u$ and obtain

$$\int_1^2 \log x\, dx = \int_0^1 \log\,(1 + u)du = \int_0^1 \log\,(1 + x)dx.$$

* The value is precise theoretically. Actually, the x_i and the $\Delta_i x$ are displayed, correct only to 8 decimal places, and the resulting value of the integral is an approximation to that extent.

With $n = 1$, we have the approximation $1 \cdot \log (1.5) = 0.40547$ With $n = 2$, we obtain $\frac{1}{2}[\log (1.211325) + \log (1.788675)] = \frac{1}{2}[(0.19171) + (0.58148)] = 0.38659$. With $n = 3$. we obtain $(0.27777778)[\log (1.11270167) + \log (1.88729833)] + (0.44444444) \log (1.5) = (0.27777778)[(0.10678) + (0.63515)] + (0.44444444) (.40547) = 0.38630$.

In this illustration we see that Gauss's formula is practically as accurate with three intervals as Simpson's rule is with ten. However, on account of the irrationality of the numbers concerned and correspondingly large number of decimals carried, the labor is possibly greater than that necessary to obtain the same accuracy by Simpson's rule. Gauss's formula should certainly be used in case the following three conditions occur, when it is of tremendous practical value.

1. *The values of $f(x)$ are determined experimentally.*

2. *The readings for the determination of $f(x)$ may be taken to correspond to any chosen values of x.*

3. *It is desired that the total number of readings be kept small, for the sake of convenience or economy.*

We now adduce, without discussion, a formula called *Euler's formula* by some and the *Euler-Maclaurin formula* by others, which states that successively better approximations to the value of $\int_a^b f(x)dx$ are given by $A, B, C, D, \ldots$, where

$$A = \frac{\Delta x}{2}[f(a) + 2f(a + \Delta x) + 2f(a + 2\Delta x) + \cdots + 2f(a + \overline{n-1}\,\Delta x) + f(b)],$$

the value by the trapezoidal rule,

$$B = A - \frac{\overline{\Delta x}^2}{12}[f'(b) - f'(a)],$$

$$C = B + \frac{\overline{\Delta x}^4}{720}[f'''(b) - f'''(a)],$$

$$D = C - \frac{\overline{\Delta x}^6}{30240}[f^{v}(b) - f^{v}(a)], \text{ etc.}$$

(The denominators under $\overline{\Delta x}^2, \overline{\Delta x}^4, \overline{\Delta x}^6, \cdots$ are values of $-\frac{1}{2!}B_1$, $+\frac{1}{4!}B_2$, $-\frac{1}{6!}B_3$, etc., where B_1, B_2, B_3, etc., are the so-called *Bernoullian numbers* and are $B_1 = 1/6$, $B_2 = 1/30$, $B_3 = 1/42$, $B_4 = 1/30$, $B_5 = 5/66$, etc.).

If we consider the value B for the illustration above, with $\Delta x = 0.2$ and, hence, $A = 0.38463$, we get, since $f(x) = \log x$ and $f'(x) = 1/x$,

$$B = A - \frac{0.04}{12}\left[\frac{1}{2} - 1\right] = 0.38463 + 0.00167 = 0.38630.$$

Again, if we consider the value of B with $\Delta x = 0.1$ and $A = 0.38588$, we obtain

$$B = A - \frac{0.01}{12}\left[\frac{1}{2} - 1\right] = 0.38588 + 0.00042 = 0.38630.$$

Both of these results agree with the actual value of the integral in the fourth decimal place.

Problems

1. Compute each of the following, (I) by the trapezoidal rule; (II) by Euler's formula as far as approximation B; (III) by the prismoidal formula; (IV) by Simpson's rule:

(a) $\int_0^1 \sqrt{1 + x^3}\, dx$, with $\Delta x = 0.1$.

(b) $\int_0^{\pi/2} \sqrt{\sin x}\, dx$, with $\Delta x = \pi/8$.

(c) $\int_3^5 \sqrt[3]{x^2 - 9}\, dx$, with $\Delta x = \frac{1}{4}$.

(d) $\int_0^1 \frac{dx}{\sqrt{4 + x^3}}$, with $\Delta x = 0.1$.

(e) $\int_0^{\pi/3} \log_{10} \cos x\, dx$, with $\Delta x = \pi/12$.

(f) $\int_0^1 e^{x^2}\, dx$, with $\Delta x = 0.1$.

(g) $\int_0^{\pi} \sqrt{1 + \sin x}\, dx$, with $\Delta x = \pi/8$.

(h) $\int_{\pi/6}^{\pi/2} \frac{\cos x\, dx}{\sqrt{x}}$, with $\Delta x = \pi/12$.

2. Compute the value of the integral of Prob. 1(f) by Gauss's formula for the cases $n = 2$; $n = 3$, and $n = 4$.

3. Compute the value of the integral of Prob. 1(e) by Gauss's formula for the cases $n = 2$, $n = 3$, and $n = 4$.

4. Find the length of the ellipse $x = 4 \cos \varphi$, $y = 2\sqrt{2} \sin \varphi$. (See Prob. 4, page 304.) *Ans.* 21.610.

5. Find the length of the loop of the curve $y^2 = x^2(1 - x)$.

6. Find the length of the hyperbola $x^2 - 2y^2 = 1$ from (1,0) to (3,2).

7. By a suitable approximation formula, find the area bounded by the curve $y = f(x)$, the x-axis, the y-axis, and the ordinate at $x = 8$, given that

$f(0) = 3, f(2) = 4, f(4) = 4.5, f(5.5) = 4.75, f(7) = 4.25, f(7.5) = 4.2$, and $f(8) = 4$. *Ans.* $33\frac{43}{60}$ (by prismoidal formula).

8. Show that the trapezoidal rule can be put in the form

$$S = f(a) + f(a + \Delta x) + f(a + 2\Delta x) + \cdots + f(b) = \frac{1}{\Delta x}\int_a^b f(x)dx + \frac{1}{2}[f(a) + f(b)],$$

and that the successive approximations by the Euler formula give

$$S = \frac{1}{\Delta x}\int_a^b f(x)dx + \frac{1}{2}[f(a) + f(b)],$$

$$S = \frac{1}{\Delta x}\int_a^b f(x)dx + \frac{1}{2}[f(a) + f(b)] + \frac{\Delta x}{12}[f'(b) - f'(a)],$$

$$S = \frac{1}{\Delta x}\int_a^b f(x)dx + \frac{1}{2}[f(a) + f(b)] + \frac{\Delta x}{12}[f'(b) - f'(a)] - \frac{\overline{\Delta x}^3}{720}[f'''(b) - f'''(a)], \text{ etc.}$$

9. Using the result of Prob. 8, find the value of the sum

(*a*) $\sin 0° + \sin 11° 15' + \sin 22° 30' + \sin 33° 45' + \cdots + \sin 180°$, to 5 decimal places;

(*b*) $e^0 + e^{.1} + e^{.2} + e^{.3} + \cdots + e^{1.9} + e^2$, to 4 decimal places; *Ans.* 68.1383.

(*c*) $\frac{1}{20^2} + \frac{1}{21^2} + \frac{1}{22^2} + \cdots + \frac{1}{39^2} + \frac{1}{40^2}$ to 4 decimal places.

CHAPTER XII

PARTIAL DERIVATIVES

91. Functions of More than One Variable. We consider, in this chapter, some properties of a function that depends on two or more independent variables. First, a

Definition. A variable u is said to be a function of x, y, z, $\cdots$, in some region, $x_1 \leqq x \leqq x_2$, $y_1 \leqq y \leqq y_2$, $z_1 \leqq z \leqq z_2$, $\ldots$, if every set of values $(x,y,z, \ldots)$ in the region defines a corresponding value, or values, of u.

If u takes on but one value for each set $(x,y,z, \ldots)$, it is said to be a *single-valued* function. Unless specified to the contrary, we shall concern ourselves only with such.

The function $u = f(x,y,z, \cdots)$ is said to be *continuous* at $x = a$, $y = b$, $z = c$, $\cdots$ if

(I) u is defined for $x = a$, $y = b$, $z = c$, $\cdots$.

(II) u approaches a limit u_1 as $x \to a$, $y \to b$, $z \to c$, $\cdots$.

(III) $u_1 = f(a,b,c, \cdots)$,

in other words, if

$$\lim_{\substack{x \to a \\ y \to b \\ z \to c}} f(x,y,z, \cdots) = f(a,b,c, \cdots).$$

As is the case of a function of one independent variable, the above definition is equivalent to the following: $u = f(x,y,z, \cdots)$ is said to be continuous at $(x = a, y = b, z = c, \cdots)$ if, given a positive quantity, ϵ, another positive quantity, δ, can be found, such that $|u - u_1| < \epsilon$ when $|x - a| < \delta$, $|y - b| < \delta$, $|z - c| < \delta$, $\cdots$, where $u_1 = f(a,b,c, \cdots)$.

To illustrate, consider the function u of three variables, $u = x^2 + 2y - z^3$. For any set of finite values, $x = a$, $y = b$, $z = c$, u is obviously defined and has the value $a^2 + 2b - c^3$. Again, as $x \to a$, $y \to b$, $z \to c$, $x^2 \to a^2$, $2y \to 2b$, $z^3 \to c^3$ (for x^2, $2y$ and z^3 are, obviously, continuous functions of x, y, z, respectively) and hence the sum $(x^2 + 2y - z^3) \to (a^2 + 2b - c^3)$. In other words, we have

$$\lim_{\substack{x \to a \\ y \to b \\ z \to c}} (x^2 + 2y - z^3) = \text{value of } (x^2 + 2y - z^3). \atop \substack{\text{at } x = a \\ y = b \\ z = c}$$

This function is, then, continuous at any set $(x = a,\ y = b,\ z = c)$, in the light of the first definition of continuity given above.

As another example, consider the function u of two variables, $u = \dfrac{xy + x^2}{x^2 + y^2}$. This function is, evidently, not defined for $x = 0$, $y = 0$. For all other sets $(x = a, y = b)$ it is defined as $\dfrac{ab + a^2}{a^2 + b^2}$ and is continuous. Now, if this function approached some limit as $x \to 0$, $y \to 0$, we could assign to the function the value of that limit at $(x = 0, y = 0)$, and the function would then be continuous for that set. Notice, however, that if (x,y) is made to approach $(0,0)$ through sequences of sets of values for which $y = 2x$, u takes on values equal to $\dfrac{2x^2 + x^2}{x^2 + 4x^2} = \dfrac{3}{5}$ and approaches $\frac{3}{5}$ as a limit as x and y both approach zero, while if (x,y) is made to approach $(0,0)$ through sequences of sets of values for which $y = 3x$, u takes on values equal to $\dfrac{3x^2 + x^2}{x^2 + 9x^2} = \dfrac{2}{5}$ and approaches $\frac{2}{5}$ as a limit as x and y now approach zero. Clearly, the function does not approach a definite limit as x and y both approach zero, and hence cannot be made continuous at $(x = 0, y = 0)$.

Problems

1. State for what sets of values of the independent variables, if any, the following functions are discontinuous:

(a) $u = x^2 + yz$.

(b) $u = e^{x+y} + \sin(xyz)$.

(c) $u = \dfrac{y + z}{x}$.

(d) $u = \dfrac{x}{y - x}$

2. Find what limit is approached by the function $u = \dfrac{xy^2 + x^3}{x^3 + 2y^3}$, as x and y both approach zero through a sequence of sets of values for which (a) $y = x$. (b) $y = -x$.

92. Partial Derivatives. Given a function z of two variables, $z = f(x,y)$, having the value z_0 at (x_0,y_0), let us consider x as a variable and equal to $x_0 + \Delta x$, where Δx is an increment, while y is held constant at y_0. If Δz is the increment in z, corresponding to the increment Δx of x, we have

$$z_0 + \Delta z = f(x_0 + \Delta x, y_0).$$

The limit

$$\lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x},$$

if it exists, is called the *partial derivative of z with respect to x at* (x_0,y_0).

Likewise, if x is held constant and equal to x_0, while y varies and is represented by $y_0 + \Delta y$, we define

$$\lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y},$$

provided that limit exists, as the *partial derivative of z with respect to y at* (x_0,y_0).

In like manner, a function of any number of independent variables can be differentiated *partially* with respect to any one of those variables. Thus if $v = f(x,y,z,u)$ and $v_0 = f(x_0,y_0,z_0,u_0)$, we define the partial derivative of v with respect to u at (x_0,y_0,z_0,u_0) as

$$\lim_{\Delta u \to 0} \frac{\Delta v}{\Delta u} = \lim_{\Delta u \to 0} \frac{f(x_0,y_0,z_0,u_0 + \Delta u) - f(x_0,y_0,z_0,u_0)}{\Delta u},$$

if the limit exists.

Since the partial derivatives so obtained clearly depend for their value, in each case, on the values of the independent variables, $x_0,y_0,z_0, \ldots$ for which they are computed, we may regard them as functions of those variables and, thus, speak of the partial derivative, say of $f(x,y)$ with respect to x for any set of values (x,y), and designate it by $f_x(x,y)$. It is also designated by the notation $\frac{\partial f(x,y)}{\partial x}$. Likewise, the notations $f_y(x,y)$ and $\frac{\partial f(x,y)}{\partial y}$ are used to represent the partial derivative of $f(x,y)$ with respect to y, for any set of values (x,y), and the symbols $f_u(x,y,z,u)$ or $\frac{\partial f(x,y,z,u)}{\partial u}$ are used to represent the partial derivative of $f(x,y,z,u)$ with respect to u for any set of values (x,y,z,u), etc.

For a given set of values, $x_0,y_0, \ldots$, these functions are then written as $f_x(x_0,y_0)$ or $\left(\frac{\partial f(x,y)}{\partial x}\right)_0$, $f_y(x_0,y_0)$ or $\left(\frac{\partial f(x,y)}{\partial y}\right)_0$, $f_u(x_0,y_0,z_0,u_0)$ or $\left(\frac{\partial f(x,y,z,u)}{\partial u}\right)_0$, etc.

Clearly, the rules of differentiation that the student has learned for a function of one independent variable apply as well in partial differentiation, for in fact we deal with such a function when all but one of the independent variables are held constant. Thus, if $u = x^2y + yz + z^2x$, we write at once

$$u_x = 2xy + z^2 \qquad (\text{or } \partial u/\partial x = 2xy + z^2),$$
$$u_y = x^2 + z, \qquad u_z = y + 2zx.$$

The student is familiar with the geometric representation of a function $z = f(x,y)$ by means of a surface, Now, if we hold one of the independent variables, say y, constant, the two equations $\begin{Bmatrix} z = f(x,y) \\ y = \text{constant} \end{Bmatrix}$ define a plane curve on that surface. The student will perceive at once that $\partial z/\partial x$ measures the slope of the tangent drawn to the curve at a point (x,y,z) (*i.e.*, it measures the value of $\tan \alpha$ in the figure); also that it represents the rate of change of z with respect to x as P moves along the curve.

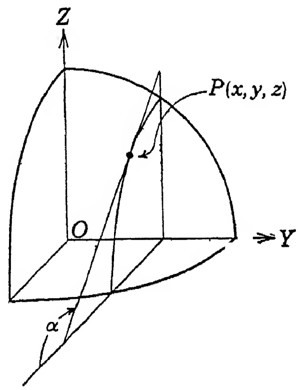

FIG. 167.

As an illustration, let the equation of the surface be $z = 2 - 2x^2 - y^2$ and the curve be $\begin{Bmatrix} z = 2 - 2x^2 - y^2 \\ y = 1, \end{Bmatrix}$, then at the point $(\frac{1}{2},1,\frac{1}{2})$ of the curve we have

$$\partial z/\partial x = (-4x)_{(x=\frac{1}{2})} = -2,$$

and this is the slope of the line tangent to the curve at that point. The equations of the tangent, furthermore, would be $\begin{Bmatrix} z - \frac{1}{2} = -2(x - \frac{1}{2}) \\ y = 1 \end{Bmatrix}$ or $\begin{Bmatrix} 2z + 4x = 3 \\ y = 1 \end{Bmatrix}$. Again, if a point moves along that curve and at $(\frac{1}{2},1,\frac{1}{2})$ its x-coordinate is increasing at the rate of 1 unit per second, then its z-coordinate is decreasing at the rate of 2 units per second (since the rate of change of z with respect to x at that point is -2).

Exercise 1. Prove that if the relation $z = f(x,y)$, when solved for x, gives $x = \varphi(y,z)$, then $\partial z/\partial x$ and $\partial x/\partial z$ are reciprocals of each other.

Exercise 2. Verify that in $x = r \cos \theta$ and $r = \sqrt{x^2 + y^2}$ the well-known relations between rectangular and polar coordinates, $\partial x/\partial r$ computed

from the first and $\partial r/\partial x$ computed from the second are not reciprocals of each other. Account for the fact.

Problems

1. Find the partial derivative indicated in each case.

(a) $z = \sin(xy)$; $\partial z/\partial x$. *Ans.* $\partial z/\partial x = y\cos(xy)$.

(b) $z = x + \sqrt{x^2 + y^2}$; $\partial z/\partial y$. *Ans.* $\partial z/\partial y = y(x^2 + y^2)^{-\frac{1}{2}}$.

(c) $u = \log(xz) + \dfrac{y}{x+z}$; $\dfrac{\partial u}{\partial z}$. *Ans.* $\dfrac{1}{z} - \dfrac{y}{(x+z)^2}$.

(d) $z = x^2 - xy + \cos x \cdot \sin y$; $\partial z/\partial x$.
Ans. $2x - y - \sin x \cdot \sin y$.

(e) $z = \dfrac{x+y}{\sqrt{x^2 - y^2}}$; $\dfrac{\partial z}{\partial y}$.

(f) $u = e^{x^2+xz-z^2}$; $\partial u/\partial z$.

2. Find the equations of the tangent to each curve below at the point indicated.

(a) $\begin{cases} z = x^2 + y^2 \\ x = 1 \end{cases}$ at (1,2,5). *Ans.* $z = 4y - 3$, $x = 1$.

(b) $\begin{cases} z - 1 = x^2 + 3y^2 \\ y = 2 \end{cases}$ at (−1,2,14).
Ans. $2x + z = 12$, $y = 2$.

(c) $\begin{cases} x^2 + y^2 + z^2 = 9 \\ x = 2 \end{cases}$ at (2,1,2). *Ans.* $y + 2z = 5$, $x = 2$.

(d) $\begin{cases} 9x^2 - 36y^2 - 4z^2 = 36 \\ y = 1 \end{cases}$ at $(\sqrt{12}, 1, -3)$.

3. Find at what rate the volume of a right circular cylinder is changing when the altitude is constant and is 6 in. while the radius of the base is decreasing at the rate of ½ in./sec. and is 4 in. at the instant in question.
Ans. 24π cu. in./sec.

4. Find the rate of change of the volume of the frustum of a cone, of base radii R and r and altitude h, with respect to R; also with respect to h.
Ans. $\dfrac{\partial v}{\partial R} = \dfrac{\pi h}{3}(2R + r)$.

5. (a) If $u = x^2y + yz^2 + z^3$, verify that $xu_x + yu_y + zu_z = 3u$.

(b) If $z = \sin\dfrac{y}{x} + \log\dfrac{x}{y}$, verify that $xz_x + yz_y = 0$.

(c) If $z = e^{y/x}(x^2 - 2y^2) + \dfrac{y^3}{x-y}$, verify that $xz_x + yz_y = 2z$.

6. Verify that the functions of Prob. 5 are homogeneous, of degree 3,0, and 2, respectively, by the

Definition. A function $f(x,y,v, \cdots)$ of any number of variables is said to be homogeneous and of degree n if $f(tx,ty,tv, \cdots) = t^n \cdot f(x,y,v, \cdots)$.

Thus, $f(x,y) = x^2 + xy\tan^{-1}\dfrac{y}{x}$ is homogeneous and of degree 2, since

$$f(tx,ty) = t^2x^2 + t^2xy \cdot \tan^{-1}\frac{ty}{tx} = t^2\left(x^2 + xy\tan^{-1}\frac{y}{x}\right) = t^2 \cdot f(x,y).$$

93. Partial Derivatives of Higher Order. Given a function u of several independent variables, say, $u = f(x,y,z)$, each of its partial derivatives, u_x, u_y, and u_z will, in general, still be a function of x, y, and z. If any of these are, in turn, differentiated partially with respect to one of the independent variables, the results are called *second partial derivatives* of u. The notation, in this case, for the nine possible second derivatives is

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y\,\partial x}, \frac{\partial^2 u}{\partial z\,\partial x}, \frac{\partial^2 u}{\partial x\,\partial y}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial z\,\partial y}, \frac{\partial^2 u}{\partial x\,\partial z}, \frac{\partial^2 u}{\partial y\,\partial z}, \frac{\partial^2 u}{\partial z^2}$$

or

$$u_{xx},\ u_{xy},\ u_{xz},\ u_{yx},\ u_{yy},\ u_{yz},\ u_{zx},\ u_{zy},\ u_{zz},$$

the first representing $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$ [or $(u_x)_x$], the second $\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x}\right)$ [or $(u_x)_y$], the fourth $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial y}\right)$ [or $(u_y)_x$], etc.

Thus, if $u = x^2y + \sin(xyz) + e^{y+z}$, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}[2xy + yz\cos(xyz)] = 2y - y^2z^2\sin(xyz);$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial}{\partial y}(u_z) = \frac{\partial}{\partial y}(xy\cos(xyz) + e^{y+z}) = -x^2yz\sin(xyz) + x\cos(xyz) + e^{y+z};$$

and $\dfrac{\partial^2 u}{\partial z\,\partial y} = \dfrac{\partial}{\partial z}(u_y) = \dfrac{\partial}{\partial z}(x^2 + xz\cos(xyz) + e^{y+z}) = -x^2yz\sin(xyz) + x\cos(xyz) + e^{y+z}$. The equality, by the way, of the last two partial derivatives, $\partial^2u/\partial y\,\partial z$ and $\partial^2u/\partial z\,\partial y$, which we notice in this instance, is not accidental. It is, in fact, in conformity with a general theorem, proved in a more advanced course in calculus, that *if the function and its first and second derivatives are continuous in the neighborhood of* $(x = a,\ y = b,\ z = c,\ \cdots)$, *the order of differentiation for second partial derivatives is immaterial, i.e.*, in that case $u_{xy} = u_{yx}$ at $(x = a,\ y = b,\ z = c,\ \cdots)$; $u_{xz} = u_{zx}$ at $(x = a,\ y = b,\ z = c,\ \cdots)$, etc.

Partial derivatives of the third and higher orders are defined in an obvious manner as

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2}\right) \qquad [\text{or } u_{xxx} = (u_{xx})_x],$$

$$\frac{\partial^3 u}{\partial y^2\,\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y\,\partial z}\right) \qquad [\text{or } u_{zyy} = (u_{zy})_y],$$

etc.

Exercise 1. Show as a corollary to the theorem stated above, (*viz.*, that the order of differentiation is immaterial for second partial derivatives) that the order of differentiation is also immaterial for partial derivatives of the third order—the hypothesis now extending to the continuity of the third-order derivatives.

Exercise 2. Extend the theorem to partial derivatives of any order, *i.e.*, prove that the value of the partial derivative depends only on the number of differentiations made with respect to each variable, and not on the order in which these are performed, provided the function and the appropriate number of partial derivatives are continuous.

Problems

1. Given a function u of three variables x, y, and z, state the number of its distinct second partial derivatives; of its distinct third partial derivatives.

2. Find z_{xx}, z_{xy}, z_{yx}, and z_{yy} for each of the following:

(*a*) $z = \text{arc tan}\left(\dfrac{y}{x}\right)$. *Ans.* $z_{xy} = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}$.

(*b*) $z = e^{x^2+y^2}$. *Ans.* $z_{xy} = 4xye^{x^2+y^2}$

(*c*) $z = \log\sqrt{x^2 + y^2}$.

(*d*) $z = \dfrac{xy^2}{1 - y^2}$. *Ans.* $z_{yx} = \dfrac{2y}{(1 - y^2)^2}$.

(*e*) $z = x \cos y - y \cos x$.

(*f*) $z = \log \sin (x - y)$.

3. Find the derivatives specified in each case for the functions below.

(*a*) $z = x^3y - x^4$; z_{xxy}, z_{xyx}, and z_{yxx}. *Ans.* $z_{xxy} = 6x$.

(*b*) $u = \sin (xy + z)$; u_{xxz}, and u_{xzy}. *Ans.* $u_{xxz} = -y^2 \cos (xy + z)$.

(*c*) $u = \tan^{-1} (xyz)$; u_{xzy}, and u_{xxz}.

(*d*) $u = \log (x^2 + y^2 + z^2)$; u_{xxy}, and u_{xyz}. *Ans.* $u_{xyz} = \dfrac{16xyz}{(x^2 + y^2 + z^2)^3}$.

4. If $z = \log\sqrt{x^2 + y^2} + \frac{1}{2}\tan^{-1}\left(\dfrac{y}{x}\right)$, show that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0$.

5. If $z = x^3y \cdot \log \dfrac{y}{x}$, show that

$$x^3\frac{\partial^3 z}{\partial x^3} + 3x^2y\frac{\partial^3 z}{\partial x^2\, \partial y} + 3xy^2\frac{\partial^3 z}{\partial x\, \partial y^2} + y^3\frac{\partial^3 z}{\partial y^3} = 24z.$$

6. If $u = e^{a\varphi} \sin (a \log r)$, show that $\dfrac{\partial^2 u}{\partial r^2} + \dfrac{1}{ar} \cdot \dfrac{\partial^2 u}{\partial r\, \partial \varphi} + \dfrac{1}{r^2} \cdot \dfrac{\partial^2 u}{\partial \varphi^2} = 0$.

94. Total Derivatives. Differentials. Given a function u of two variables, x and y, let us start with two fixed values of x and y that determine the value of u as $u = f(x,y)$ and assign increments Δx and Δy to the two independent variables, obtaining thereby an increment Δu, defined by

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x,y).$$

This may be written as

$$\Delta u = f(x + \Delta x, y + \Delta y) - f(x, y + \Delta y) + f(x, y + \Delta y) - f(x,y).$$

By the theorem of the mean for functions of one variable, the first two terms are equivalent to $f_x(\xi, y + \Delta y) \cdot \Delta x$*, where $x < \xi < x + \Delta x$. Likewise, the second pair of terms are equivalent to $f_y(x,\eta) \cdot \Delta y$, where $y < \eta < y + \Delta y$. Thus we have

$$\Delta u = f_x(\xi, y + \Delta y) \cdot \Delta x + f_y(x,\eta) \cdot \Delta y.$$

Assuming now the continuity of the two functions f_x and f_y at (x,y), we have

$$\lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} f_x(\xi, y + \Delta y) = f_x(x,y), \qquad \lim_{\Delta y \to 0} f_y(x,\eta) = f_y(x,y).$$

In other words,

$$f_x(\xi, y + \Delta y) = f_x(x,y) + \epsilon_1, \qquad \text{where} \qquad \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \epsilon_1 = 0$$

and

$$f_y(x,\eta) = f_y(x,y) + \epsilon_2, \qquad \text{where} \qquad \lim_{\Delta y \to 0} \epsilon_2 = 0.$$

The result is now

$$\begin{aligned} \Delta u &= [f_x(x,y) + \epsilon_1]\Delta x + [f_y(x,y) + \epsilon_2]\Delta y, \\ &= f_x(x,y)\Delta x + f_y(x,y)\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y, \end{aligned}$$

or, what is the same,

$$\Delta u = u_x\,\Delta x + u_y\,\Delta y + \epsilon_1\,\Delta x + \epsilon_2\,\Delta y. \tag{128}$$

As in the case of one independent variable, we define the *differentials*, dx and dy, as arbitrary increments, *i.e.*, by

$$\begin{aligned} dx &= \Delta x, \\ dy &= \Delta y, \end{aligned}$$

and define, further, the *total differential of the function* as

$$du = u_x\,\Delta x + u_y\,\Delta y,$$

or

$$du = u_x\,dx + u_y\,dy. \tag{129}$$

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* The notation $f_x(\xi, y + \Delta y)$, of course, denotes the value of $f_x(x,y)$ when x is replaced by ξ, and y by $y + \Delta y$.

Let now x and y be each a continuous function of another independent variable t (making u a continuous function of t), where the fixed values x, y, and u, correspond to a fixed value t and the increments Δx, Δy, and Δu are caused by an arbitrary increment Δt of t. Formula (128) still remains true, and divided through by Δt, it becomes

$$\frac{\Delta u}{\Delta t} = u_x \cdot \frac{\Delta x}{\Delta t} + u_y \cdot \frac{\Delta y}{\Delta t} + \epsilon_1 \cdot \frac{\Delta x}{\Delta t} + \epsilon_2 \cdot \frac{\Delta y}{\Delta t}$$

hence,

$$\lim_{\Delta t \to 0}\left(\frac{\Delta u}{\Delta t}\right) = u_x \cdot \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) + u_y \cdot \lim_{\Delta t \to 0}\left(\frac{\Delta y}{\Delta t}\right) + \lim_{\Delta t \to 0} \epsilon_1 \cdot \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) + \lim_{\Delta t \to 0} \epsilon_2 \cdot \lim_{\Delta t \to 0}\left(\frac{\Delta y}{\Delta t}\right)$$

or

$$D_t u = u_x \cdot D_t x + u_y \cdot D_t y, \tag{130}$$

since $\Delta x \to 0$ and $\Delta y \to 0$ as $\Delta t \to 0$ and, therefore $\lim_{\Delta t \to 0} \epsilon_1 = 0$ and $\lim_{\Delta t \to 0} \epsilon_2 = 0$. The derivative $D_t u$, in (130), is called the *total derivative* of u with respect to t. It may, of course, be expressed as well by the notation du/dt. Note that in this case (x and y not independent variables, but functions of t) we no longer have $dx = \Delta x$ and $dy = \Delta y$. In fact, $dx = D_t x \cdot \Delta t$ and $dy = D_t y \cdot \Delta t$, expressions which, in general, are different from Δx and Δy. However, multiplying (130) through by dt, we have

$$D_t u \cdot dt = u_x \cdot D_t x \cdot dt + u_y \cdot D_t y \cdot dt,$$

or

$$du = u_x \cdot dx + u_y \cdot dy$$

i.e., formula (129) holds, whether x and y are independent variables, or dependent upon another independent variable.

Exercise 1. Prove, when $\Delta x = dx$ and $\Delta y = dy$, that the difference between the increment and the differential of u, *viz.*,

$$\Delta u - du = \epsilon_1 \cdot \Delta x + \epsilon_2 \, \Delta y,$$

is an infinitesimal of higher order than Δx and Δy, when the latter are infinitesimals.*

* An infinitesimal ξ, depending on two or more infinitesimals, $\alpha, \beta, \gamma, \ldots$, is said to be of the same order as $\alpha, \beta, \gamma, \ldots$ if

$$\lim_{\alpha, \beta, \cdots \to 0} \frac{\xi}{\sqrt{\alpha^2 + \beta^2 + \gamma^2 + \cdots}} = k$$

where k is a finite number not zero, and of higher order if that limit is zero.

On account of the result of this exercise, the differential du may be taken as an approximation to the value of Δu, the quality of the approximation improving, of course, the smaller Δx and Δy are in numerical value.

Exercise 2. If u is a function of x and y while x and y are each functions of t, prove the formula

$$D_t^2u = u_{xx} \cdot (D_tx)^2 + 2u_{xy} \cdot D_tx \cdot D_ty + u_{yy} \cdot (D_ty)^2 + u_x \cdot D_t^2x + u_y \cdot D_t^2y.$$

HINT: Differentiate (130) with respect to t, remembering that u_x and u_y are each functions of x and y, and that, hence, Eq. (130) applies to them.

The argument gone through applies to a function of any number of variables, and so for any function $u = f(x,y,z, \cdots)$ we have

$$\Delta u = u_x \cdot \Delta x + u_y \cdot \Delta y + u_z \cdot \Delta z + \cdots + \epsilon_1 \cdot \Delta x + \epsilon_2 \cdot \Delta y + \epsilon_3 \cdot \Delta z + \cdots,$$

with

$$\lim_{\Delta x, \Delta y, \Delta z, \cdots \to 0} \epsilon_i = 0 \qquad (i = 1,2,3, \cdots).$$

We define the differential of u as

$$du = u_x \cdot dx + u_y \cdot dy + u_z \cdot dz + \cdots$$

and the total derivative with respect to t, if x, y, z, . . . are functions of t, as

$$D_tu = u_x \cdot D_tx + u_y \cdot D_ty + u_z \cdot D_tz + \cdots.$$

If the variables, x, y, z, . . . are functions of several variables, say, $x = f_1(r,s,t)$, $y = f_2(r,s,t)$, $z = f_3(r,s,t)$, $\cdots$, the above formulas still remain true, except that the derivatives are now partial. Thus we write

$$\frac{\partial u}{\partial r} = u_x \cdot \frac{\partial x}{\partial r} + u_y \cdot \frac{\partial y}{\partial r} + u_z \cdot \frac{\partial z}{\partial r} + \cdots,$$

$$\frac{\partial u}{\partial s} = u_x \cdot \frac{\partial x}{\partial s} + u_y \cdot \frac{\partial y}{\partial s} + u_z \cdot \frac{\partial z}{\partial s} + \cdots,$$

etc.

Exercise 3. Prove *Euler's theorem on homogeneous functions:* If

$$f(x,y,z, \ldots)$$

is homogeneous and of degree n, then $xf_x + yf_y + zf_z + \cdots = nf$. (See Prob. 6, page 348, for the definition of a homogeneous function.)

HINT: In the relation $f(tx,ty,tz, \cdots) = t^n \cdot f(x,y,z, \cdots)$, put $u = tx$, $v = ty$, $w = tz$, $\cdots$, and differentiate partially with respect to t, applying (130). In the resulting relation,

$$\frac{\partial f}{\partial u} \cdot x + \frac{\partial f}{\partial v} \cdot y + \frac{\partial f}{\partial w} \cdot z + \cdots = nt^{n-1} \cdot f(x,y,z, \cdots),$$

put $t = 1$, making $u = x$, $v = y$, $w = z$, $\cdots$.

Note that Prob. 5, page 348, is made up of instances of Euler's theorem on homogeneous functions.

Problems

1. State the value of du in each case below.

(a) $u = x^2y + \frac{y}{x}$. *Ans.* $\left(2xy - \frac{y}{x^2}\right) dx + \left(x^2 + \frac{1}{x}\right)dy$.

(b) $u = x^2y + 3xz^2$. *Ans.* $(2xy + 3z^2)dx + x^2\,dy + 6xz\,dz$.

(c) $u = e^{\sin(x-y)}$.

(d) $u = e^r(\cos\varphi + 2\sin\varphi)$.

(e) $u = xyz \cdot \tan(x + y + z)$.

(f) $u = (xy)^z$.

Ans. $z(xy)^{z-1}(ydx + xdy) + (xy)^z(\log x + \log y)dz$.

2. Find D_tu in each case without expressing u as a function of t.

(a) $u = x^2 + y^2 - z^2$; $x = 2t$, $y = t^2$, $z = t$.

Ans. $4x + 4yt - 2z$.

(b) $u = xy^2 + yz^2 + zx^2$; $x = \log t$, $y = \sin t$, $z = e^t$.

(c) $u = \tan^{-1}\frac{yz}{x}$; $x = 1 + t^2$, $y = \sqrt{t}$, $z = 1 - t$.

Ans. $\frac{1}{x^2 + y^2z^2}\left(-2yzt + \frac{xz}{2\sqrt{t}} - xy\right)$.

(d) $u = (xyz)^v$; $x = \cos t$, $y = \tan t$, $z = \sinh t$, $v = \cosh t$.

3. Find the partial derivative indicated in each case, without expressing the function in terms of the independent variables.

(a) $z = e^{y/x}$; $x = r^2 + s^2$, $y = r^2 - s^2$; $\left(\frac{\partial z}{\partial r}\right)$. *Ans.* $\frac{2zr}{x^2}(x - y)$.

(b) $z = e^x(\sin 2y + 3\cos 2y)$; $x = u + v - w$, $y = u - v + w$; $(\partial z/\partial v)$. *Ans.* $e^x(7\sin 2y + \cos 2y)$.

(c) $u = \cos^{-1}\left(\frac{x+y}{z}\right)$; $x = r^2 + s^2 - t^2$, $y = s^2 + t^2 - r^2$, $z = t^2 + r^2 - s^2$; $\left(\frac{\partial u}{\partial s}\right)$.

(d) $v = (xy)^{uz}$; $x = r + s + t$, $y = r^2 + s^2 + t^2$, $z = rst$, $u = r + s - t$; $\left(\frac{\partial v}{\partial t}\right)$.

4. Find the derivative indicated in each case, without expressing the function in terms of the independent variables.

(a) $z = \log(xy)$; $x = 1 + t^2$, $y = \sqrt{t}$; (D_t^2z).

Ans. $\frac{2}{x} - \frac{4t^2}{x^2} - \frac{1}{4y^2t} - \frac{1}{4yt^{3/2}}$.

(b) $z = e^{x-y}$; $x = \sin 2t$, $y = \cos 2t$; (D_t^2z).

Ans. $4z[(\cos 2t + \sin 2t)^2 + \cos 2t - \sin 2t]$.

(c) $u = x^2 + 2y^2 + 3z^2$; $x = 2 - t^3$, $y = 1 + t - t^2$, $z = \sqrt{1+t}$; (D_t^2u).

(d) $u = xy^2z^3$; $x = 2r^2 - s$, $y = r^3 + s$, $z = r^2 - s^2$; $(\partial^2u/\partial r^2$ and $\partial^2u/\partial s\partial r)$.

(e) $z = x\sqrt{y}$; $x = u^2 - v^2$, $y = u^2 + v^2$; $(\partial^3z/\partial u^3$ and $\partial^3z/\partial u^2\partial v)$.

5. Prove the following and verify each by an example:

(a) If $z = f(xy)$, then $xz_x - yz_y = 0$. (HINT: Put $u = xy$).

(b) If $z = f(xy^2)$, then $2xz_x - yz_y = 0$.

(c) If $z = f(ax + by) + g(ax - by)$, then $b^2z_{xx} - a^2z_{yy} = 0$.

6. If u and v are functions of x and y for which $u_x = v_y$ and $u_y = -v_x$, show that $u_r = v_\theta/r$, and $v_r = -u_\theta/r$, where $x = r\cos\theta$, and $y = r\sin\theta$.

7. Find the difference between du and Δu in each of the following cases:

(a) $u = xyz$; $x = 3$; $y = 4$; $z = 1$; $\Delta x = 0.01$, $\Delta y = 0.02$, $\Delta z = 0.01$.
Ans. 0.001202.

(b) $u = x^2 + yz$; $x = 2$, $y = 2$, $z = 3$; $\Delta x = 0.02$, $\Delta y = -0.01$, $\Delta z = 0.01$.
Ans. 0.0003.

8. The sides of a rectangle are found, by measurement, to be 4 ft. and 6 ft. If there is a possible error of 0.02 ft. in each side, find, by means of differentials, the approximate value of the greatest possible error in the computed area; also the approximate relative error. (NOTE: The latter is expressed by dA/A, where A is the area.) *Ans.* $\frac{1}{5}$ sq. ft.; $\frac{1}{120}$.

9. The dimensions of a rectangular box are found, by measurement, to be 4 ft., 6 ft., and 8 ft. If there is a possible error of 0.03 ft. in each dimension, find, by means of differentials, the approximate value of the greatest possible error in the computed volume; also the approximate relative error.

10. Find the approximate value of $(3.02^2 + 3.99^2)^{3/2}$. HINT: Set

$$z = (x^2 + y^2)^{3/2};\ x = 3,\ y = 4;\ \Delta x = 0.02,\ \Delta y = -0.01, \text{ and find } dz.$$

11. Find the approximate value of $\sin(29° 59' 30'' + 45° 1' 00'')$.

12. By measurement, two sides of a triangle are found to be 20 ft. and 30 ft. and to include a 30° angle. If the possible error in measuring the first side is 0.1 ft. and in the other is 0.15 ft., while the possible error in measuring the angle is 1°, find the approximate value of the greatest possible error in the computed area; also the approximate relative error.

13. For a homogeneous function $f(x,y,z)$ of degree n, prove:

$$x^2f_{xx} + y^2f_{yy} + z^2f_{zz} + 2xyf_{xy} + 2yzf_{yz} + 2zxf_{zx} = n(n-1)f.$$

Verify for the function $x^3 + y^2z - zx^2$. HINT: See Exercise 3 above. Differentiate with respect to t twice, before putting $t = 1$.

95. Differentiation of Implicit Functions. Given a relation among a number of variables, say $f(x,y,z,u) = 0$, suppose we wish to find the value of $\partial u/\partial x$. This implies that we construe the given relation to define u as a function of x, y, and z, and that, in the process of obtaining the derivative desired, y and z are held constant, hence $dy = dz = 0$. Now,

$$df = f_x\,dx + f_y\,dy + f_z\,dz + f_u\,du,$$

and since for all the sets of values (x,y,z,u) that satisfy the given

relation, f is constant, its total differential equals zero. Hence

$$df = f_x dx + f_y dy + f_z dz + f_u du = 0.$$

By the equations $dy = dz = 0$, this reduces to

$$f_x\,dx + f_u\,du = 0,$$

whence

$$\frac{\partial u}{\partial x} = -\frac{f_x}{f_u}, \qquad (f_u \neq 0)$$

(since $\partial u/\partial x$ is nothing else than du/dx when the remaining variables are held constant). In like manner,

$$\frac{\partial u}{\partial y} = -\frac{f_y}{f_u}, \qquad (f_u \neq 0)$$

etc. We are, thus, enabled to find the partial derivatives of u (or, clearly, of any other of the variables) without solving the given relation explicitly for it. To illustrate, consider the relation $x^3y + yzu^2 - xu^3 + 5 = 0$. We find, say, $\partial u/\partial y$, as

$$\begin{aligned}\frac{\partial u}{\partial y} &= -\frac{(x^3y + yzu^2 - xu^3 + 5)_y}{(x^3y + yzu^2 - xu^3 + 5)_u}\\ &= -\frac{x^3 + zu^2}{2yzu - 3xu^2}.\end{aligned}$$

Given, now, two simultaneous relations

$$\begin{aligned} f(x,y,z,u) &= 0,\\ g(x,y,z,u) &= 0,\end{aligned}$$

let it be required to find the value of $\partial u/\partial x$. Note that this is ambiguous, without some hypothesis as to what variables are held constant in the process of partial differentiation. For we could construe the two given equations as defining u and z each as a function of x and y (in which case y is held constant in the process of finding u_x), or as defining u and y each as a function of x and z (in which case z is held constant in the process of finding u_x). Let us say, for the sake of definiteness, that it is y that is held constant, and exhibit that fact by using the notation $\dfrac{\partial u(x,y)}{\partial x}$ for the derivative we are seeking. Instead of solving the two

given equations for u and z as functions of x and y, we form the total differentials

$$\begin{aligned} df &= f_x\,dx + f_y\,dy + f_z\,dz + f_u\,du = 0, \\ dg &= g_x\,dx + g_y\,dy + g_z\,dz + g_u\,du = 0. \end{aligned}$$

(The sets of values of x, y, z, and u for which the last two equations are true are, of course, only those for which the two given equations are true.) If y is held constant, $dy = 0$, and from the resulting equations,

$$\begin{aligned} f_x\,dx + f_z\,dz + f_u\,du &= 0, \\ g_x\,dx + g_z\,dz + g_u\,du &= 0, \end{aligned}$$

we obtain, solving these for du and dz,

$$\left.\begin{aligned} du &= -\frac{f_z g_x - f_x g_z}{f_z g_u - f_u g_z}dx, \\ dz &= -\frac{f_x g_u - f_u g_x}{f_z g_u - f_u g_z}dx. \end{aligned}\right. \qquad (f_z g_u - f_u g_z \neq 0)$$

From the first of these results we find the partial derivative sought as

$$\frac{\partial u(x,y)}{\partial x} = -\frac{f_z g_x - f_x g_z}{f_z g_u - f_u g_z}.$$

The second would give us

$$\frac{\partial z(x,y)}{\partial x} = -\frac{f_x g_u - f_u g_x}{f_z g_u - f_u g_z}.$$

For example, let us find $\partial z(x,y)/\partial x$ from the relations

$$\begin{aligned} x \sin u + y \log z &= 0, \\ u \sin x + z \log y &= 0. \end{aligned}$$

It will be better to apply the method just employed rather than to rely on the formulas obtained above. We have, then, treating these two equations as $f(x,y,z,u) = 0$, $g(x,y,z,u) = 0$,

$$df = \sin u \cdot dx + \log z \cdot dy + \frac{y}{z} \cdot dz + x \cos u \cdot du = 0,$$

$$dg = u \cos x \cdot dx + \frac{z}{y} \cdot dy + \log y \cdot dz + \sin x \cdot du = 0.$$

Since y is being held constant, we set dy equal to zero and display the resulting equations as

$$\frac{y}{z}\cdot dz + x\cos u\cdot du = -\sin u\cdot dx,$$
$$\log y\cdot dz + \sin x\cdot du = -u\cos x\cdot dx.$$

Solving these for dz, we obtain

$$dz = -\frac{\sin x\sin u - ux\cos x\cos u}{\frac{y\sin x}{z} - x\cos u\log y}dx,$$

whence

$$\frac{\partial z(x,y)}{\partial x} = \frac{uzx\cos x\cos u - z\sin x\sin u}{y\sin x - zx\cos u\log y}.$$

NOTE: The quantity $f_z g_u - f_u g_z$, which occurs in the denominators of the expressions for $\frac{\partial u(x,y)}{\partial x}$ and $\frac{\partial z(x,y)}{\partial x}$, is called the *Jacobian of the two functions $f(x,y,z,u)$ and $g(x,y,z,u)$ with respect to z and u.* It is commonly designated by the symbol $\partial(f,g)/\partial(z,u)$. The number of variables is immaterial. If we had the pair of functions $f(x,y,z)$ and $g(x,y,z)$, or the pair $F(x,y)$ and $G(x,y)$, the Jacobian of f and g with respect, say, to x and y would be $f_x g_y - f_y g_x$ [designated by $\partial(f,g)/\partial(x,y)$], while the Jacobian of F and G with respect to x and y would be $F_x G_y - F_y G_x$ [designated by $\partial(F,G)/\partial(x,y)$]. The notation for Jacobians suggests that in some way they are a generalization of the derivative. Indeed, they share a number of properties in common with derivatives. Let us verify one. It is known that if y is a function of x, then $D_x y$ and $D_y x$ are reciprocals of each other, provided both exist.

A similar property for Jacobians is stated in the first of the following three exercises. In each of these exercises the continuity of the functions involved and of their first partial derivatives is assumed.

Exercise 1. Prove that if u and v are each functions of x and y, and $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$, then $\frac{\partial(u,v)}{\partial(x,y)}\cdot\frac{\partial(x,y)}{\partial(u,v)} = 1$. HINT: From $\begin{Bmatrix} du = u_x\,dx + u_y\,dy \\ dv = v_x\,dx + v_y\,dy \end{Bmatrix}$, obtain $x_u = v_y/J$, $y_u = -v_x/J$, etc.; where J designates $\partial(u,v)/\partial(x,y)$. Now compute the Jacobian $\partial(x,y)/\partial(u,v) = x_u y_v - x_v y_u$ by substituting the values of the partial derivatives just found.

Exercise 2. Prove that in the case of the relation $f(x,y) = 0$ between x and y

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \qquad (f_y \neq 0)$$

and

$$\frac{dx}{dy} = -\frac{f_y}{f_x} \qquad (f_x \neq 0).$$

Exercise 3. Prove that if z is defined as a function of x and y by the relations $x = f(u,v)$, $y = g(u,v)$, $z = h(u,v)$, then

$$z_x = \frac{h_u g_v - h_v g_u}{f_u g_v - f_v g_u} \qquad z_y = \frac{h_v f_u - h_u f_v}{f_u g_v - f_v g_u} \qquad (f_u g_v - f_v g_u \neq 0).$$

Problems

1. Find dy/dx in each of the following:

(a) $x + y \sin x = 0$. *Ans.* $-\dfrac{1 + y \cos x}{\sin x}$.

(b) $x^2y - \log(xy) = 2$. *Ans.* $\dfrac{2x^2y^2 - y}{x - x^3y}$.

(c) $e^{x+y} = x - y$.

(d) $\sin^{-1}(xy^2) + x + y = 0$.

2. Find the derivative indicated in each case below.

(a) $xy^2 + yz^2 + zx^2 = 3$, (z_x). *Ans.* $-\dfrac{y^2 + 2xz}{x^2 + 2yz}$.

(b) $e^{x+y+z} + xyz = 1$, (z_y). *Ans.* $-\dfrac{e^{x+y+z} + xz}{e^{x+y+z} + xy}$.

(c) $x \sin y + y \cos z - xyz = 0$, (y_x).

Ans. $\dfrac{yz - \sin y}{x \cos y + \cos z - xz}$.

(d) $\log(xy^2z^3) - x - y - z + 2u = 0$, (u_z).

3. Find the equations of tangent to the curve obtained by intersecting the surface $x^2 - 2xz + y^2 = 5$ with the plane $y = 2$, drawn at the point $(1,2,0)$. *Ans.* $z = x - 1$, $y = 2$.

4. Find the equations of the tangent drawn at the point $(1,1,1)$ to the surface $(xy)^{1/2} + (yz)^{1/2} + (zx)^{1/2} = 3$ and parallel to the yz-plane.

Ans. $y + z = 2$, $x = 1$.

5. Prove that if z is a function of y defined by $f(x,y) = 0$, $g(x,z) = 0$, then $D_yz = f_yg_x/f_xg_z$. Verify for $x - y^2 = 0$, $x - z^3 = 2$.

6. Prove that, when x, y, and z are related by an equation $f(x,y,z) = 0$, then $\dfrac{\partial x}{\partial y} \cdot \dfrac{\partial y}{\partial z} \cdot \dfrac{\partial z}{\partial x} = -1$. Verify for $x^2y - 2yz^2 + zx^2 = 3$.

7. Without solving the given relations for x and y in terms of z, find D_zx and D_zy

(a) when x,y,z satisfy the two equations $\begin{cases} x^2 + y^2 + z^2 = 9, \\ x + y + z = 1; \end{cases}$

Ans. $D_zx = \dfrac{y - z}{x - y}$.

(b) when x,y,z satisfy the two equations $\begin{cases} x^2 - xz + y = 3, \\ y^2 - yz + x = 5. \end{cases}$

8. (*a*) Given $\begin{cases} x^2 - yz + u^2 = 0, \\ y^2 + xz + u = 2, \end{cases}$ find $\dfrac{\partial u(x,y)}{\partial x}$. *Ans.* $-\dfrac{2x^2 + yz}{y + 2xu}$.

(*b*) Given $\begin{cases} 2x + \log y - uz = 0, \\ \sin (xu) + yz = 0, \end{cases}$ find $\dfrac{\partial z(x,y)}{\partial y}$.

9. Verify the property proved in Exercise 1 for $\begin{cases} u = 2x + 3y \\ v = 3x - 4y. \end{cases}$

10. Show that if u and v are functions of r and s, while r and s are functions of x and y, then

$$\frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)}.$$

To what property of ordinary derivatives is this analogous?

11. Verify the property proved in Prob. 10 for

$$\begin{cases} u = r^2 + s^2, \\ v = r^2 - s^2, \end{cases} \qquad \begin{cases} r = x + \log y, \\ s = \log (xy). \end{cases}$$

96. Tangent Plane and Normal Line to a Surface. We have already seen in Sec. 92 of this chapter that the tangent to

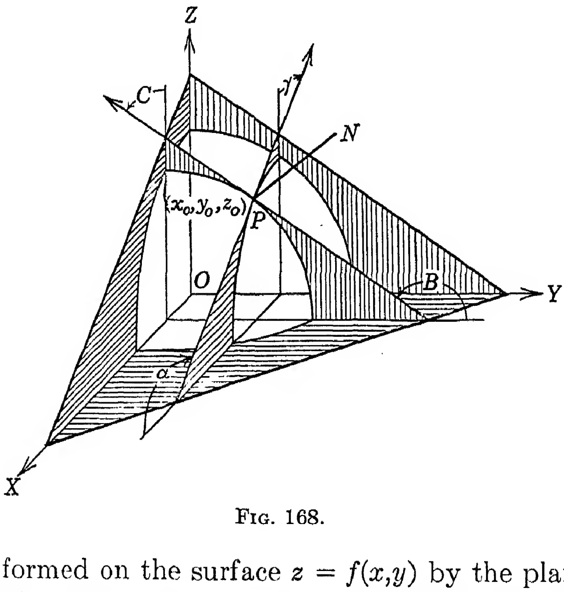

FIG. 168.

the curve formed on the surface $z = f(x,y)$ by the plane $y = y_0$, drawn at the point (x_0,y_0,z_0), has the slope $(z_x)_0$ [we use this notation to designate the value of z_x computed at (x_0,y_0,z_0)]. That is, $\tan \alpha = (z_x)_0$ in Fig. 168. From this

$$\cos \alpha = \pm \frac{1}{\sqrt{1 + (z_x)_0{}^2}},$$

and let us assume the tangent so oriented that $\cos \alpha = \dfrac{1}{\sqrt{1 + (z_x)_0{}^2}}$.

Now $\gamma = \alpha - 90°$ and $\beta = 90°$, where α,β,γ are the direction angles of the tangent. Hence, the direction cosines of the tangent are

$$\frac{1}{\sqrt{1 + (z_x)_0^2}}, \qquad 0, \qquad \frac{(z_x)_0}{\sqrt{1 + (z_x)_0^2}}$$

(since $\cos \gamma = \cos (\alpha - 90°) = \sin \alpha$). In like manner, the slope of the tangent to the curve formed on the surface by the plane $x = x_0$ drawn at the point (x_0,y_0,z_0), (*i.e.*, $\tan B$) is $(z_y)_0$, and we may assume the tangent so oriented that $\cos B = \dfrac{1}{\sqrt{1 + (z_y)_0^2}}$, whence $\cos C = \dfrac{(z_y)_0}{\sqrt{1 + (z_y)_0^2}}$; also $\cos A = 0$, where A, B, C are the direction angles of this tangent. Its direction cosines are, then

$$0, \qquad \frac{1}{\sqrt{1 + (z_y)_0^2}}, \qquad \frac{(z_y)_0}{\sqrt{1 + (z_y)_0^2}}.$$

We define now a line as *normal* to a surface at a point P if it is perpendicular to the tangent of every curve passing through P and lying on the surface. In Exercise 3 of Sec. 97 we shall prove that all such tangents indeed lie in a plane. If the line PN in the figure is the normal to the surface at P, it must, in particular, be perpendicular to the two tangents we have just considered. If we call its direction numbers l, m, n we have

$$l + n(z_x)_0 = 0,$$
$$m + n(z_y)_0 = 0,$$

(recalling Exercise 11, Sec. 35). Hence, the *direction numbers of the normal* may be taken as $(z_x)_0$, $(z_y)_0$, -1. The equations of the normal are, then

$$\frac{x - x_0}{(z_x)_0} = \frac{y - y_0}{(z_y)_0} = \frac{z - z_0}{-1}.$$

The plane passing through the point P of the surface, perpendicular to the normal, is called the *tangent plane* (with P as the point of tangency). It is, indeed, the plane which contains all the tangents at P to curves on the surface through P. From the known direction numbers of the normal to the surface, which is also the normal to the tangent plane, we obtain as *the equation of the plane tangent to the surface* $z = f(x,y)$ *at* P (x_0,y_0,z_0)

$$(z_x)_0(x - x_0) + (z_y)_0(y - y_0) - (z - z_0) = 0.$$

Consider now a point Q $(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ on the surface. The line through Q, parallel to OZ, has the equations $\begin{matrix} x = x_0 + \Delta x, \\ y = y_0 + \Delta y. \end{matrix}$ If we put these values into the equation of the tangent plane at P (x_0,y_0,z_0) for x and y, we obtain $z = z_0 + (z_x)_0 \Delta x + (z_y)_0 \Delta y = z_0 + dz$ (since here, x and y are independent variables and $\Delta x = dx$, $\Delta y = dy$) for the point R where the line through Q and parallel to OZ meets the tangent plane. In other words, the distance, along the line through Q and parallel to OZ,

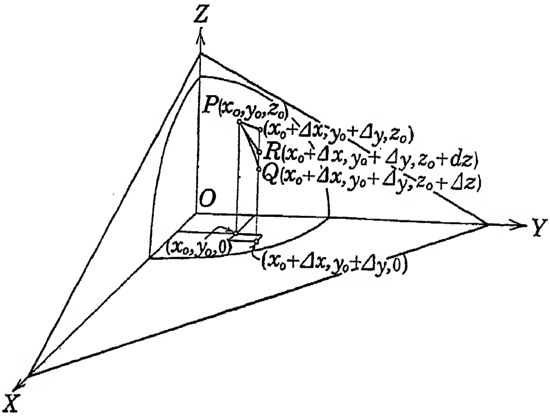

FIG. 169.

between the surface and the plane tangent to the surface at P is $\Delta z - dz$. We thus have a geometric interpretation for the differential dz. (Compare the geometric meaning of dy in the case $y = f(x)$ is the equation of a curve in the xy-plane.)

Exercise 1. Prove that the direction numbers of the normal to the surface $f(x,y,z) = 0$ at the point P (x_0,y_0,z_0) may be taken as $(f_x)_0$, $(f_y)_0$, $(f_z)_0$.

Exercise 2. Prove that the direction numbers of the normal at a point of the surface $\begin{cases} x = f(u,v), \\ y = g(u,v), \\ z = h(u,v), \end{cases}$ may be taken as $h_u g_v - h_v g_u$, $f_u h_v - f_v h_u$, $g_u f_v - g_v f_u$, all computed for that point (see Exercise 3, Sec. 95).

Exercise 3. Prove that if at a point on the surface $f(x,y,z) = 0$, f_x, f_y, and f_z all vanish, no normal line or tangent plane is defined at that point. (Such a point is called a *singular* point of the surface.)

To illustrate, let us find the equations of the normal line and tangent plane at the point $(1, -2,2)$ for the surface $x^2 + y^2 + z^2 = 9$. Here $f_x = 2x$, and $(f_x)_0 = 2$; $f_y = 2y$, and

$(f_y)_0 = -4$; $f_z = 2z$, and $(f_z)_0 = 4$. By exercise 1, the equations of the normal line are, then, $\frac{x-1}{2} = \frac{y+2}{-4} = \frac{z-2}{4}$ an the equation of the tangent plane is

$$2(x-1) - 4(y+2) + 4(z-2) = 0.$$

Problems

1. Find the equations of the normal line and the tangent plane for each surface at the point indicated.

(a) $z = x^2 + 2y^2 - 4$ at $(2,1,2)$.

Ans. $\frac{x-2}{4} = \frac{y-1}{4} = \frac{z-2}{-1}$; $4x + 4y - z = 10$.

(b) $z = x^2 + 2xy$ at $(1,3,7)$.

Ans. $\frac{x-1}{8} = \frac{y-3}{2} = \frac{z-7}{-1}$; $8x + 2y - z = 7$.

(c) $z = x^{1/2} + y^{1/2}$ at $(1,1,2)$.

(d) $x^2 + 2y^2 - z^2 = 5$ at $(-1,2,-2)$.

(e) $xy^2 + yz^2 - zx^2 + 18 = 0$ at $(0,-2,3)$.

(f) $x = 2y - z^2 - 3$ at $(-3,0,0)$.

Ans. $2x + y + 6 = 0$, $z = 0$; $x - 2y + 3 = 0$.

(g) $y = 3xz - z^2 + 2$ at $(2,7,1)$.

Ans. $\frac{2-x}{3} = y - 7 = \frac{1-z}{4}$; $3x - y + 4z = 3$.

(h) $x^{1/2} + (-y)^{1/2} + z^{1/2} = 4$ at $(4,-1,1)$.

2. Find the angles between the two normals to the two surfaces $x^2 + xy - z^2 = 8$ and $x - y + xz = 0$ at their common point $(3,0,-1)$.

Ans. $\cos^{-1}(\pm 3/7\sqrt{10})$.

3. Show that the equation of the tangent plane at the point (x_0,y_0,z_0) of the surface

(a) $x^2 + y^2 - 3z^2 = 1$ is $xx_0 + yy_0 - 3zz_0 = 1$;

(b) $xy + yz - z^2 = 3$ is $\frac{1}{2}(xy_0 + yx_0) + \frac{1}{2}(yz_0 + zy_0) - zz_0 = 3$.

4. Find the singular point of the cone $x^2 + y^2 - 2z^2 = 0$. Show that every tangent plane of the cone passes through that point.

Ans. $(0,0,0)$.

5. Find the equation of the plane tangent to the surface

$$x^2 - y^2 + 2xy + 3z = a$$

and parallel to the xy-plane.

6. Show that the plane $5x + 8y + 2z + 16 = 0$ is tangent to the surface $x^2 + 2xz - 2y^2 + 16 = 0$, and find the point of tangency. *Ans.* $(2,-4,3)$.

7. Show that the line $\begin{cases} y = 2z - 15 \\ x = 2z - 15 \end{cases}$ is normal to the surface

$$x^{2/3} + y^{2/3} + z^{2/3} = 6$$

and find the point at which it is the normal. *Ans.* $(1,1,8)$.

8. Show that the volume of the tetrahedron bounded by the coordinate planes and any plane tangent to the surface $xyz = a$ is constant.

9. Find the equations of the normal line and the tangent plane for the surface

$$(a) \begin{cases} x = u^2 + v \\ y = u + v^2 \text{ at the point where } u = 1,\ v = 2; \\ z = u^2 + v^2 \end{cases}$$

Ans. $\dfrac{x-3}{4} = \dfrac{y-5}{6} = \dfrac{z-5}{-7}$; $4x + 6y - 7z = 7$.

$$(b) \begin{cases} x = u - v^3 \\ y = uv + v^2 \text{ at the point where } u = 0,\ v = 1. \\ z = u^2 + uv \end{cases}$$

97. Space Curves. It will be convenient, in treating space curves, *i.e.*, curves that do not lie in a plane, to assume their equations in the parametric form $x = f(t)$, $y = g(t)$, $z = h(t)$. We take up, first,

I. *The Tangent Line at a Point P* (x_0, y_0, z_0) *of the Curve.* This is defined, as in the case of a plane curve, as *the limiting position of the secant line through the given point and a neighboring point of the curve, as the latter approaches the given point along the curve.* If the neighboring point has the coordinates $(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$, the direction numbers of the secant PQ may be taken as Δx, Δy, Δz, or, what is the same, $\dfrac{\Delta x}{\Delta t}, \dfrac{\Delta y}{\Delta t}, \dfrac{\Delta z}{\Delta t}$, where Δt is the increment of the parameter t to which Δx, Δy, and Δz correspond. From the definition of the tangent line, its direction numbers may be taken as the limits of these quantities as $\Delta t \to 0$. In other words, *a set of direction numbers for the tangent at a point* (x_0,y_0,z_0) *are*

$$\left(\frac{dx}{dt}\right)_0, \quad \left(\frac{dy}{dt}\right)_0, \quad \left(\frac{dz}{dt}\right)_0$$

that is, $f'(t_0)$, $g'(t_0)$, $h'(t_0)$, the zero subscript indicating, as usual, that the derivatives in question are computed at the point (x_0,y_0,z_0) corresponding to the value t_0 of t.

II. *The normal plane* to the above curve at P (x_0,y_0,z_0) is defined as the plane through the given point and perpendicular to the tangent line at P. Its equation is, obviously,

$$(x - x_0)\cdot\left(\frac{dx}{dt}\right)_0 + (y - y_0)\cdot\left(\frac{dy}{dt}\right)_0 + (z - z_0)\cdot\left(\frac{dz}{dt}\right)_0 = 0.$$

III. *The Osculating Plane.* Consider, first, any plane whatever passing through a point P (x_0,y_0,z_0) of the above curve. Its

equation is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$, where A, B, and C are any set of real numbers not all zero. Let us examine the distance from a neighboring point of the curve Q $(x_0 + \Delta x, y_0 + \Delta y, z_0 + \Delta z)$ to that plane. By the formula of Sec. 42, that distance equals

$$\frac{|A\ \Delta x + B\ \Delta y + C\ \Delta z|}{\sqrt{A^2 + B^2 + C^2}}$$

By Taylor's theorem,

$$\Delta x = f(t_0 + \Delta t) - f(t_0) = \Delta t \cdot f'(t_0) + \frac{\overline{\Delta t}^2}{2!}f''(t_0) + \frac{\overline{\Delta t}^3}{3!}f'''(t_1), \qquad (t_0 < t_1 < t_0 + \Delta t),$$

$$\Delta y = g(t_0 + \Delta t) - g(t_0) = \Delta t \cdot g'(t_0) + \frac{\overline{\Delta t}^2}{2!}g''(t_0) + \frac{\overline{\Delta t}^3}{3!}g'''(t_2), \qquad (t_0 < t_2 < t_0 + \Delta t),$$

$$\Delta z = h(t_0 + \Delta t) - h(t_0) = \Delta t \cdot h'(t_0) + \frac{\overline{\Delta t}^2}{2!}h''(t_0) + \frac{\overline{\Delta t}^3}{3!}h'''(t_3), \qquad (t_0 < t_3 < t_0 + \Delta t).$$

Substituting these into the above expression for the distance, we obtain for it

$$\frac{\Delta t[Af'_0 + Bg'_0 + Ch'_0] + \frac{\overline{\Delta t}^2}{2!}[Af''_0 + Bg''_0 + Ch''_0] + \frac{\overline{\Delta t}^3}{3!}[Af'''_1 + Bg'''_2 + Ch'''_3]}{\sqrt{A^2 + B^2 + C^2}}$$

(where the derivatives $f'(t_0)$, $f''(t_0)$, . . . have been written as f'_0, f''_0, . . .) from which the quotient, distance/Δt, reduces to

$$\frac{[Af'_0 + Bg'_0 + Ch'_0] + \frac{\Delta t}{2!}[Af''_0 + Bg''_0 + Ch''_0] + \frac{\overline{\Delta t}^2}{3!}[Af'''_1 + Bg'''_2 + Ch'''_3}{\sqrt{A^2 + B^2 + C^2}}.$$

If, now, Δt is allowed to approach zero, the distance we are considering also approaches zero, while the ratio, distance/Δt, approaches

$$\frac{|Af'_0 + Bg'_0 + Ch'_0|}{\sqrt{A^2 + B^2 + C^2}}$$

For random values of A, B, C this is, in general, different from zero (unless $f'_0 = g'_0 = h'_0 = 0$, in which case we are dealing with a *singular* point of the curve). Hence, the distance from a neighboring point Q of the curve to a random plane through a point P of the curve is an infinitesimal of the same order as Δt, the increment in the parameter from the point P to the point Q.

If the plane in question contains not only the point P but also the tangent to the curve at P, the coefficients A, B, C satisfy the condition

$$Af'_0 + Bg'_0 + Ch'_0 = 0,$$

and for the distance considered we have

$$\lim_{\Delta t \to 0}\left(\frac{\text{distance}}{\Delta t}\right) = 0, \quad \lim_{\Delta t \to 0}\left(\frac{\text{distance}}{\overline{\Delta t}^2}\right) = \frac{|\frac{1}{2}[Af''_0 + Bg''_0 + Ch''_0]}{\sqrt{A^2 + B^2 + C^2}}$$

Since the second of these limits is a quantity not zero in general, the distance from the point Q to a random plane through the tangent line at the point P is an infinitesimal of the second order with respect to Δt.

If, now, the coefficients A, B, C are subjected to the further condition

$$Af''_0 + Bg''_0 + Ch''_0 = 0,$$

we obtain

$$\lim_{\Delta t \to 0} \frac{\text{distance}}{\overline{\Delta t}^2} = 0, \qquad \lim_{\Delta t \to 0} \frac{\text{distance}}{\overline{\Delta t}^3} = \frac{[Af'''_1 + Bg'''_2 + Ch'''_3]}{3!\sqrt{A^2 + B^2 + C^2}}$$

This last limit is in general not zero (*i.e.*, not for A, B, C that satisfy the two conditions imposed above), and the distance discussed is an infinitesimal of the third order with respect to Δt.

Note that making the distance from a neighboring point Q of the curve to the plane passing through P an infinitesimal of higher order at each step amounts to requiring that the plane fit the curve closer and closer at and near P. Note also that the conditions we have imposed on the plane, *viz.*, passing through P, containing the tangent drawn at P and having coefficients A, B, and C that satisfy the relation $Af''_0 + Bg''_0 + Ch''_0 = 0$, are all the conditions we can impose on one plane and are enough to precisely determine the plane. We have thus argued out the existence (save at exceptional points) of a definite and unique

plane the extent of whose closeness to the curve near P is measured by the fact that the distance from a neighboring point of the curve to it (the plane) is an infinitesimal of order at least three relative to the increment in the parameter value (and hence, relative to the increments of the coordinates). This plane is called the *osculating plane of the curve at the point P.* Its equation is thus

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0,$$

where

$$\begin{cases} Af'_0 + Bg'_0 + Ch'_0 = 0, \\ Af''_0 + Bg''_0 + Ch''_0 = 0. \end{cases}$$

Exercise 1. Show that the last two equations determine the ratios of A, B, and C as

$$A:B:C = [g'_0 \cdot h''_0 - h'_0 \cdot g''_0]:[h'_0 \cdot f''_0 - f'_0 \cdot h''_0]:[f'_0 \cdot g''_0 - g'_0 \cdot f''_0]$$

and, hence, that the equation of the osculating plane is

$$[g'_0 \cdot h''_0 - h'_0 \cdot g''_0](x - x_0) + [h'_0 \cdot f''_0 - f'_0 \cdot h''_0](y - y_0) \\ + [f'_0 \cdot g''_0 - g'_0 \cdot f''_0](z - z_0) = 0.$$

IV. *The Binormal.* This name is given to *that normal of the curve at P (x_0,y_0,z_0) which is perpendicular to the osculating plane at P.* Its direction numbers can, obviously, be taken, as

$$[g'_0 \cdot h''_0 - h'_0 \cdot g''_0], \quad [h'_0 \cdot f''_0 - f'_0 \cdot h''_0], \quad [f'_0 \cdot g''_0 - g'_0 \cdot f''_0].$$

V. *The Principal Normal.* By this term we designate *that normal of the curve at P (x_0,y_0,z_0) which lies in the osculating plane at P.* It is thus, the line of intersection of the osculating plane and the normal plane and is perpendicular to both the tangent line and the binormal. As equations of the principal normal we may hence display the pair

$$\begin{cases} f'_0(x - x_0) + g'_0(y - y_0) + h'_0(z - z_0) = 0, \\ [g'_0 \cdot h''_0 - h'_0 \cdot g''_0](x - x_0) + [h'_0 \cdot f''_0 - f'_0 \cdot h''_0](y - y_0) + \\ \qquad [f'_0 \cdot g''_0 - g'_0 \cdot f''_0](z - z_0) = 0. \end{cases}$$

If we call, for brevity, the sets of direction numbers displayed above for the tangent and the binormal a, b, c and l, m, n, respectively, while we designate a set of direction numbers of the principal normal by u, v, w, we have

$$ua + vb + wc = 0,$$
$$ul + vm + wn = 0.$$

From these, we readily obtain

$$u:v:w = (bn - cm):(cl - an):(am - bl).$$

VI. *The Rectifying Plane.* By this is designated *the plane through P (x_0,y_0,z_0) perpendicular to the principal normal at P.* It, therefore, contains the tangent line and the binormal at P. Its equation, obviously, is

$$u(x - x_0) + v(y - y_0) + w(z - z_0) = 0,$$

where u, v, w are the above direction numbers of the principal normal.

We have thus described, for each point of a space curve, a unique system of three mutually perpendicular planes, the normal plane, the osculating plane and the rectifying plane, which intersect in pairs in three mutually perpendicular lines, the tangent line, the binormal, and the principal normal. The adjoining figure is intended to aid the student in comprehending the configuration, known in differential geometry as *the moving trihedral.*

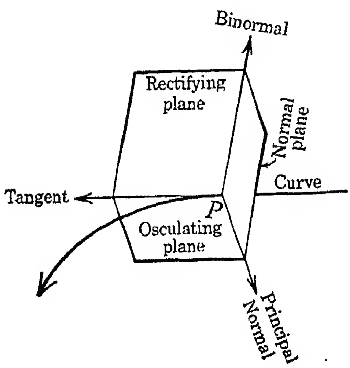

FIG. 170.

To illustrate, let us find a set of direction numbers for each of these lines and the equation of each of these planes (spoken of as the *principal directions* and *principal planes*) for that point on the helix

$$\begin{cases} x = a\cos t, \\ y = a\sin t, \\ z = bt, \end{cases}$$

at which $t = \pi/2$, *i.e.*, at the point $(0,a,b\pi/2)$. Here

$$\begin{array}{lll} f(t) = a\cos t, & f'(t) = -a\sin t, & f''(t) = -a\cos t, \\ g(t) = a\sin t, & g'(t) = a\cos t, & \quad '(t) = -a\sin t, \\ h(t) = bt, & h'(t) = b, & h''(t) = 0, \end{array}$$

$$\begin{array}{lll} f_0 = 0, & f'_0 = -a, & f''_0 = 0, \\ g_0 = a, & g'_0 = 0, & g''_0 = -a, \\ h_0 = \dfrac{b\pi}{2}, & h'_0 = b, & h''_0 = 0. \end{array}$$

Hence (*a*) A set of direction numbers for the tangent line is $-a, 0, b$.

(*b*) A set of direction numbers for the binormal is $[0 + ab]$, $[0]$, $[a^2 - 0]$ or $b, 0, a$.

(*c*) Calling the direction numbers of the principal normal u, v, w and recognizing that it is perpendicular to the tangent and binormal, we have

$$\begin{cases} -au + bw = 0, \\ bu + aw = 0. \end{cases}$$

These equations are satisfied only if $u = w = 0$, hence we exhibit a possible set of direction numbers of the principal normal as 0, 1, 0. It is, therefore parallel to OY.

Had we wished to examine the principal normal at any point of the curve, we would have set up the equations

$$-ua \sin t + va \cos t + wb = 0$$
$$ub \sin t - vb \cos t + wa = 0,$$

which are satisfied by

$$u:v:w = \cos t:\sin t:0.$$

From this, by setting $t = \pi/2$, we obtain the former result 0, 1, 0.

From the direction numbers of the three principal directions it is easy to set down the equations of the three principal planes.

(*d*) The equation of the osculating plane is

$$bx + a\left(z - \frac{b\pi}{2}\right) = 0.$$

(*e*) The equation of the normal plane is

$$-ax + b\left(z - \frac{b\pi}{2}\right) = 0.$$

(*f*) The equation of the rectifying plane is $y - a = 0$.

Exercise 2. Prove that if a curve is represented by two equations $f(x,y,z) = 0$, $g(x,y,z) = 0$, a set of direction numbers for its tangent at a point (x_0,y_0,z_0) is

$$(f_y g_z - f_z g_y)_0, \qquad (f_z g_x - f_x g_z)_0, \qquad (f_x g_y - f_y g_x)_0.$$

Exercise 3. Prove that all the tangents at a point P (x_0,y_0,z_0) of a surface, (*i.e.*, the tangents to all the curves on the surface passing through that point) lie in a plane.

HINT: Any given curve on the surface $f(x,y,z) = 0$ may be construed as the intersection of the given surface with some other surface $g(x,y,z) = 0$, hence as satisfying the two equations $f(x,y,z) = 0$, $g(x,y,z) = 0$.

Exercise 4. Prove that the direction cosines of the tangent to the curve $x = f(t)$, $y = g(t)$, $z = h(t)$, at the point (x_0,y_0,z_0) are $(dx/ds)_0$, $(dy/ds)_0$, $(dz/ds)_0$, where $ds = \sqrt{\overline{dx}^2 + \overline{dy}^2 + \overline{dz}^2}$.

Exercise 5. If l, m, n, the direction cosines of a line, are functions of a parameter t, and $\Delta\theta$ designates the angle between the direction (l,m,n) and $(l + \Delta l, m + \Delta m, n + \Delta n)$, prove that

$$\left(\frac{d\theta}{dt}\right)^2 = \left(\frac{dl}{dt}\right)^2 + \left(\frac{dm}{dt}\right)^2 + \left(\frac{dn}{dt}\right)^2.$$

HINT: From the relations $l^2 + m^2 + n^2 = 1$,

$(l + \Delta l)^2 + (m + \Delta m)^2 + (n + \Delta n)^2 = 1$, and $\cos(\Delta\theta) = l(l + \Delta l) + m(m + \Delta m) + n(n + \Delta n)$,

obtain

$$\cos(\Delta\theta) = 1 - \tfrac{1}{2}[\overline{\Delta l}^2 + \overline{\Delta m}^2 + \overline{\Delta n}^2],$$

whence

$$\left(\frac{\Delta l}{\Delta t}\right)^2 + \left(\frac{\Delta m}{\Delta t}\right)^2 + \left(\frac{\Delta n}{\Delta t}\right)^2 = \left(\frac{\sin\frac{\Delta\theta}{2}}{\Delta\theta/2}\right)^2 \cdot \left(\frac{\Delta\theta}{\Delta t}\right)^2.$$

Let $\Delta t \to 0$, and the result follows readily.

Exercise 6.* The curvature of a space curve is defined, as in the case of a plane curve, as $d\theta/ds$, where $\Delta\theta$ is the angle between the tangent at a given point and the tangent at a neighboring point (*i.e.*, as is commonly put, the curvature is the rate of turning of the tangent per unit length of the curve). Prove that the curvature of a curve is given by

$$k = \sqrt{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2 + \left(\frac{d^2z}{ds^2}\right)^2}.$$

(The reciprocal of the curvature, usually denoted by ρ, is called the *radius of curvature.*)

Exercise 7. The torsion of a space curve is defined as $d\varphi/ds$, where $\Delta\varphi$ is the angle between the binormal at a given point and the binormal at a neighboring point (*i.e.*, the torsion is the rate of turning of the binormal, or of the osculating plane, per unit length of the curve). Prove that the torsion of a curve is given by $\sqrt{(dl/ds)^2 + (dm/ds)^2 + (dn/ds)^2}$, where l, m, n are the direction cosines of the binormal. (The reciprocal of the torsion, usually designated by τ, is called the *radius of torsion.*)

Problems

1. Find a set of direction numbers for the tangent and binormal of the curve

$$\begin{cases} x = t\cos t \\ y = t\sin t \\ z = at \end{cases}$$

at the point where $t = \pi/2$. Also find the equations of the osculating plane and the normal plane. *Ans.* $(-\pi/2, 1, a)$; $(\pi a/2, -2a, \pi^2/4 + 2)$.

2. Find direction numbers for the principal normal, and the equation of the rectifying plane, for the data of Prob. 1.

$$\textit{Ans.}\quad \left(\frac{\pi^2}{4} + 2 + 2a^2, \frac{\pi a^2}{2} + \frac{\pi^3}{8} + \pi, \frac{\pi a}{2}\right).$$

3. Show that the curvature of the helix $\begin{cases} x = a\cos t \\ y = a\sin t \\ z = bt \end{cases}$ is constant and equals $a^2 +$

4. Show that the torsion of the helix in Prob. 3 is constant and equals $\overline{a^2 + b^2}$

5. Given the curve $\begin{cases} y = t^2, \\ z = \frac{4}{3}t^{3/2} \end{cases}$ at the point where $t = 1$,

(*a*) find the equations of the tangent line;

$$\textit{Ans.}\quad x - 1 = \frac{y-1}{2} = \frac{z - \frac{4}{3}}{2}$$

(*b*) find the equations of the binormal;

$$\textit{Ans.}\quad \frac{x-1}{-2} = \frac{y-1}{-1} = z - $$

(*c*) find the equations of the principal normal;

$$\textit{Ans.}\quad \frac{x-1}{2} = \frac{y-1}{-2} \qquad z - \frac{4}{3}.$$

(*d*) find the equation of the normal plane;

(*e*) find the equation of the osculating plane;

(*f*) find the equation of the rectifying plane;

(*g*) find the curvature and torsion. *Ans.* $\frac{1}{9}, \frac{1}{9}$.

6. Find the equations of the tangent line and normal plane for the curve $\begin{matrix} y = x^2 + 1 \\ z = y^2 - 1 \end{matrix}$ at the point $(-1,2,3)$.

$$\textit{Ans.}\quad \text{Tang. line: } x + 1 = \frac{y-2}{-2} = \frac{z-3}{-8}.$$

7. Show that the curve of Prob. 6 is represented parametrically by $\begin{matrix} x = t \\ y = t^2 + 1, \\ z = t^4 + 2t^2 \end{matrix}$. Find the equations of the osculating plane and the binormal at the point stated in Prob. 6.

8. Find the equations of the tangent line and the normal plane for the curve

(*a*) $\begin{matrix} x + 2y - z^2 = 1 \\ yz + x = 2 \end{matrix}$ at the point $(2,0,1)$;

$$\textit{Ans.}\quad \text{Tang. line: } \frac{x-2}{\;} = \frac{-y}{\;} = 1 - z.$$

(*b*) $\begin{cases} x^2 + y^2 - z = 8 \\ x - y^2 + z^2 = -2 \end{cases}$ at the point $(2,-2,0)$.

9. Show that the osculating plane of a plane curve is the plane in which the curve lies, and that its torsion is zero.

10. Show that a curve whose curvature at every point is zero is a straight line.

11. Given the curve $\begin{cases} x = 1 - t^2 \\ \quad = t(1 - t^2)^{1/2} \\ \quad = t \end{cases}$ show that every normal plane passes through the origin.

98. Directional Derivative. Let a function $f(x,y)$ be defined for a certain range of values of x and y (*i.e.*, in a certain region in the xy-plane), and let P (x,y) be a point in that region. The rate of change of the value of the function per unit length of arc in a given direction is defined as the *directional derivative* of the function $f(x,y)$ in that direction. In other words, the directional derivative of $f(x,y)$, at P, in the direction of the curve PQ (Fig. 171) is df/ds, where ds is the element of arc of the curve PQ. Now

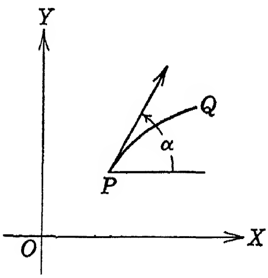

FIG. 171.

$$\frac{df}{ds} = f_x\frac{dx}{ds} + f_y\frac{dy}{ds} = f_x \cos \alpha + f_y \sin \alpha,$$

where α is the inclination of the tangent line of PQ to OX. Since the values of f_x and f_y are fixed if the point P is kept fixed, the directional derivative, for a particular function at a particular point, is a function of α, the angle that determines the direction of the curve in question. We may represent it by $D(\alpha)$. Desired, now, to find in what direction the function changes most rapidly, *i.e.*, to find the value of α that makes $D(\alpha)$ a maximum. To that end, we differentiate $D(\alpha)$, and obtain

$$D'(\alpha) = -f_x \sin \alpha + f_y \cos \alpha.$$

Setting this equal to zero, we obtain $\tan \alpha = f_y/f_x$. This gives two values of α, say, α_1 and $\alpha_2 = \alpha_1 + \pi$, such that

$$\sin \alpha_1 = \frac{f_y}{\sqrt{f_x^2 + f_y^2}}, \qquad \cos \alpha_1 = \frac{f_x}{\sqrt{f_x^2 + f_y^2}}$$

$$\sin \alpha_2 = \quad -f_y \qquad \cos \alpha_2 = \frac{-f_x}{\sqrt{f_x^2 + f_y^2}}.$$

By $D''(\alpha) = -f_x \cos \alpha - f_y \sin \alpha$, we get

$$D''(\alpha_1) = -\sqrt{f_x^2 + f_y^2}, \qquad D''(\alpha_2) = \sqrt{f_x^2 + f_y^2}.$$

Hence, $D(\alpha)$ has a maximum value for $\alpha = \alpha_1$ and a minimum value for $\alpha = \alpha_2$. The values of $D(\alpha)$ in these two directions are $\pm\sqrt{f_x^2 + f_y^2}$. It appears that the function increases most rapidly in the direction determined by $\tan \alpha = f_y/f_x$, $\sin \alpha = f_y/\sqrt{f_x^2 + f_y^2}$, and that its rate of increase in that direction is $\sqrt{f_x^2 + f_y^2}$. This (maximum) value of the directional derivative is called the *gradient* or *normal derivative* of the function $f(x,y)$ at the point P.

Note that the direction in which the directional derivative is zero (the function stationary) is given by

$$f_x \cos \alpha + f_y \sin \alpha = 0,$$

hence by $\tan \alpha = -f_x/f_y$, and that this direction is perpendicular to the one found above, giving the gradient.

To illustrate, consider the function $f(x,y) = xy + y^2$ and the point (1,2). We have $f_x = y = 2$, $f_y = x + 2y = 5$. Along any direction, the value of the directional derivative at that point is $2 \cos \alpha + 5 \sin \alpha$. Along a curve through (1,2), for which, say, $\alpha = 30°$, it has the value $\sqrt{3} + 5/2$. The function is stationary for the direction $\tan \alpha = -2/5$. The value of the gradient is $\sqrt{4 + 25} = \sqrt{29}$, and the directional derivative attains that value along any curve through the point (1,2) for which $\tan \alpha = 5/2$ and $\sin \alpha = 5/\sqrt{29}$.

We introduce the notion of a *contour line* for a given function $f(x,y)$. By this is meant a locus in the xy-plane for all points of which the function has the same value. If $f(x,y)$ is a single-valued function there will be one such curve passing through every point of the plane for which the function is defined. If the value of the function at a point P is f_0, the equation of the contour line passing through it is $f(x,y) = f_0$. The direction of its tangent at P is given by $\tan \alpha = dy/dx = -f_x/f_y$ and of its normal by $\tan \alpha = f_y/f_x$. The first is precisely the direction found above for which the function is stationary, while the second is the direction of the most rapid increase of the function, *viz.*, the direction of the gradient. Thus, *the direction of the gradient of the function at a point P is along the normal to the contour line of the function passing through*

that point. Hence, the alternative name for the gradient, the *normal derivative.*

Exercise 1. Prove that the value of the directional derivative in a direction making an angle φ with the direction of the gradient is

$$\sqrt{f_x^2 + f_y^2} \cdot \cos \varphi.$$

HINT: In the adjoining figure, let PN be the normal to the contour line through P (hence, the direction of the gradient). The directional derivative in the direction of PM is $f_x \cos (\alpha + \varphi) + f_y \sin (\alpha + \varphi)$. Expand this, and employ the relations $\cos\alpha = f_x/\sqrt{f_x^2 + f_y^2}$, $\sin\alpha = f_y/\sqrt{f_x^2 + f_y^2}$. To make a complete proof, consideration should also be made of the case in which the inclination of PM is $\alpha - \varphi$.

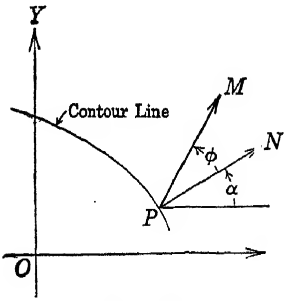

FIG. 172.

The notion of a directional derivative is extended to a function of three independent variables, and it is easily argued out that its value at a point P along a curve whose tangent at P has the direction angles α,β,γ is

$$\frac{df}{ds} = f_x \cos \alpha + f_y \cos \beta + f_z \cos \gamma.$$

The extension of the notion of contour lines takes the form of *contour surfaces,* surfaces at all points of which the function has the same value. The contour surface passing through a given point P has for the direction cosines of its normal

$$\cos \alpha = \frac{f_x}{\sqrt{f_x^2 + f_y^2 + f_z^2}} \qquad \cos \beta = \frac{f_y}{\sqrt{f_x^2 + f_y^2 + f_z^2}},$$
$$\cos \gamma = \frac{f_z}{\sqrt{f_x^2 + f_y^2 + f_z^2}}$$

and along that normal the value of the directional derivative is readily found to be $\sqrt{f_x^2 + f_y^2 + f_z^2}$. This, again, is called the gradient, or normal derivative, of the function $f(x,y,z)$ at P.

Exercise 2. Prove that the value of the directional derivative in a direction that makes an angle φ with the gradient is $\sqrt{f_x^2 + f_y^2 + f_z^2} \cdot \cos \varphi$. From that, show that the maximum value of the directional derivative at a given point in space is that of the gradient.

Directional derivatives have applications in studies such as the flow of heat through a metal due to a difference in temperature,

the flow of electricity through a conductor due to a difference in potential, and the flow of liquids and gases due to a difference in pressure. If the temperature, potential or pressure at a point P is a function of the coordinates of that point, the direction of flow at that point is that of the gradient. The contour lines for this function are known as *isothermals* in the case of heat flow, and *equipotential lines* in the case of electricity.

Problems

1. Interpret f_x and f_y for a function of two variables, also f_x, f_y, and f_z for a function of three variables, as directional derivatives.

2. For each function below, find the directional derivative for the point and direction specified, also the magnitude and direction of the gradient at that point.

(a) $x^2 - y^3$; $(-1,2)$; $(\alpha = 45°)$.

Ans. $-7\sqrt{2}$; $2\sqrt{37}$; $\tan^{-1} 6$.

(b) xe^y; $(2,0)$; $(\alpha = 60°)$. *Ans.* $\frac{1}{2} + \sqrt{3}$; $\sqrt{5}$; $\tan^{-1} 2$.

(c) $\log (xy)$; $(1,1)$; $(\alpha = -30°)$.

(d) $(x - y)^{\frac{1}{2}}$; $(5,1)$; $(\alpha = -45°)$.

(e) xyz; $(1,2,-1)$; $(\alpha = 60°, \beta = 45°, \gamma = 120°)$.

Ans. $-2 - \dfrac{\sqrt{2}}{2}$; 3; $\left(\cos^{-1} - \dfrac{2}{3}, \cos^{-1} - \dfrac{1}{3}, \cos^{-1}\dfrac{2}{3}\right)$

(f) $x^2 + y^2 - z^2$; $(0,1,3)$; $(\alpha = 60°, \beta = 45°, \gamma = 60°)$.

3. For each directional derivative called for in Prob. 2 verify the property set forth in Exercises 1 and 2.

4. Find the directional derivative of the function $x^2 + xy + y^2$ at the point $(1,-2)$, in the direction of the curve $x^2 - 2y = 5$. *Ans.* $\pm 3\sqrt{2}/2$.

5. Find the directional derivative of the function $e^x \log y$, at the point $(0,1)$, in the direction of the curve $x^2 + 2xy - y^2 + 1 = 0$.

6. If a point P moves on the curve $x = t$, $y = t^2 + 1$, $z = 2t - 3$, in the direction of increasing t, find, at the point where $t = 1$, at what rate its distance from

(a) the origin is changing per unit arc of the curve.

Ans. $1/\sqrt{6}$.

(b) the z-axis is changing per unit arc of the curve.

Ans. $\frac{1}{3}\sqrt{5}$.

7. Find the directional derivative of the function xy^2z^3 at the point $(1,2,2)$, in the direction of the curve $2x + z = 4$, $x^2 + y^2 + z^2 = 9$.

Ans. $32/\sqrt{29}$.

8. (a) For a function of two variables, show that the sum of the squares of the directional derivatives along any two perpendicular directions is constant at any point and equals the square of the gradient.

(b) For the point $(1,-2)$, find at what rate its distance from the origin is changing, per unit arc of curve, when it moves along the curve

$x^2 + y^2 = 5$, also when it moves along the line $2x + y = 0$. Verify, for these, the property proved in (*a*).

9. Show that the gradient of the function $f(r,\theta)$ at the point whose polar coordinates are (r,θ) is $[f_r^2 + (f_\theta/r)^2]^{1/2}$.

10. A plane area of metal is bounded by two concentric circles of radii 7 in. and 12 in. The temperature decreases uniformly from 212° at the inner circle to 32° at the outer. Express the temperature as a function of x and y. Find the isothermal curves and the lines of flow. At a point $3\sqrt{10}$ units from the center find the rate of change of the temperature per unit length of arc along the curve $y = x^2$. *Ans.* $684/\sqrt{370}$.

11. A heat-conducting medium occupies all points in three-dimensional space for which $x \geqq 0$. Owing to constant heating along the z-axis, the temperature at the point (x,y,z) reaches a steady value given by

$$100(1 + x^2 + y^2)^{-1}.$$

Find, at (0,7,0), the rates of change of the temperature per unit length of arc along curves whose tangents have the direction ratios, (1:0:0), (0:1:0), (0:0:1), and (1:1:0). Also find the direction of the gradient at that point. Prove that the lines of flow have the equations $z = c_1$, $y = c_2x$.

12. An infinite plane conducts electricity from the point (2,0), where the potential is 110 volts, to the point (0,0), where the potential is zero. The potential at any point is found to be a function $f(u)$ where $u = \dfrac{(x-2)^2 + y^2}{x^2 + y^2}$. Find the direction of the gradient at the points (1,0), (1,1), (−1,0), and (0,1). Also show that any circle through the points (0,0) and (2,0) has, at each of its points, the direction of the gradient for that point.

99. Envelopes. Consider a one-parameter family of curves, $f(x,y,\alpha) = 0$, of which α is the parameter. It may happen that there exists a curve that is tangent to every member of the family and that, conversely, every one of its points is such a point of tangency. Such a curve is called the *envelope* of the given family. If there are several such curves, all of them, taken together, are said to constitute the envelope. Thus, the family consisting of all lines tangent to a given curve has an envelope, *viz.*, that curve.

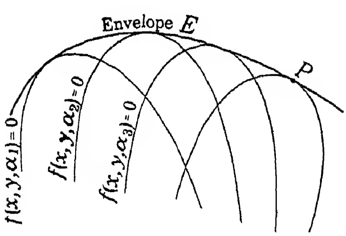

FIG. 173.

Since each point P of the envelope E may be taken to correspond to a definite member of the family, *viz.*, the one it touches, and hence to a definite value of the parameter α, we may assume the equations of E in the parametric form $\begin{cases} x = g(\alpha) \\ y = h(\alpha) \end{cases}$, and the problem

of finding the equation of the envelope, if it exists, for the given family $f(x,y,\alpha) = 0$, is that of finding the functions $g(\alpha)$ and $h(\alpha)$, explicitly or in some implicit form.

If P is a point on the envelope, the slope of the tangent to the envelope is $h'(\alpha)/g'(\alpha)$. Also, the slope of the corresponding member of the family is $-f_x(x,y,\alpha)/f_y(x,y,\alpha)$. Since the two curves are tangent to each other, we have $\dfrac{h'(\alpha)}{g'(\alpha)} = \dfrac{-f_x(x,y,\alpha)}{f_y(x,y,\alpha)}$, or

$$f_x(x,y,\alpha) \cdot g'(\alpha) + f_y(x,y,\alpha) \cdot h'(\alpha) = 0 \tag{131}$$

(x, y, and α all being those of the point of tangency, P). Now, by $f(x,y,\alpha) = 0$, we have, for every point of every member of the family, $f_x\,dx + f_y\,dy + f_\alpha\,d\alpha = 0$, or $f_x\dfrac{dx}{d\alpha} + f_y\dfrac{dy}{d\alpha} + f_\alpha = 0$. At the point P, $dx/d\alpha = g'(\alpha)$ and $dy/d\alpha = h'(\alpha)$. Hence, at that point $f_x \cdot g'(\alpha) + f_y \cdot h'(\alpha) + f_\alpha = 0$. In view of (131), $f_\alpha = 0$. Hence, at that point, we have the two equations satisfied, $\begin{cases} f(x,y,\alpha) = 0 \\ f_\alpha(x,y,\alpha) = 0. \end{cases}$ If, thus, an envelope exists, then for each point (x_1,y_1) of it there will be a value α_1, of α, such that x_1, y_1, α_1, will satisfy this pair of equations. We may, in any given case, choose to leave them as they stand, to solve for x and y, each in terms of α, thus obtaining $g(\alpha)$ and $h(\alpha)$, or to eliminate α, and obtain an equation for the envelope in the form $F(x,y) = 0$.

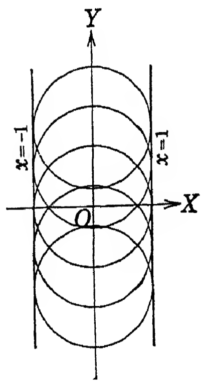

FIG. 174.

To illustrate, consider the equation

$$x^2 + (y - \alpha)^2 = 1,$$

representing a family of circles of radius 1 and with centers on the y-axis. The two equations found above to determine the envelope are, in this case,

$$x^2 + (y - \alpha)^2 = 1, \qquad \text{and} \qquad -2(y - \alpha) = 0.$$

It is simple to eliminate α from this pair of equations, obtaining $x^2 = 1$. A glance at Fig. 174 will satisfy one that the lines $x = \pm 1$ indeed form the envelope.

Exercise 1. Apply the above method to the family of parabolas,

$$y^2 - 3(x - \alpha) = 0,$$

and show that the resulting pair of equations has no locus. Draw several parabolas and explain.

Consider, now, the family of curves $(x - \alpha)^2 = y^2(y + 1)$. The pair of equations that determine the envelope becomes, for this family, $\begin{cases}(x - \alpha)^2 - y^2(y + 1) = 0, \\ x - \alpha = 0,\end{cases}$ which reduces to $y^2(y + 1) = 0$, representing two straight lines. A glance at the figure, with several members of the family sketched in, shows that the line $y = -1$ is an envelope while the line $y = 0$ is not.

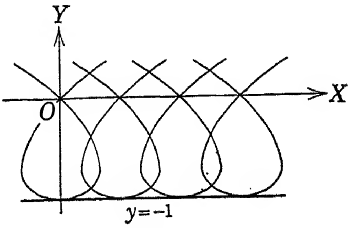

FIG. 175.

This last example shows that the equations that we obtain in this manner may represent some other locus besides the envelope. To determine, in any given case, whether the result found represents any locus other than the envelope, one checks by means of a figure or by comparison of the slopes. The latter method, used in the last example, would show the slope of any member of the family at any point (x,y) to be $\dfrac{dy}{dx} = \dfrac{2(x - \alpha)}{3y^2 + 2y}$. At the point $(\alpha,-1)$, where it meets the line $y = -1$, its slope is seen to be zero, the same as that of the line. At the point $(\alpha,0)$, where it meets the x-axis, the slope is indeterminate. We may resolve the indeterminacy by setting $x - \alpha = \pm y\sqrt{y + 1}$. The slope then reduces to $\pm\dfrac{2\sqrt{y + 1}}{3y + 2} = \pm 1$. There are, thus, two tangents to the curve at the point $(\alpha,0)$, and neither of them is the x-axis. This rejects the x-axis as an envelope.

The student may well inquire how an extraneous locus (the x-axis in the above example) happens to be represented by the equations that are intended to represent the envelope. A hint to the answer lies in the observation that, in the above example, the x-axis is the locus of points of the given family of curves, where $f_x = 2(x - \alpha) = 0$ and $f_y = -(3y^2 + 2y) = 0$. To complete the discussion the student is asked to turn to the following exercise:

Exercise 2. Prove that if each curve of the family $f(x,y,\alpha) = 0$ possesses a point at which $f_x = f_y = 0$, then the locus of those points is a part of the locus of the pair of equations $\begin{cases}f(x,y,\alpha) = 0, \\ f_\alpha(x,y,\alpha) = 0.\end{cases}$

Problems

1. Find the envelope of each one-parameter family given below, if it exists. Sketch a few curves of the family and the envelope. Distinguish such extraneous loci as may appear.

(a) $(x - \alpha)^2 + (y - 1)^2 = 4$. *Ans.* $(y + 1)(y - 3) = 0$.

(b) $\dfrac{x}{\alpha} + y\alpha = 1$. *Ans.* $4xy = 1$.

(c) $x^2 - 2x + 4(y - \alpha) = 0$. *Ans.* No envelope.

(d) $x = -\alpha y + \alpha^2$. *Ans.* $y^2 + 4x = 0$.

(e) $x \cos \alpha + y \sin \alpha - 1 = 0$. *Ans.* $x^2 + y^2 = 1$.

(f) $(y + \alpha)^2 + x^3(x - 1) = 0$.

(g) $(y + \alpha)^2 + x^4(x - 1) = 0$.

(h) $y + \alpha x - 2\alpha - \alpha^3 = 0$. *Ans.* $27y^2 = 4(x - 2)^3$.

2. A line of constant length l moves with its ends on the two axes. Find the curve it envelops (*i.e.*, the envelope of the one-parameter family of lines described). *Ans.* $x^{2/3} + y^{2/3} = l^{2/3}$.

HINT: Write the equation of the family of lines as

$$x \cos \alpha + y \sin \alpha - p = 0.$$

Show that the two parameters, α and p, are related by $p = \pm l \sin \alpha \cdot \cos \alpha$ (the $+$ sign for lines across the first or third quadrants, the $-$ sign for lines across the second and fourth quadrants). The equation of the family is, then, $x \cos \alpha + y \sin \alpha \pm l \sin \alpha \cdot \cos \alpha = 0$. The resulting equations of the envelope will be found to be $\begin{cases} x = l \sin^3 \alpha, \\ y = l \cos^3 \alpha. \end{cases}$ Now eliminate α.

3. Find the envelope of the family of lines such that both intercepts are positive and their sum is a constant, equal to c.

Ans. $x^{1/2} + y^{1/2} = c^{1/2}$.

4. Find the envelope of the family of circles that

(a) Have the double ordinates of the parabola $y^2 = 2px$ for diameters. *Ans.* $y^2 = 2px + p^2$.

HINT: If (u,v) and $(u, -v)$ are the ends of a diameter, $v^2 = 2pu$.

(b) Pass through the origin and have their centers on the curve $y^2 = 2px$.

5. Find the envelope of the family of ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$,

(a) Whose area is a constant, equal to c. HINT: The area of an ellipse, with semiaxes a and b, is πab; *Ans.* $yx = \pm c/2\pi$.

(b) For which the line joining the extremities of the major and minor axes is of constant length c.

6. Obtain the evolute of the parabola $y = x^2$ as the envelope of its normals. HINT: The equation of a normal of the parabola at the point (u,v) is

$$y - v = -\frac{1}{2u}(x - u),$$

where $v = u^2$. *Ans.* $27x^2 = 2(2y - 1)^3$.

7. Obtain the evolute of the ellipse $\begin{matrix} x = a \cos \varphi \\ y = b \sin \varphi \end{matrix}$ as the envelope of its normals. HINT: The equation of the normal, at a point corresponding to the parameter value φ, is

$$-b \sin \varphi = \frac{a \tan \varphi}{b}(x - a \cos \varphi).$$

Ans. $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$.

8. Obtain the evolute of the four-cusped hypocycloid $\begin{cases} x = a \cos^3 \varphi \\ y = a \sin^3 \varphi \end{cases}$ as the envelope of its normals. *Ans.* $(x + y)^{\frac{2}{3}} + (x - y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

9. Obtain the evolute of any curve $y = f(x)$ as the envelope of its normals. From the result show that the center of curvature corresponding to the point (u,v) of the curve $y = f(x)$ is the point where the evolute touches the normal drawn at the point (u,v). (Compare with formulas on page 216.)

100. Theorem of the Mean. Taylor's Formula. Given a function $f(x,y)$, continuous, along with its first partial derivatives, in the neighborhood of the point (a,b), consider the value of $f(a + h, b + k)$, where h and k are arbitrary increments of the independent variables. To be able to avail ourselves of the Theorem of the Mean for the functions of one variable, let us introduce an independent variable t, and define x and y by $x = a + th$, $y = b + tk$. Then

$$f(x,y) = f(a + th, b + tk) = F(t).$$

The function $F(t)$ is continuous in the interval $0 \leqq t \leqq 1$, if we assume $f(x,y)$ to be continuous for $a \leqq x \leqq a + h, b \leqq y \leqq b + k$ [since $F(0) = f(a,b)$ and $F(1) = f(a + h, b + k)$]. Now, for any value of t between 0 and 1, we have,

$$F(t) = F(0) + tF'(\theta t), \qquad 0 < \theta < 1.$$

Again,

$$\begin{aligned} F'(t) = D_tF(t) = D_tf(x,y) &= f_x(x,y) \cdot D_tx + f_y(x,y) \cdot D_ty \\ &= h \cdot f_x(x,y) + k \cdot f_y(x,y) \\ &= h \cdot f_x(a + th, b + tk) + k \cdot f_y(a + th, b + tk), \end{aligned}$$

and hence

$$F'(\theta t) = h \cdot f_x(a + \theta th, b + \theta tk) + k \cdot f_y(a + \theta th, b + \theta tk),$$

from which

$$\begin{aligned} f(a + th, b + tk) = f(a,b) + t[hf_x(a &+ \theta th, \\ b + \theta tk) &+ kf_y(a + \theta th, b + \theta tk)]. \end{aligned}$$

If we now set $t = 1$, the last equality becomes

$$f(a + h, b + k) = f(a,b) + hf_x(a + \theta h, b + \theta k) + kf_y(a + \theta h, b + \theta k), (0 < \theta < 1).$$

This is the theorem of the mean for a function of two variables.

If the second partial derivatives of $f(x,y)$ exist and are continuous in the interval considered, then $F''(t)$ will exist, be continuous and be given by

$$\begin{aligned} F''(t) &= D_t[hf_x(x,y) + kf_y(x,y)] \\ &= h[f_{xx}(x,y) \cdot D_tx + f_{xy}(x,y) \cdot D_ty] + k[f_{xy}(x,y) \cdot D_tx + f_{yy}(x,y) \cdot D_ty] \\ &= h^2f_{xx}(x,y) + 2hkf_{xy}(x,y) + k^2f_{yy}(x,y) \\ &= h^2f_{xx}(a + th, b + tk) + 2hkf_{xy}(a + th, b + tk) + k^2f_{yy}(a + th, b + tk). \end{aligned}$$

By Taylor's formula for a function of one variable, we have

$$F(t) = F(0) + tF'(0) + \frac{t^2}{2!}F''(\varphi t), \qquad (0 < \varphi < 1),$$

and this becomes, on putting $t = 1$,

$$\begin{aligned} f(a + h, b + k) = f(a,b) + hf_x(a,b) + kf_y(a,b) + \frac{1}{2!}[h^2f_{xx}(a + \varphi h, b + \varphi k) + 2hkf_{xy}(a + \varphi h, b + \varphi k) + \\ k^2f_{yy}(a + \varphi h, b + \varphi k)], \qquad (0 < \varphi < 1). \quad (132) \end{aligned}$$

Exercise 1. Prove that if the third partial derivatives of $f(x,y)$ exist and are continuous in the interval $a \leqq x \leqq a + h$, $b \leqq y \leqq b + k$, then

$$\begin{aligned} f(a + h, b + k) &= f(a,b) + hf_x(a,b) + kf_y(a,b) \\ &+ \frac{1}{2!}[h^2f_{xx}(a,b) + 2hkf_{xy}(a,b) + k^2f_{yy}(a,b)] \qquad (133) \\ &+ \frac{1}{3!}[h^3f_{xxx}(a + \rho h, b + \rho k) + 3h^2kf_{xxy}(a + \rho h, b + \rho k) \\ &+ 3hk^2f_{xyy}(a + \rho h, b + \rho k) + k^3f_{yyy}(a + \rho h, b + \rho k)], \qquad (0 < \rho < 1). \end{aligned}$$

The results embodied in the theorem of the mean, formulas (132) and (133), are forms of *Taylor's formula* for a function of two variables. As in the case of a function of one variable, granting the continuity of the partial derivatives of $f(x,y)$ of still higher orders, the expansion for $f(a + h, b + k)$ by Taylor's formula can be carried out to still more terms than occur in (133).

We may now put $a = x$, $b = y$, and display the law of the mean and Taylor's formula as

$$f(x + h, y + k) = f(x,y) + hf_x(x + \theta h, y + \theta k) + kf_y(x + \theta h, y + \theta k), \quad (134)$$

$$f(x + h, y + k) = f(x,y) + [hf_x(x,y) + kf_y(x,y)] + \frac{1}{2!}[h^2 f_{xx}(x + \theta h, y + \theta k) + 2hkf_{xy}(x + \theta h, y + \theta k) + k^2 f_{yy}(x + \theta h, y + \theta k)], \quad (135)$$

$$f(x + h, y + k) = f(x,y) + [hf_x(x,y) + kf_y(x,y)] + \frac{1}{2!}[h^2 f_{xx}(x,y) + 2hkf_{xy}(x,y) + k^2 f_{yy}(x,y)] + \frac{1}{3!}[h^3 f_{xxx}(x + \theta h, y + \theta k) + 3h^2 kf_{xxy}(x + \theta h, y + \theta k) + 3hk^2 f_{xyy}(x + \theta h, y + \theta k) + k^3 f_{yyy}(x + \theta h, y + \theta k)], \quad (136)$$

etc., in each of which $0 < \theta < 1$.

As an example, consider $f(x,y) = (1 - x + y)^{1/2}$. We have

$$f_x = -\tfrac{1}{2}(1 - x + y)^{-1/2}, \qquad f_y = \tfrac{1}{2}(1 - x + y)^{-1/2},$$
$$f_{xx} = -\tfrac{1}{4}(1 - x + y)^{-3/2}, \qquad f_{xy} = \tfrac{1}{4}(1 - x + y)^{-3/2},$$
$$f_{yy} = -\tfrac{1}{4}(1 - x + y)^{-3/2}$$
$$f_{xxx} = -\tfrac{3}{8}(1 - x + y)^{-5/2}, \qquad f_{xxy} = \tfrac{3}{8}(1 - x + y)^{-5/2},$$
$$f_{xyy} = -\tfrac{3}{8}(1 - x + y)^{-5/2}, \qquad f_{yyy} = \tfrac{3}{8}(1 - x + y)^{-5/2}.$$

Hence, the expansions of $[1 - (x + h) + (y + k)]^{1/2}$ by the theorem of the mean and by Taylor's formula as given in (135) and (136) are, respectively,

$$[1 - (x + h) + (y + k)]^{1/2} = [1 - x + y]^{1/2} - \tfrac{1}{2}(h - k)[1 - (x + \theta h) + (y + \theta k)]^{-1/2},$$

$$[1 - (x + h) + (y + k)]^{1/2} = [1 - x + y]^{1/2} - \tfrac{1}{2}(h - k)[1 - x + y]^{-1/2} - \tfrac{1}{8}(h^2 - 2hk + k^2)[1 - (x + \theta h) + (y + \theta k)]^{-3/2},$$

$$[1 - (x + h) + (y + k)]^{1/2} = [1 - x + y]^{1/2} - \tfrac{1}{2}(h - k)[1 - x + y]^{-1/2} - \tfrac{1}{8}(h^2 - 2hk + k^2)[1 - x + y]^{-3/2} - \tfrac{1}{16}(h^3 - 3h^2k + 3hk^2 - k^3)[1 - (x + \theta h) + (y + \theta k)]^{-5/2},$$

the values of θ in the succeeding expressions not being necessarily the same but included between 0 and 1 in each case.

If, in particular, $x = 1$, $y = 1$, $h = 0.1$, $k = 0.2$, the above expansions become approximations for

$$[1 - (1.1) + (1.2)]^{1/2} = (1.1)^{1/2}$$

of the form

$$(1.1)^{1/2} = (1)^{1/2} - \tfrac{1}{2}(-0.1)[1 + (0.1)\theta]^{-1/2} = 1 + \tfrac{1}{20}[1 + (0.1)\theta]^{-1/2},$$

$$(1.1)^{1/2} = (1)^{1/2} - \tfrac{1}{2}(-0.1)(1)^{-1/2} - \tfrac{1}{8}(0.01)[1 + (0.1)\theta]^{-3/2} = 1 + \tfrac{1}{20} - \tfrac{1}{800}[1 + (0.1)\theta]^{-3/2},$$

$$(1.1)^{1/2} = (1)^{1/2} - \tfrac{1}{2}(-0.1)(1)^{-1/2} - \tfrac{1}{8}(0.01)(1)^{-3/2} - \tfrac{1}{16}(-0.001)[1 + (0.1)\theta]^{-5/2}$$
$$= 1 + \frac{1}{20} - \frac{1}{800} + \frac{1}{16{,}000}[1 + (0.1)\theta]^{-5/2}.$$

If we ignore the last term in each expression, we obtain for successive values of $(1.1)^{1/2}$ the numbers 1, 1.05, 1.04875, readily verified as steadily better approximations by reference to a table of square roots.

Problems

1. For $f(x,y) = x \log (y + 1)$, verify the following equalities:

$$x \log (y + 1) = 0 + x \log (\theta y + 1) + \frac{\theta xy}{\theta y + 1}, \quad (0 < \theta < 1).$$

$$x \log (y + 1) = 0 + x \cdot 0 + y \cdot 0 + \frac{1}{2}\left(x^2 \cdot 0 + \frac{2xy}{\theta y + 1} - \frac{\theta xy^2}{(\theta y + 1)^2}\right), \quad (0 < \theta < 1).$$

$$x \log (y + 1) = 0 + x \cdot 0 + y \cdot 0 + \tfrac{1}{2}(x^2 \cdot 0 + 2xy + y^2 \cdot 0) + \frac{1}{6}\left(x^3 \cdot 0 + 3x^2y \cdot 0 - \frac{3xy^2}{(\theta y + 1)^2} + \frac{2\theta xy^3}{(\theta y + 1)^3}\right), \quad (0 < \theta < 1).$$

From these we readily conclude that for small values of x and y, an approximation of $x \log (y + 1)$ is xy.

2. For $f(x,y) = \sin x \cos y$, verify the equality

$$\sin x \cos y = 0 + x \cdot 1 + y \cdot 0 + \tfrac{1}{2}(x^2 \cdot 0 + 2xy \cdot 0 + y^2 \cdot 0) + \tfrac{1}{6}[-x^3 \cos (\theta x) \cos (\theta y) + 3x^2y \sin (\theta x) \sin (\theta y) - 3xy^2 \cos (\theta x) \cos (\theta y) + y^3 \sin (\theta x) \sin (\theta y)], \quad (0 < \theta < 1),$$

and hence, that for small values of x and y an approximation to $\sin x \cos y$ is x.

3. Obtain an approximation to e^{x+y} in the form

$$e^{x+y} = 1 + x + y + \tfrac{1}{2}(x^2 + 2xy + y^2) + \tfrac{1}{6}(x^3 + 3x^2y + 3xy^2 + y^3).$$

4. Obtain an approximation to $e^x \cos y$ in the form

$$e^x \cos y = 1 + x + \tfrac{1}{2}(x^2 - y^2) + \tfrac{1}{6}(x^3 - 3xy^2).$$

5. Obtain the approximate value of

(a) $e^{0.3}$ by setting $x = 0.1$, $y = 0.2$ in Prob. 3. *Ans.* 1.3495.

(b) $e^{0.1} \cos 1°$ by using the result of Prob. 4.

6. (a) Obtain an approximation to $\sqrt{1 + x^2 + y^2}$ in the form of a polynomial in x and y as far as terms of the second degree.

(b) From the result in (a) obtain the approximate value of $\sqrt{1.05}$ by setting $x = 0.1$, $y = 0.2$.

7. For a function $f(x,y,z)$ of three variables, obtain the relations

$$f(x + h, y + k, z + l) = f(x,y,z) + hf_x(x + \theta h, y + \theta k, z + \theta l) + kf_y(x + \theta h, y + \theta k, z + \theta l) + lf_z(x + \theta h, y + \theta k, z + \theta l), \quad (0 < \theta < 1),$$

$$\begin{aligned} f(x + h, y + k, z + l) &= f(x,y,z) + hf_x(x,y,z) + kf_y(x,y,z) + lf_z(x,y,z) \\ &+ \tfrac{1}{2}[h^2 f_{xx}(x + \theta h, y + \theta k, z + \theta l) + k^2 f_{yy}(x + \theta h, y + \theta k, z + \theta l) \\ &+ l^2 f_{zz}(x + \theta h, y + \theta k, z + \theta l) + 2hk f_{xy}(x + \theta h, y + \theta k, z + \theta l) \\ &+ 2kl f_{yz}(x + \theta h, y + \theta k, z + \theta l) + 2lh f_{zx}(x + \theta h, y + \theta k, z + \theta l)], \end{aligned} \quad (0 < \theta < 1)$$

101. Maximum and Minimum Values of a Function of Two Variables. As in the case of a function of one variable, a function $f(x,y)$ is said to have a maximum (minimum) value at (a,b) if $f(a + h, b + k) - f(a,b)$ is negative (positive) for all values of h and k sufficiently near zero, say, $0 < h^2 + k^2 < p^2$, where p is some real number not zero. Now, a necessary condition for a maximum (minimum) value of $f(x,y)$ at (a,b) may be found in the fact that if we put $k = 0$ and deal with $f(x,b)$, a function of x alone, this function must have a maximum (minimum) value for $x = a$, a condition for which, if $f_x(x,b)$ is continuous at $x = a$, is $f_x(a,b) = 0$. By a like argument, we obtain the necessary condition $f_y(a,b) = 0$ if $f_y(a,y)$ is continuous at $y = b$. The continuity of the function and the existence of its first partial derivatives in the neighborhood of (a,b) being assumed, we thus arrive at a necessary condition for a maximum (minimum) value of $f(x,y)$ as

$$\begin{cases} f_x(a,b) = 0, \\ f_y(a,b) = 0. \end{cases} \tag{137}$$

To seek, now, a sufficient condition for a maximum or minimum value of $f(x,y)$ at (a,b) we make use of the directional derivative

$$f_x(x,y) \cos \alpha + f_y(x,y) \sin \alpha$$

at a point (x,y) and in the direction α, where the point (x,y) is at a distance p from (a,b) in the direction α, *i.e.*, at $(a + p \cos \alpha, b + p \sin \alpha)$. Thus evaluated, the directional derivative is

$$f_x(a + p \cos \alpha, b + p \sin \alpha) \cdot \cos \alpha + f_y(a + p \cos \alpha, b + p \sin \alpha) \cdot \sin \alpha. \tag{138}$$

If $f(x,y)$ is continuous at (a,b) and if, for all values of α and sufficiently small positive values of p, the expression (138) is negative and if, also, $f(x,y)$ is defined at (a,b), then $f(x,y)$ has a maximum value at (a,b). If, for all values of α and sufficiently small positive values of p, the expression (138) is positive and if, also, $f(x,y)$ is defined at (a,b), then $f(x,y)$ has a minimum value at (a,b).

The proof that the condition, as stated, is sufficient to insure a maximum or minimum value of $f(x,y)$ at (a,b) lies in the fact that in the one case $f(x,y)$ is decreasing as (x,y) leaves (a,b) in any direction, while in the other case $f(x,y)$ is increasing as (x,y) leaves (a,b) in any direction.

Note that the above condition for a maximum or minimum value of $f(x,y)$ at (a,b) is sufficient if $f_x(x,y)$ and $f_y(x,y)$ are continuous in the deleted neighborhood of (a,b) even if those partial derivatives are not continuous at (a,b). In the light of this fact, Eqs. (137), being neither sufficient nor necessary [in case f_x and f_y are not continuous at (a,b)], are chiefly useful for locating critical points (a,b) at which to apply the test involving (138). Other critical points representing possible maximum or minimum values of $f(x,y)$ are those at which f_x and f_y are both discontinuous or one is discontinuous while the other is zero. Thus, Eqs. (137) may be augmented to include

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = \infty, \end{cases} \qquad \begin{cases} f_x(x,y) = \infty \\ f_y(x,y) = 0, \end{cases}$$
$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0, \end{cases} \qquad \begin{cases} f_x(x,y) = \infty \\ f_y(x,y) = \infty. \end{cases} \tag{139}$$

Any point (a,b) which satisfies any one of these pairs of conditions should be tested to see whether, for it, the expression (138) is definitely positive or definitely negative for all values of α when p is a sufficiently small positive number.

To illustrate, consider the function $f(x,y) = x^2 + xy + y^2 - 3y$. Here

$$f_x(x,y) = 2x + y, \qquad f_y(x,y) = x + 2y - 3.$$

The equations $\begin{cases} 2x + y = 0 \\ x + 2y - 3 = 0 \end{cases}$ are satisfied by $(-1,2)$. With $a = -1$, $b = 2$, expression (138) becomes

$$[2(-1 + p\cos\alpha) + (2 + p\sin\alpha)] \cdot \cos\alpha + [(-1 + p\cos\alpha) + 2(2 + p\sin\alpha) - 3] \cdot \sin\alpha$$

which reduces to

$$p(2 + \sin 2\alpha).$$

This quantity is positive regardless of α when p is positive. Hence $f(x,y)$ has a minimum at $(-1,2)$. Geometrically this means that the surface $z = x^2 + xy + y^2 - 3y$ has its lowest point at $(-1,2,-3)$.

Problems

1. Examine the following functions for maximum and minimum values:

(a) $2x^2 - 3xy + 2y^2 - 11x + 10y$. *Ans.* Min. at $(2,-1)$.

(b) $x^3 + y^3 - 12y$.

(c) $x^3 + y^3 + 3xy$. *Ans.* Max. at $(-1,-1)$.

(d) $4xy^2 - 2x^2y - x$. *Ans.* No max. or min.

(e) $x(1 + \sqrt{1 - y^2}) + y(1 + \sqrt{1 - x^2})$.

(f) $a - x^{2/3} - y^{2/3}$. *Ans.* Max. at $(0,0)$.

(g) $x^4 + y^4$. *Ans.* Min. at $(0,0)$.

(h) $x^4 + y^3$. *Ans.* No min. and no max.

2. Find the point in the plane $x + 2y + z = 6$, nearest the origin. *Ans.* $(1,2,1)$.

3. Find the shortest distance between the lines $\begin{cases} y = 2x \\ z = x + 1 \end{cases}$ and $\begin{cases} y = x - 1 \\ z = x. \end{cases}$

4. The perimeter of a triangle given as equal to a, find the lengths of the sides of the triangle which gives the maximum area. *Ans.* Each, $a/3$.

5. A rectangular parallelepiped has three of its faces in the coordinate planes and one vertex in the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ where a, b, and c are positive. Find its dimensions to give the greatest volume. *Ans.* $(a/3, b/3, c/3)$.

6. Find the dimensions of the rectangular parallelepiped of the largest volume that can be inscribed in the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

7. A tent has the shape of a cylinder surmounted by a cone. Find its dimensions to require the least amount of canvas, for a given volume a.

8. Determine the point within a triangle such that the sum of the squares of its distances from the vertices may be a minimum. *Ans.* Center of gravity of the triangle.

9. Divide 94 into three parts such that one-half the product of one pair, plus one-third the product of another pair, plus one-fourth the product of the third pair may sum to a maximum value. *Ans.* 40, 42, 12.

10. (a) Given a particle of mass 3, placed at $(1,0)$, a particle of mass 2, placed at $(-1,2)$, and a particle of mass 4, placed at $(3,-1)$, find the point in the plane about which the moment of inertia of the system is least. Verify that the point found is the center of gravity of the three particles.

(b) Verify that in the case of any number of particles $m_1, m_2, \ldots m_n$, placed at (x_1,y_1), (x_2,y_2), $\ldots$ (x_n,y_n), respectively, their center of gravity is the point about which the moment of inertia of the n particles is least.

11. Show that a necessary condition for $f(x,y,z)$ to have a maximum or minimum value at (a,b,c) is

$$\begin{cases} f_x(a,b,c) = 0, \\ f_y(a,b,c) = 0, \\ f_z(a,b,c) = 0, \end{cases}$$

the continuity of the function and of its first partial derivatives in the neighborhood of (a,b,c) being assumed.

12. Adapt the test for a maximum or minimum value of $f(x,y)$ at a given point (a,b), given in terms of the directional derivative, for a function of three variables, say $f(x,y,z)$ at the point (a,b,c).

13. Employ the conditions developed in Probs. 11 and 12 to find points in space at which the following functions have maximum or minimum values.

(a) $x^2 + y^2 + z^2$. *Ans.* Min. at (0,0,0).

(b) $1 - x^{2/3} - y^{2/3} - z^{2/3}$. *Ans.* Max. at (0,0,0).

(c) $x^4 + y^5 + z^6$. *Ans.* None.

102. Exact Differentials.

An expression of the form

$$M(x,y)dx + N(x,y)dy$$

is called an *exact differential* if it is the total differential of some function $z(x,y)$, *i.e.*, if a function $z(x,y)$ exists such that

$$dz = M(x,y)dx + N(x,y)dy.$$

When such is the case, if we hold y constant—hence, $dy = 0$—we obtain $dz = M(x,y)dx$, or $\dfrac{dz}{dx}_{y=\text{constant}} = M(x,y)$, *i.e.*,

$$z_x = M(x,y). \tag{140}$$

By a like argument

$$z_y = N(x,y). \tag{141}$$

If the functions $M(x,y)$, $N(x,y)$ and their first partial derivatives are continuous, then the second partial derivatives of z are continuous. Hence, $\dfrac{\partial}{\partial y}(z_x) = \dfrac{\partial}{\partial x}(z_y)$, and from (140) and (141) we arrive at

$$M_y(x,y) = N_x(x,y) \tag{142}$$

as *a necessary condition for $M(x,y)dx + N(x,y)dy$ to be an exact differential.*

Thus, the expression $(2x - 4y)dx + (3x + 2y)dy$ is not an exact differential, since, for it,

$$M(x,y) = 2x - 4y, \qquad N(x,y) = 3x + 2y,$$
$$M_y(x,y) = -4, \qquad N_x(x,y) = 3,$$

and $M_y \neq N_x$.

Let, now, an expression $M(x,y)dx + N(x,y)dy$ be given for which (142) holds. Should this be an exact differential, and equal to dz, it would entail

$$z_x = M(x,y), \qquad z_y = N(x,y).$$

From the first of these,

$$z = \int M(x,y)dx = f(x,y) + g(y),$$

where $f(x,y)$ is obtained by integrating $M(x,y)dx$ as a function of x alone, *i.e.*, with y held constant. Hence $f_x(x,y) = M(x,y)$. Here $g(y)$, an arbitrary function of y, serves as the constant of integration. We wish to select this function $g(y)$ so that z, defined as $f(x,y) + g(y)$ will meet with the requirements, $z_x = M$, $z_y = N$. The first is met, and the second takes the form

$$z_y = f_y(x,y) + g'(y) = N(x,y),$$

or

$$g'(y) = N(x,y) - f_y(x,y).$$

Such a function $g(y)$ can evidently be found, if, and only if, $N(x,y) - f_y(x,y)$, the right-hand member of the last equation, is a function of y alone. To show that this is true (under the condition $M_y = N_x$) is to show that the partial derivative of the expression with respect to x is zero. Now

$$\begin{aligned}\frac{\partial}{\partial x}[N(x,y) - f_y(x,y)] &= N_x(x,y) - f_{yx}(x,y) = M_y(x,y) - f_{yx}(x,y)\\ &= \frac{\partial}{\partial y}[M(x,y) - f_x(x,y)] = \frac{\partial}{\partial y}(0) = 0.\end{aligned}$$

We have, thus, argued out that (142) *is also a sufficient condition for $M(x,y)dx + N(x,y)dy$ to be an exact differential.*

The steps employed in the proof just completed suggest also the manner in which to find the function z whose total differential is a given exact differential $M(x,y)dx + N(x,y)dy$.

For example, given $(2xy + 2y^2 - 2x)dx + (x^2 + 4xy - 4y)dy$, we satisfy ourselves, by $M_y = 2x + 4y = N_x$, that it is an exact differential, and hence that there exists a function $z(x,y)$ such that

$$z_x = 2xy + 2y^2 - 2x,$$
$$z_y = x^2 + 4xy - 4y.$$

Integrating the first, holding y constant, we have

$$z = \int (2xy + 2y^2 - 2x)dx = x^2y + 2y^2x - x^2 + g(y).$$

By comparing the result of this with the second of the above conditions, we have

$$x^2 + 4xy + g'(y) = z_y = x^2 + 4xy - 4y.$$

Hence $g'(y) = -4y$, so that $g(y) = -2y^2 + C$, and

$$z = x^2y + 2xy^2 - x^2 - 2y^2 + C.$$

Exercise 1. Prove that if $du = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$, then $u_x = P(x,y,z)$, $u_y = Q(x,y,z)$, $u_z = R(x,y,z)$. ($Pdx + Qdy + Rdz$ is then said to be an *exact differential.*)

Exercise 2. Prove that the relations $P_y = Q_x$, $Q_z = R_y$, $R_x = P_z$, are necessary conditions that $P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ be an exact differential.

Exercise 3. Prove that the relations stated in Exercise 2 are also sufficient conditions for an exact differential.

To illustrate the case of an exact differential in the case of three variables, consider the expression

$$(2xy + z^2)dx + (x^2 + 2yz)dy + (y^2 + 2xz)dz.$$

Here, $P_y = 2x = Q_x$, $Q_z = 2y = R_y$, $R_x = 2z = P_z$, and hence, by Exercises 2 and 3, we have an exact differential, *i.e.*, there exists a function $u(x,y,z)$ such that

$$du = (2xy + z^2)dx + (x^2 + 2yz)dy + (y^2 + 2xz)dz,$$

and, furthermore, by Exercise 1,

$$u_x = 2xy + z^2, \qquad u_y = 2yz + x^2, \qquad u_z = 2xz + y^2.$$

By the first of these, $u = \int (2xy + z^2)dx = x^2y + z^2x + f(y,z)$. Hence, differentiating this result partially and comparing with the above conditions,

$$x^2 + f_y(y,z) = u_y = x^2 + 2yz,$$
$$2xz + f_z(y,z) = u_z = 2xz + y^2.$$

From these we have

$$f_y = 2yz, \qquad f_z = y^2.$$

By the first of these $f = \int(2yz)dy = y^2z + g(z)$, and, comparing this with the second, we have

$$y^2 + g'(z) = f_z = y^2.$$

Thus, $g'(z) = 0$ and $g(z) = C$, making $f = y^2z + C$, and, finally

$$u = x^2y + z^2x + y^2z + C$$

Problems

Show that the following are exact differentials, and find, in each case, the function of which the given expression is the total differential.

1. $(2x - y)dx + (3y - x)dy.$ *Ans.* $x^2 - xy + \frac{3}{2}y^2 + C.$

2. $(y^2 - 2xy + y)dx + (2xy - x^2 + x)dy.$ *Ans.* $xy^2 - x^2y + xy + C.$

3. $(\sin y + y \cos x)dx + (x \cos y + \sin x)dy.$

4. $\left(2xe^{x^2+y^2} + \frac{1}{x}\right)dx + \left(2ye^{x^2+y^2} + \frac{1}{y}\right)dy.$ *Ans.* $e^{x^2+y^2} + \log(xy) + C.$

5. $\left(\frac{1}{y^2} + \frac{4y}{x^3} + \frac{1}{x^2y}\right)dx + \left(\frac{1}{xy^2} - \frac{2x}{y^3} - \frac{2}{x^2}\right)dy.$

6. $y\left(\frac{1}{\sqrt{1 - x^2y^2}} + \frac{1}{x^2 + y^2}\right)dx + x\left(\frac{1}{\sqrt{1 - x^2y^2}} - \frac{1}{x^2 + y^2}\right)dy.$

7. $(2x + yz)dx + (2y + zx)dy + (2z + xy)dz.$
Ans. $x^2 + y^2 + z^2 + xyz + C.$

8. $(\sin y + z \sec^2 x)dx + (x \cos y + \cos z)dy + (\tan x - y \sin z)dz.$
Ans. $x \sin y + y \cos z + z \tan x + C.$

CHAPTER XIII

MULTIPLE INTEGRALS—LINE INTEGRALS

103. Regions. We shall deal, in many places in this chapter, with the concept of a *region* in a plane or in space. By a region in a plane we shall mean all of that portion of that plane which is inclosed by some finite continuous closed curve. By a region in space we shall mean all of that portion of space which is included within some finite continuous closed surface.

We shall speak, upon occasion, of *dividing a region into subregions*, meaning that the subregions are nonoverlapping and that every point in the given region is either interior to one of the subregions or upon one of the boundaries. See Fig. 176, for an example which shows a region R of the xy-plane which has been partitioned into four subregions, R_1, R_2, R_3, and R_4. We shall also, at times, speak of allowing the number of subregions, into which a given region has been divided, to increase beyond bound, *while the area of each region is made to approach zero.* By that statement we shall understand, not only that the measure of the area of the subregion approaches zero, but also that it takes place in such a way that the radius of the least circle inclosing the subregion approaches zero, *i.e., the largest diameter of the subregion approaches zero.* Similarly, the bare statement that *the volumes of the subregions* into which a region of space has been divided *approach zero,* shall be construed to mean that the *greatest diameter in each subregion approaches zero.*

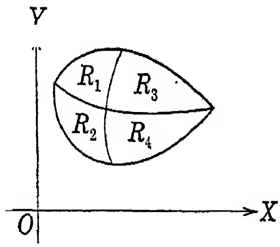

FIG. 176.

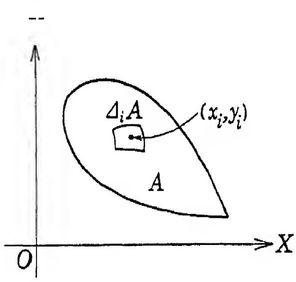

FIG. 177.

104. Double Integrals. Let a function $f(x,y)$ be defined and continuous over a region A (Fig. 177), including its boundary. Let the region be partitioned, in any manner whatever, into a

number of subregions, of area $\Delta_1 A, \Delta_2 A, \ldots, \Delta_n A$, and the function be computed for one point chosen at random in each of the subregions. Finally, let us compute the sum

$$S = f(x_1,y_1)\Delta_1 A + f(x_2,y_2)\Delta_2 A + \cdots + f(x_n,y_n)\Delta_n A,$$

where (x_i,y_i) is a point selected in the region of area $\Delta_i A$. It is evident that the value of S above will depend on the mode of partition of the region A (the number of subregions and their shape), as well as on the choice of points, (x_i,y_i) in each subregion, after the partition has been made. Now let the number of subregions be permitted to increase indefinitely, while the area of each is made to approach zero in the sense agreed upon in the preceding article. The sum S just defined will then approach a limit, in view of the following fundamental

Theorem. *Given a function $f(x,y)$, continuous over a region A of the xy-plane, as well as on the boundary of the region, the sum $S = \sum_{i=1}^{n} f(x_i,y_i)\Delta_i A$—where $\Delta_i A$ is the area of a subregion of A and (x_i,y_i) is any point within or on the boundary of that subregion—approaches a limit, as the number of subregions is made to increase beyond bound, while simultaneously the area of each is made to approach zero, this limit being independent of the mode of partitioning A into subregions and of the choice of the point (x_i,y_i) for each subregion.*

This limit, in other words, $\lim\limits_{\substack{n\to\infty \\ \Delta_i A \to 0}} \sum_{i=1}^{n} f(x_i,y_i)\Delta_i A$, is defined as the *double integral* of the function $f(x,y)$ over the region A. The student will not fail to note the similarity between the double integral just defined and the definite integral of $f(x)$ over the interval $a \leqq x \leqq b$, as defined by the fundamental theorem of integral calculus, page 287. As in that case, we shall here also forego the actual proof of the existence of the limit just stated. We shall content ourselves, instead, with the following observation.

Let us erect a z-axis and set up the surface $z = f(x,y)$, where $f(x,y)$ is the given function. A cylindrical surface through the boundary of the region A and with elements parallel to the z-axis will intercept a portion B of that surface, as shown in Fig. 178. As we partition A into subregions, $\Delta_i A$, these will project vertically

upon the surface, dividing it into parts $\Delta_i B$. Now, if $f(x_i,y_i)$ is positive, $f(x_i,y_i)\Delta_i A$ will represent the volume of a cylindrical solid of base $\Delta_i A$ and of altitude P_iQ_i, where P_i and Q_i have the coordinates $(x_i, y_i, 0)$ and $[x_i, y_i, f(x_i,y_i)]$. This volume is, in general, not identical with that of the column bounded below by the base $\Delta_i A$ and above by the portion of surface $\Delta_i B$. The sum $S = \sum_1^n f\,(x_i,y_i)\Delta_i A$ will, then, represent a total volume not, in general, identical with the volume of the column bounded below by A and above by B. It is tolerably clear, however, that as the area $\Delta_i A$ of a subregion is assigned an increasingly smaller value the volume $f(x_i,y_i)\Delta_i A$ will represent more and more closely the volume of the column from the subregion $\Delta_i A$ to the portion $\Delta_i B$ of the surface, and consequently, as all the $\Delta_i A$ are assigned ever smaller values, the sum S will represent ever more nearly the volume from A to B,—the volume under the surface B, as it is commonly put. As, finally, the number of subregions of A is increased beyond bound, while their areas are let approach zero, the sum S will approach a limit, *viz.*, the number representing this volume.

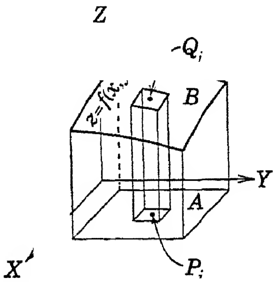

FIG. 178.

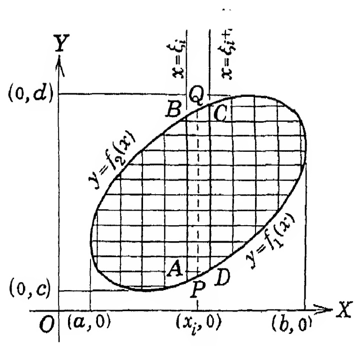

FIG. 179.

To compute the double integral, we may proceed as follows. Divide the area A into subregions by drawing lines parallel to OY at equal intervals of length Δx and lines parallel to OX at equal intervals of length Δy. Let us first sum the products $f(x_i,y_i)\Delta_i A$ over the strip of area $ABCD$, bounded by the lines AB $(x = \xi_i)$, CD $(x = \xi_i + \Delta x)$, and the arcs AD and BC. In that strip draw any line PQ, parallel to OY, and form the sum

$$S_i = f(x_i,y_1)\cdot\Delta y\cdot\Delta x + f(x_i,y_2)\cdot\Delta y\cdot\Delta x + \qquad\qquad + f(x_i,y_n)\cdot\Delta y\cdot\Delta x$$

where the points (x_i,y_i) at which the function is computed are

all on the line PQ, and $\Delta y \cdot \Delta x$ is the area of each of the rectangular subregions in the strip between the lines AB and CD. Then, as Δy approaches zero, the sum S_i approaches the limit

$$\Delta x \cdot \int_{y=f_1(x_i)}^{y=f_2(x_i)} f(x_i,y)dy.$$

(We are assuming here that a line parallel to OY crosses the boundary of A in not more than two points, and that y, explicitly in terms of x, is given by $y = f_1(x)$ for the lower part of the boundary and by $y = f_2(x)$ for the upper part. If a region that does not meet these requirements is such that it can be divided into a finite number of parts, each of which does meet them, then the double integral over the entire region can be computed as the sum of the double integrals over the sundry parts.) In this integration, $f(x_i,y)$ enters, of course, as a function of y alone, since x is held fast to the value, x_i, which it has on the line PQ.

Clearly, $\int_{y=f_1(x_i)}^{y=f_2(x_i)} f(x_i,y)dy$ will amount to some function of x_i alone, say $\varphi(x_i)$, and, thus, the products $f(x_i,y_i) \cdot \Delta y \cdot \Delta x$ over the strip $ABCD$ have a sum whose limit is $\varphi(x_i)\Delta x$ as Δy is made to approach zero.* The sum of products over the entire region is, evidently, to be obtained by adding the results for all such strips. In other words,

$$S = \varphi(x_1)\Delta x + \varphi(x_2)\Delta x + \cdots + \varphi(x_m)\Delta x,$$

where m is the number of the strips and, as Δx approaches zero, this sum approaches

$$\int_a^b \varphi(x)dx,$$

where $x = a$ and $x = b$, are, respectively, the least and greatest values of x on the boundary of the given region.

We have thus obtained the double integral of the function $f(x,y)$ over the region as

* To be exact, $\varphi(x_i) \cdot \Delta x$ represents the limit of the sum of those products, not over the strip $ABCD$, but over a rectangle bounded by the four lines, $x = \xi_i, x = \xi_i + \Delta x$, $y = f_1(x_i)$, and $y = f_2(x_i)$, where $\xi_i \leqq x_i \leqq \xi_i + \Delta x$, *i.e.*, a rectangle with base Δx and altitude PQ. The discrepancy is immaterial in view of the fact that we are about to let Δx approach zero, thus erasing the distinction between the strip $ABCD$ and the rectangle just described.

$$\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)dy \cdot dx.$$

To illustrate, let us evaluate the double integral of the function $x^2 + y^2$ over the region (Fig. 180) bounded by the parabola $y = x^2$ and the line $y = 2x$. By the above argument, this is represented by

$$\int_0^2 \int_{x^2}^{2x} (x^2 + y^2)dy\, dx$$

We first evaluate the inner integral, with x held constant, and obtain

$$x^2y + \frac{y^3}{3}\Big]_{x^2}^{2x} = \left[x^2 \cdot 2x + \frac{(2x)^3}{3}\right] - \left[x^2 \cdot x^2 + \frac{(x^2)^3}{3}\right]$$
$$= 2x^3 + \frac{8}{3}x^3 - x^4 - \frac{1}{3}x^6 = \frac{14x^3 - 3x^4 - x^6}{3}.$$

This is the $\varphi(x)$ pointed out above, and $\varphi(x) \cdot \Delta x$ now forms one term of a sum S whose limit is sought as the proposed double integral. The limit of the sum of such terms is

$$\int_0^2 (\tfrac{14}{3}x^3 - x^4 - \tfrac{1}{3}x^6)dx = \tfrac{7}{6}x^4 - \tfrac{1}{5}x^5 - \tfrac{1}{21}x^7\Big]_0^2$$
$$= \tfrac{56}{3} - \tfrac{32}{5} - \tfrac{128}{21} = \tfrac{216}{35}.$$

Note that the double integration is actually performed as if we had $\int_0^2 \left(\int_{x^2}^{2x} (x^2 + y^2)dy\right)dx$, and that, in general $\int_a^b \int_{f_1(x)}^{f_2(x)} f(x,y)dy\, dx$ is computed as $\int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)dy\right)dx$.

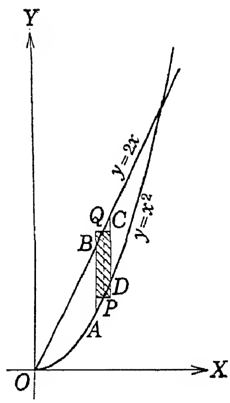

FIG. 180.

Exercise 1. If a line parallel to OX crosses the region A in not more than two points, and if x, explicitly in terms of y, is given by $x = g_1(y)$ for the left-hand part of the boundary and by $x = g_2(y)$ for the right-hand part, show that the value of the double integral of $f(x,y)$ over the region A is also given by $\int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dx\;\; dy$, where c and d are the least and greatest values of y, respectively, for the region.

Note that, in the integral of this exercise, the inner integration is performed with y held constant and yields a function, $\varphi(y)$, of y alone. The force of the two integral signs now is as $\int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)dx\right)dy.$

Exercise 2. Evaluate the double integral computed in the illustration above by the method of Exercise 1.

It may happen that the nature of the function $f(x,y)$ or the shape of the region over which its double integral is to be taken, is such that a system of polar coordinates may be used to advantage. In such a case the area A may be divided into subregions by drawing lines through the pole separated by equal angles $\Delta\theta$, and circles with center at the pole, at intervals Δr.

We must first express the area ΔA of one of the subregions, say $ABCD$ as shown in Fig. 181 (note that the subregions are not all equal in area). If

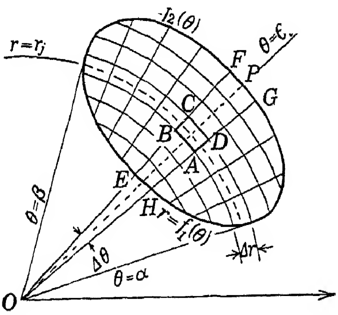

FIG. 181.

$$OA = OB = r,\ OC = OD = r + \Delta r,$$

and angle $AOB = \Delta\theta$, then the area of $ABCD$ is given by

$$\frac{1}{2}(r + \Delta r)^2 \cdot \Delta\theta - \frac{1}{2}r^2 \cdot \Delta\theta = \left(r \cdot \Delta r + \frac{\overline{\Delta r}^2}{2}\right) \cdot \Delta\theta.$$

Let the function to be integrated be $f(r,\theta)$. To sum the products $f(r_i,\theta_i)\Delta_i A$ over the strip $EFGH$, let us hold θ constant, say, equal to the value θ_i which it has along the line OP. The value of this sum depends, not only upon the θ_i and the $\Delta\theta$, which we hold fixed for the present, but also upon the Δr and the choice of the r_i. If we let Δr approach zero, however, this sum will approach a limit not dependent upon the r_i, a limit which we may write as

$$\lim_{\substack{n\to\infty \\ \Delta r\to 0}} \sum_{j=1}^{n}\left[f(r_j,\theta_i)\left(r_j \cdot \Delta r + \frac{\overline{\Delta r}^2}{2}\right)\right] \cdot \Delta\theta,$$

where we are computing the value of the function $f(r,\theta)$ for the least value of r in each subregion, *viz.*, r_j, the value of r for the smaller circle bounding the subregion. Now, by Duhamel's theorem, page 232, this limit is equal to

$$\lim_{\substack{n\to\infty \\ \Delta r\to 0}} \sum_{j=1}^{n}[f(r_j,\theta_i) \cdot (r_j\Delta r)] \cdot \Delta\theta,$$

since the summands, $u_j = f(r_j,\theta_i) \cdot (r_j\Delta r)$ and

$$v_j = f(r_j,\theta_i) \cdot (r_j\Delta r + \tfrac{1}{2}\overline{\Delta r}^2)$$

satisfy the hypotheses of that theorem. Hence, as Δr approaches zero, the products $f(r_j,\theta_i)\Delta_i A$ summed over the strip $EFGH$ have as their limit

$$\int_{r=f_1(\theta_i)}^{r=f_2(\theta_i)} f(r,\theta_i)\cdot r dr \cdot \Delta\theta,^*$$

where we have assumed that a line through the pole will meet the boundary of the given region in not more than two points, and that the loci of these points are given by $r = f_1(\theta)$ and $r = f_2(\theta), f_1(\theta) \leqq f_2(\theta)$.

This last integral is, clearly the product of $\Delta\theta$ by some function of θ evaluated at θ_i, say, $\varphi(\theta_i)$. The limit we seek is that of the sum of such products as this, or

$$\lim_{\substack{m\to\infty\\ \Delta\theta\to 0}} \sum_{i=1}^{m}\varphi(\theta_i)\Delta\theta = \int_\alpha^\beta \varphi(\theta)d\theta = \int_\alpha^\beta \int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)r dr\, d\theta,$$

where, again here, the notation has the force of

$$\int_\alpha^\beta \left[\int_{f_1(\theta)}^{f_2(\theta)} f(r,\theta)r\; dr\right]d\theta.$$

Exercise 3. If every circle with center at the pole meets the boundary of a region in not more than two points at which $\theta = f_1(r)$ and $\theta = f_2(r)$, $f_1(r) \leqq f_2(r)$, prove that the double integral of $f(r,\theta)$ over the region is given by

$$\int_{r_1}^{r_2}\int_{f_1(r)}^{f_2(r)} f(r,\theta)r\; d\theta\; dr,$$

where r_1 and r_2 are, respectively, the least and greatest values of r over the region and where the notation has the force

$$\int_{r_1}^{r_2}\left[\int_{f_1(r)}^{f_2(r)} f(r,\theta)r\; d\theta\right]dr,$$

in which r is held constant for the inside integration.

To illustrate, let us compute the double integral of the function $r \sin\theta$ over the area ABC (Fig. 182) bounded by the initial line,

* Here, as in the case of rectangular coordinates, if the boundaries of the given region do not happen to be circles about the pole, we fail to sum over the exact strip $EFGH$ by an error which disappears as we later allow $\Delta\theta$ to approach zero, *i.e.*, has no effect upon the ultimate limit sought.

the circle $r = 2$ and the circle $r = 4 \cos \theta$. Its value, by the above, is

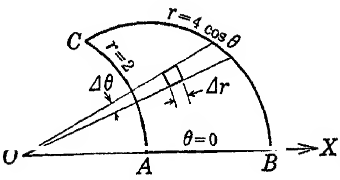

FIG. 182.

$$\int_0^{\pi/3}\int_2^{4\cos\theta} r \sin\theta\, r\, dr\, d\theta.$$

The first integration yields

$$\frac{r^3}{3} \sin\theta \Big]_2^{4\cos\theta} = \frac{64 \cos^3\theta}{3} \sin\theta - \frac{8}{3}\sin\theta.$$

Proceeding to the second integration, we obtain

$$\int_0^{\pi/3} (\tfrac{64}{3}\cos^3\theta \sin\theta - \tfrac{8}{3}\sin\theta)d\theta = -\tfrac{16}{3}\cos^4\theta + \tfrac{8}{3}\cos\theta \Big]_0^{\pi/3}$$
$$= (-\tfrac{1}{3} + \tfrac{4}{3}) - (-\tfrac{16}{3} + \tfrac{8}{3}) = \tfrac{11}{3}.$$

Problems

1. Compute each double integral below. Identify, in each case, the function $f(x,y)$ or $f(r,\theta)$, as the case may be, and the region over which it is inte ted.

(a) $\int_1^2\int_0^2 xy\, dx\, dy.$ *Ans.* 3.

(b) $\int_0^3\int_x^{2x} y^2\, dy\, dx.$ *Ans.* $\frac{189}{4}$.

(c) $\int_0^3\int_x^{4x-x^2} (x + y)dy\, dx.$ *Ans.* $\frac{351}{20}$.

(d) $\int_{\pi/6}^{\pi/2}\int_0^2 r \cos\theta \cdot r\, dr\, d\theta.$ *Ans.* $\frac{4}{3}$.

(e) $\int_0^{\sqrt{2}}\int_{\sin^{-1}(-r/2)}^{\cos^{-1}(r/2)} r\, d\theta\, dr.$ *Ans.* $\frac{\pi}{2}$.

(f) $\int_{1/2}^1\int_{x^2}^1 \log\left(\frac{y}{x^2}\right) dy\, dx.$ *Ans.* $\frac{19}{24} - \log 2$.

2. In each case below, integrate the given function over the given region.

(a) $f(x,y) = x$. Quadrilateral bounded by the lines, $y = 1$, $x = 0$, $y = 3$, and $y = x$. *Ans.* $4\frac{1}{3}$.

(b) $f(x,y) = x^2 + y^2$. Quadrilateral bounded by the lines, $x = 2$, $y = 0$, $x = 3$, and $y = 2x$. *Ans.* $75\frac{5}{6}$.

(c) $f(x,y) = e^{x+y}$. Trapezoid bounded by the lines $x = 1$, $y = 0$, $x = 2$, and $y = x$.

(d) $f(r,\theta) = r$. The half crescent bounded by the circles $r = 3$ and $r = 6 \sin\theta$ and included between the lines $\theta = \frac{1}{6}\pi$, $\theta = \frac{1}{2}\pi$. *Ans.* $27\sqrt{3} - 3\pi$.

(e) $f(r,\theta) = r \sin\theta$. The circular sector in the first quadrant bounded by the lines $\theta = 0$, $\theta = \frac{1}{3}\pi$, and the circle $r = 4$.

(f) $f(r,\theta) = \sin(r\theta)$. The area in the first quadrant bounded by the straight lines $\theta = \pi/6$ and $\theta = \pi/3$ and by the spirals $r\theta = \pi$ and $r\theta = 2\pi$ where $0 < \theta < \pi/2$. *Ans.* -9.

3. Compute $\int_0^a \int_0^{\sqrt{a^2-y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy$. KEY: It is advantageous here to change to polar coordinates. The region of integration is, evidently, the area in the first quadrant of the circle $x^2 + y^2 = a^2$, and the function to be integrated is equivalent to e^r. Hence the double integral becomes

$$\int_0^{\pi/2} \int_0^a e^r \cdot r\, dr\, d\theta.$$

4. Compute the following by changing to polar coordinates as in Prob. 3.

(*a*) $\int_0^{\pi/2} \int_0^{\sqrt{\pi^2/4-x^2}} \sin \sqrt{x^2 + y^2}\, dy\, dx$. *Ans.* $\pi/2$.

(*b*) $\int_0^1 \int_y^1 e^{x^2}\, dx\, dy$. *Ans.* $\dfrac{e-1}{2}$.

(*c*) $\int_0^{\sqrt{\pi/2}} \int_y^{\sqrt{\pi/2}} \cos (x^2) dx\, dy$.

5. Compute $\int_0^{\sqrt{\pi/3}} \int_y^{\sqrt{\pi/3}} \tan (x^2) dx\, dy$. HINT: It is to advantage here to change the order of integration. The region of integration is, evidently, the triangle bounded by the lines $y = 0$, $x = y$, and the line $x = \sqrt{\pi/3}$. We compute, then, the double integral as $\int_0^{\sqrt{\pi/3}} \int_0^x \tan (x^2) dy\, dx$.

6. Compute the following by changing the order of integration, if necessary (see Prob. 5).

(*a*) $\int_0^1 \int_{2y}^2 e^{x^2}\, dx\, dy$. *Ans.* $\frac{1}{4}(e^4 - 1)$.

(*b*) $\int_0^1 \int_{\sqrt[3]{y}}^1 \sqrt{x^4 + 1}\, dx\, dy$. *Ans.* $\frac{1}{6}(2\sqrt{2} - 1)$.

(*c*) $\int_0^{\sqrt[4]{\pi/2}} \int_{y^2}^{\sqrt{\pi/2}} y \sin (x^2) dx\, dy$.

105. Applications of Double Integrals. In the chapter on definite integrals we made use of integration in computing the values of various quantities, such as areas, volumes, moments of inertia, etc. The problems treated in that chapter were such that a proper choice of an element (of area, volume, etc.) enabled us to compute the quantity sought by means of a single integration. This, however, is not always the case. We frequently meet problems in which the element must be so chosen that two integrations need be carried out to obtain the result, *i.e.*, where we are forced to deal with a double integral.

To illustrate, consider the moment of inertia of the area AOB bounded as in Fig. 183, about an axis through the origin perpendicular to the xy-plane. Here we are unable to formulate at once the moment of inertia for either an element like I, parallel to OX,

or an element like II, parallel to OY, since the distance from the origin throughout either element is variable. We take, therefore, an element of area $\Delta x \, \Delta y$, of density δ and mass $\delta \, \Delta x \, \Delta y$. If we assume this mass concentrated at a point (x,y) situated within the element of area, the moment of inertia, about the origin, of that mass is $\delta(x^2 + y^2)\Delta x \, \Delta y$. The sum of the moments of inertia of all such elements of mass is an approximation to the moment of inertia sought, an approximation, moreover, which is made ever closer by taking the elements of area smaller and smaller and allowing their number to become correspondingly large. It is clear, from this, that the desired moment of inertia is the limit approached by that sum as the process is continued, *i.e.*, that to obtain the moment of inertia which we are seeking we must find the double integral of the function $\delta(x^2 + y^2)$ over the given area. The result is represented by either one of the expressions

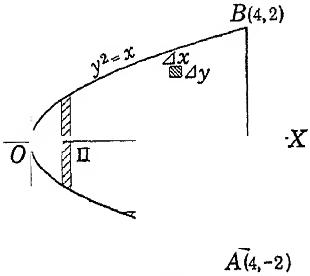

FIG. 183.

$$\delta \int_{-2}^{2} \int_{y^2}^{4} (x^2 + y^2) dx \, dy \qquad \text{or} \qquad \delta \int_0^4 \int_{-\sqrt{x}}^{\sqrt{x}} (x^2 + y^2) dy \, dx,$$

assuming that δ is constant throughout the given area.

Exercise 1. Show that either one of the above integrals yields the result $\frac{8576}{105}\delta$.

As another example, consider the pressure against the area bounded by the parabola $x^2 - 4x = 4y$ and the line $x = 2y$, immersed in a liquid of weight 50 lb./cu. ft., if the xy-plane is assumed to be vertical and the surface of the liquid is at the level $y = 5$, where all dimensions are in feet.

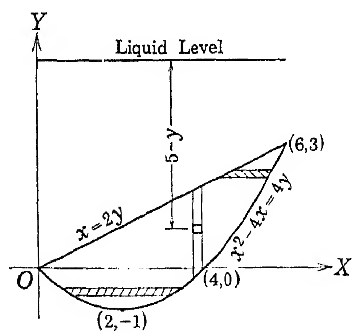

FIG. 184.

This pressure could be computed by a single integration if we first formulated the pressure on a horizontal strip of area, *i.e.*, the pressure on an element of area $(x_2 - x_1) \cdot \Delta y$. Note, however, that for the part of the area above OX, x_1 is the abscissa of a point on the line $x = 2y$ and for the part below OX,

it is the abscissa of a point on the parabola. We cannot, consequently, express $x_2 - x_1$ as an algebraic function of y which would obtain for the entire area we are considering. We could, to be sure, obtain by a single integration the pressure upon the area above OX, and then again, by another single integration, obtain the pressure upon the area below OX. We elect, instead, to proceed as follows:

The pressure upon the element of area $\Delta x\ \Delta y$, all assumed to lie at the depth of the point (x,y) located within that element, is readily seen to be $50(5 - y)\Delta x\ \Delta y$ pounds. If all such elements of pressure are added, the limit of the sum obtained, as the size of the elements is made to approach zero while their number increases beyond all bound, is the pressure which we desire. In other words the double integral of the function $50(5 - y)$ over the given area furnishes us the total pressure on that area.

Exercise 2. Carry out the integration of the last paragraph above and thus show that the pressure on the given area is 1980 lb. Note that this integration is more readily effected by integrating first with respect to y.

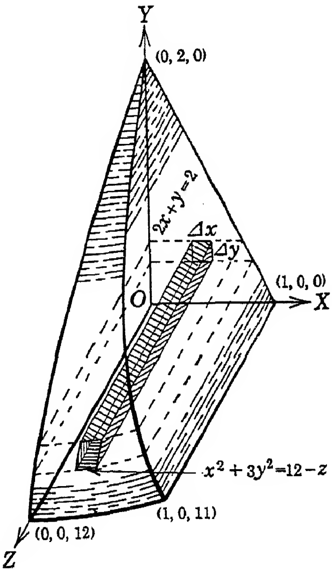

FIG. 185.

As still another example, let us compute the volume in the first octant bounded by the coordinate planes, the surface $x^2 + 3y^2 = 12 - z$, and the plane $2x + y = 2$.

In an attempt to obtain this volume by a single integration we should meet with difficulty. We choose, therefore, as element of volume, a column parallel to OZ, and standing on the rectangular base $\Delta x\ \Delta y$. The volume of this column is $z\ \Delta x\ \Delta y$ or $(12 - x^2 - 3y^2)\Delta x\ \Delta y$, or, rather, is approximated to by that expression if (x,y) are the coordinates, in the xy-plane, of a point within the base of the column. Hence, the limit of the sum of such expressions as their number increases indefinitely while the areas $\Delta x\ \Delta y$ approach zero, is the volume we proposed to find. The result is, then, that of integrating the function $(12 - x^2 - 3y^2)$ over the triangle in the xy-plane bounded by the x-axis, the y-axis and the line $2x + y = 2$.

Exercise 3. Perform the integration indicated above and thus show that the desired volume is $\frac{5}{6}$.

Exercise 4. Show that the double integrals $\iint dx\,dy$ and $\iint r\,dr\,d\theta$, computed over a closed area, represent the numerical value of that area.

Exercise 5. Show that the double integrals $\iint(x^2 + y^2)dx\,dy$ and

$$\iint r^3\,dr\,d\theta,$$

computed over a closed area, represent the moment of inertia of that area about an axis through the origin perpendicular to the plane of the area.

Exercise 6. Show that $\bar{x} = \dfrac{\iint x\,dx\,dy}{\text{area}} = \dfrac{\iint r^2 \cos\theta\,dr\,d\theta}{\text{area}}$ and

$$\bar{y} = \frac{\iint y\,dx\,dy}{\text{area}} = \frac{\iint r^2 \sin\theta\,dr\,d\theta}{\text{area}},$$

where the double integrals are computed over the area which appears in all denominators, and $\bar{x}$ and $\bar{y}$ are coordinates of the center of gravity of that area.

Exercise 7. Show that the expressions for the area, employed in single integration, *viz.*, $\int_{x=a}^{x=b} y\,dx$ and $\frac{1}{2}\int_{\theta=\alpha}^{\theta=\beta} r^2\,d\theta$, are equivalent to the expressions $\int_a^b \int_0^y dy\,dx$ and $\int_\alpha^\beta \int_0^r r\,dr\,d\theta$, employed in computing the same area by double integration. Illustrate by a figure.

Problems

1. By double integration, compute the area

(*a*) Bounded by the curve $x^2 = y^3$ and the line $y = x$. *Ans.* $\frac{1}{10}$.

(*b*) Inside the ellipse $x^2 + 3y^2 = 12$ and one branch of the hyperbola $x^2 - y^2 = 8$.

(*c*) Bounded by the parabolas $x^2 - 6x + 9y = 0$ and $2x^2 - 12x + 9y = 0$. *Ans.* 4.

(*d*) Inside the circles $x^2 - 4x + y^2 = 0$ and $x^2 - 4y + y^2 = 0$.

(*e*) In the first quadrant, bounded by the line $4x = 3y$, the circle $x^2 + y^2 = 25$, and the parabola $3x^2 = 16y$.

(*f*) In the first quadrant, bounded by the line $\pi y = 3\sqrt{3}x$, the curve $y = \cos x$, and the curve $2y = \sin x$. *Ans.* $\dfrac{\sqrt{5}}{2} - 1 + \dfrac{\pi\sqrt{3}}{24}$.

(*g*) Outside the circle $r = 2$, and inside the circle $r = 4\cos\theta$.

(*h*) Outside the cardioid $r = 2(1 + \sin\theta)$ and inside the circle $r = 6\sin\theta$.

(*i*) In the first quadrant, bounded by the parabola $r = \sec^2\dfrac{\theta}{2}$, the rose $r = \cos 3\theta$, and the line $2\theta = \pi$. *Ans.* $\dfrac{4}{3} - \dfrac{\pi}{24}$.

(*j*) In the first quadrant, bounded by the polar axis, the spiral $r - 1 = 2\sqrt{\theta}$, and the line $\theta = 1$.

(*k*) Between the lemniscate $r^2 = 2\cos 2\theta$ and the line $2r\cos\theta = \sqrt{3}$. *Ans.* $\sqrt{3}/4$.

(*l*) Between the circle $r = -\sin\theta$ and the loop of the curve $r = 1 - 2\sin\theta$.

2. Compute the moment of inertia and the radius of gyration of the area

(*a*) Between the y-axis and the parabola $x + y^2 = 4$, about the origin. *Ans.* $\frac{1664}{35}\delta$.

(*b*) Inside the circle $x^2 - 4x + y^2 = 0$, about the origin. *Ans.* $24\pi\delta$.

(*c*) Between the x-axis and the parabola $x^2 - 4x + 4y = 0$, about the x-axis.

(*d*) Between the parabolas $y^2 = 2x$ and $y^2 + 4x = 12$, about the y-axis. *Ans.* $\frac{864}{35}\delta$.

(*e*) Between the parabola $y^2 - 3x = 9$ and the line $x + y = 3$, about the y-axis.

(*f*) Inside the ellipse $x^2 + 2y^2 = 6$ and the parabola $x^2 + 4y = 0$, about the x-axis.

(*g*) Of a right triangle of sides $1, 2, \sqrt{5}$, about the hypotenuse. *Ans.* $\frac{2}{15}\delta$.

(*h*) Described in (*g*), about the vertex of the right angle. *Ans.* $\frac{5}{6}\delta$.

(*i*) Of a circle of radius a about a tangent, if the density at any point is twice the distance of that point from the diameter perpendicular to that tangent. *Ans.* $\frac{16}{5}a^5$.

3. Find the dimensions of a rectangle of perimeter $4p$ in order that its moment of inertia about a vertex be a maximum. *Ans.* $a = b = p$.

4. Compute the moment of inertia and the radius of gyration of

(*a*) The area of the circle $r = 2$, about the pole. *Ans.* $8\pi\delta$.

(*b*) The area of the circle $r = 2\sin\theta$, about the pole. *Ans.* $\dfrac{3\pi}{2}\delta$.

(*c*) The area of the cardioid $r = 1 - \cos\theta$, about the polar axis.

(*d*) The area in the first loop of the curve $r = \sin 2\theta$, about the polar axis. *Ans.* $\dfrac{3\pi}{128}\delta$.

(*e*) The area inside the circle $r = 6\sin\theta$ and outside the circle $r = 3$, about the line $\theta = 90°$. *Ans.* $\left(\dfrac{81\pi}{12} + \dfrac{243\sqrt{3}}{16}\right)\delta$.

(*f*) The area inside the circle $r = \cos\theta$ and outside the lemniscate $r^2 = \cos 2\theta$, about the line $\theta = 90°$.

(*g*) The smaller area, in the circle of radius a, between a chord of length $a\sqrt{3}$ and the circumference, about that chord.

(*h*) The area described in (*g*), about the midpoint of the chord.

5. Find the center of gravity of the area

(*a*) Between the parabola $x^2 + y = 4$ and the line $y - x = 2$. *Ans.* $\bar{y} = 12/5$.

(*b*) Between the parabolas $y = x^2 + 13$ and $y = 2x^2 + 4$.

(*c*) Inside the parabola $4x^2 = 9y$ and the circle $x^2 + y^2 = 25$.

(d) Between the parabolas $x^2 - 4x = 2y$ and $x^2 - 2x + 2y = 8$. *Ans.* $\bar{y} = \frac{5}{4}$.

(e) Inside the circles $x^2 + y^2 = 25$ and $3x^2 - 25x + 3y^2 = 0$.

(f) Between the parabola $2y^2 - 12y + 9x = 0$ and the line $3x + 2y = 0$. *Ans.* $\bar{x} = -\frac{9}{5}$.

(g) In the first quadrant, bounded by the y-axis and the curves $y = \sin x$, $y = \cos x$.

(h) In the first quadrant, bounded by the circle $x^2 + y^2 = 16$, the parabola $y^2 = 6x$, and the line $x = y\sqrt{3}$.

6. Find the center of gravity of each of the following areas. (Find $\bar{x}$, $\bar{y}$ and from that the polar coordinates of the center of gravity; see Exercise 6 of this section.)

(a) Upper half of the circle $r = 4\cos\theta$. *Ans.* $\bar{y} = 8/3\pi$.

(b) Inside the limaçon $r = 2 + \cos\theta$.

(c) Inside the first loop of the rose $r = \sin 3\theta$. *Ans.* $\bar{r} = 81\sqrt{3}/80\pi$.

(d) Upper area bounded by the lemniscate $r^2 = -2\cos 2\theta$ and the line $2r\sin\theta = \sqrt{3}$.

(e) Common to the two circles $r = \cos\theta$ and $r = \sin\theta$.

(f) The area bounded by the first spire of the spiral $r = 2\theta$ and the line $\theta = 90°$, the density at any point being twice the distance of that point from the pole.

7. From a circle of radius 2, in which the density at any point is proportional to the distance from the center, the area of a circle of radius 1, tangent to the given circle, is removed. Locate the center of gravity of the remaining area.

Ans. $\dfrac{12}{15\pi - 10}$ units from the center.

8. Compute the liquid pressures upon the following areas if, in each case, the xy-plane is assumed vertical, and the liquid is water, weighing $\frac{1}{32}$ ton/cu. ft., and all coordinates are in feet.

(a) Bounded by the parabola $3y = 6x - x^2$ and the line $y = 0$. Water surface at $y = 4$. *Ans.* 1.05 tons.

(b) Upper half of circle $x^2 + y^2 - 4x = 0$. Water surface at $y = 3$. *Ans.* $\dfrac{9\pi - 8}{48}$ tons.

(c) Bounded by the parabola $y = x^2 - 4x$ and $y = 0$. Water surface at $y = 0$.

(d) Bounded by curves $2y = 5x - x^2$ and $x + y = 0$. Water surface at $y = 6$. *Ans.* 16,121/2560 tons.

(e) Bounded by curves $3x = y^2 - 3y$ and $3x + 2y^2 = 18$. Water surface at $y = 4$.

(f) Bounded by the lower half of the circle $x^2 + y^2 = 6x$ and the parabola $x^2 - 6x + 3y = 0$. Water surface at $y = 5$.

Ans. $\dfrac{225\pi + 636}{320}$ tons.

(g) A trapezoid with parallel sides of lengths 5 and 7 ft. and vertical, lower side 3 ft. long and horizontal, and topmost vertex at the water surface.

(*h*) An elliptic disk of semiaxes 5 ft., vertical, and 3 ft., horizontal, the top of the disk being 1 ft. below the surface. *Ans.* $45\pi/16$ tons.

9. Compute the following volumes:

(*a*) In the first octant, bounded by the coordinate planes, the surface $x^2 + y^2 = 4 - z$, and the plane $x + 2y = 2$. *Ans.* $1\%_6$.

(*b*) In the first octant, bounded by the coordinate planes, the cylinder $x^2 + y^2 = 9$ and the plane $2x + y + 3z = 12$.

(*c*) In the first octant, bounded by the planes $z = 0$, $x = 0$, $x + 2y + 3z = 12$, and $x = y$.

(*d*) Bounded by the cylinders $y^2 = x$ and $x^2 = y$, the surface $x^2 + y^2 + z = 2$, and the plane $z = 0$. *Ans.* $5\%_{105}$.

(*e*) Bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 3y$, $z = y$.

(*f*) In the first octant, bounded by the cylinder $x^2 + y^2 = 16$ and the planes $3x + 2y + 12z = 24$, $z - y = 8$.

(*g*) Common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + az = a^2$, and lying in the first octant. *Ans.* $3\pi a^3/16$.

(*h*) Inside the cylinder $x^2 = ay$, the paraboloid $x^2 + z^2 = 4ay$, and bounded by the plane $y = 3a$.

(*i*) Bounded by the surfaces $2x^2 + y^2 = z$, $x^2 + y^2 = 2y$, and $z = 0$.

(*j*) Bounded by the paraboloid $2x^2 + y^2 = z$ and the plane $z - x = 3$.

(*k*) Common to the paraboloids $x^2 + 2z^2 = 4 - y$, and $2x^2 + z^2 = 3y$.

10. Compute the following volumes (they are treated to advantage by cylindrical coordinates. The cylindrical element of volume with base $r\,dr\,d\theta$ in the xy-plane and altitude z, where $z = f(r,\theta)$, is, then, $zr\,dr\,d\theta$, and the volume sought becomes $\iint zr\,dr\,d\theta$, the integral extending over the entire base. Likewise, a cylindrical element may be of the form $yr\,dr\,d\theta$, where $r\,dr\,d\theta$ is an element of area in the xz-plane, etc.):

(*a*) Common to the sphere $x^2 + y^2 + z^2 = 36$ and the cylinder $x^2 + y^2 = 6x$.

HINT: In cylindrical coordinates, the two surfaces are $r^2 + z^2 = 36$ and $r = 6\cos\theta$. The volume is, then, $4\int_0^{\pi/2}\int_0^{6\cos\theta} r\sqrt{36 - r^2}\,dr\,d\theta$.

(*b*) Inside the sphere $x^2 + y^2 + z^2 = 16$ and the cylinder

$$x^2 + y^2 = 9.$$

Ans. $\pi(256 - 28\sqrt{7})/3$.

(*c*) Outside the paraboloid $x^2 + y^2 = 25 - 5z$, inside the cylinder $x^2 + y^2 = 16$, and beneath the plane $z = 5$.

(*d*) Bounded by the paraboloid $x^2 + y^2 = 4z$, the cylinder $x^2 + y^2 = 8y$, and the xy-plane. *Ans.* 96π.

(*e*) Inside the cone $x^2 + y^2 - z^2 = 0$ and the sphere $x^2 + y^2 + z^2 = 18$.

(*f*) Bounded by the cylinder $y^2 + z^2 = 4z$, the plane $2x = 4 + z$, and the yz-plane.

(g) Common to a sphere of radius a and a cylinder of radius $\frac{1}{2}a$ having an element passing through the center of the sphere.

Ans. $\dfrac{2a^3}{9}(3\pi - 4)$.

(h) Common to a sphere of radius a and a cone of revolution whose semivertical angle is 30°, whose axis of revolution passes through the center of the sphere, and whose vertex is in the surface of the sphere.

Ans. $\frac{7}{12}\pi a^3$.

11. Find the mass of that part of the cone $x^2 + y^2 = (5 - z)^2$ between the vertex and the xy-plane, if the density at any point is proportional to its distance from the axis of the cone.

12. Find the attraction of a circular disk of radius a upon a unit mass situated at a distance h above the center of the disk, if the density at any point of the disk is proportional to its distance from the center.

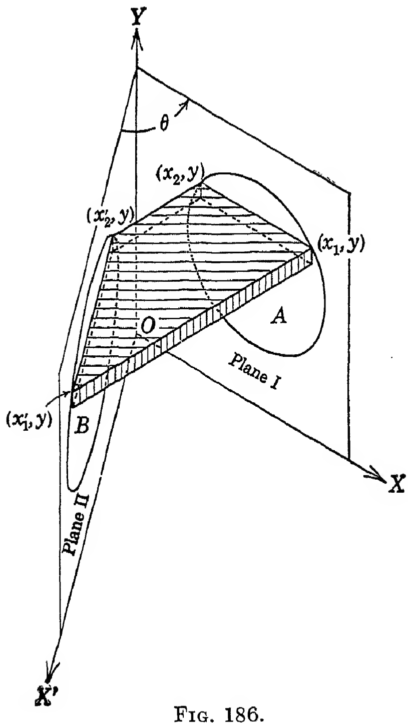

FIG. 186.

106. Areas of Surfaces.

We prove the following

Lemma. If an area B lying in a plane II *is projected orthogonally into an area A in a plane* I, *then* $A = B \cos\theta$, *where* θ *is the angle between the planes* I *and* II.

Proof. Assume the line of intersection of the two planes as a y-axis, and draw in each plane a line meeting OY at right angles in the point O, to serve as the x-axis in its plane. An element of area of the type $(x_1 - x_2)\Delta y$ in plane I is the orthogonal projection of a corresponding element $(x'_1 - x'_2)\Delta y$ in plane II. The areas A and B may be represented as integrals in the form

$$A = \int(x_1 - x_2)dy, \qquad B = \int(x'_1 - x'_2)dy,$$

where the limits on y are the same in both cases. Now $x_1 - x_2 = (x'_1 - x'_2)\cos\theta$, so that

$$\begin{aligned} A = \int(x_1 - x_2)dy &= \int(x'_1 - x'_2)\cdot\cos\theta\cdot dy \\ &= \cos\theta\int(x'_1 - x'_2)dy = B\cos\theta. \qquad Q.E.D. \end{aligned}$$

We turn, now, to the problem of finding the area of a portion S of a surface whose equation is $z = f(x,y)$. Let S be projected orthogonally into A, upon the xy-plane. An element of area $\Delta x\, \Delta y$ of A is the orthogonal projection of an element ΔS of S.* Through any point (x,y,z) of ΔS, say P, let a plane be drawn tangent to S. The lines parallel to OZ which project $\Delta x\, \Delta y$ into ΔS will intercept on that tangent plane a parallelogram (not shown in the figure). If the normal of S at P makes an angle γ with OZ, this, by definition, is also the angle between the tangent plane at P and the xy-plane, and hence, by the lemma just proved, the area of that parallelogram is $\Delta x\, \Delta y/\cos \gamma$. Now

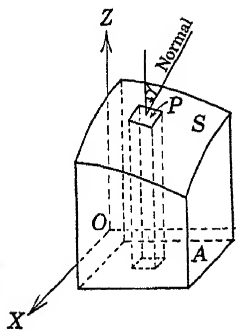

FIG. 187.

$$\cos \gamma = \frac{1}{\sqrt{z_x{}^2 + z_y{}^2 + 1}},$$

computed, of course, at the point P, and the area of the parallelogram is, then,

$$\sqrt{1 + z_x{}^2 + z_y{}^2}\, \Delta x\, \Delta y.$$

Let now the subregions $\Delta x\, \Delta y$ of A approach zero by letting all the intervals Δx and Δy tend to zero as a limit. The elements ΔS on S, and with them the intercepted portions (parallelograms) on the tangent planes will simultaneously tend to zero. We *define* the area of S as *the limit of the sum of the intercepted elements on the tangent planes,* as their number increases beyond bound and their areas all approach zero as a limit. But this is nothing else than the double integral of the function $\sqrt{1 + z_x{}^2 + z_y{}^2}$ over the area A. Hence, an expression for the area of S is

$$S = \iint \sqrt{1 + z_x{}^2 + z_y{}^2}\, dx\, dy$$

with limits suggested by the boundary of A.

The continuity of the functions z_x and z_y, and hence of $\sqrt{1 + z_x{}^2 + z_y{}^2}$, at all points of S is assumed, to justify the existence of the limit of $\Sigma\sqrt{1 + z_x{}^2 + z_y{}^2}\, \Delta x\, \Delta y$. Geometrically, this amounts to assuming that there is a tangent plane at every point of S.

* This assumes that no normal of S is parallel to the xy-plane. It is also assumed that every parallel to OZ meets S only once, which assigns a unique element ΔS corresponding to the element $\Delta x\, \Delta y$ of A.

Exercise 1. If no normal of S is parallel to the yz-plane, prove that an expression for the area of S is $S \quad \iint \sqrt{1 + x_y^2 + x_z^2}\, dy\, dz$, the double integral extending over the projection of S upon the yz-plane.

Exercise 2. If no normal of S is parallel to the zx-plane, prove that an expression for the area of S is $S = \iint \sqrt{1 + y_z^2 + y_x^2}\, dz\, dx$, the double integral extending over the projection of S upon the zx-plane.

Exercise 3. Prove that an expression for the area of the surface S, expressed in the form $z = f(r,\theta)$ is $\iint \sqrt{r^2 + r^2 z_r^2 + z_\theta^2}\, dr\, d\theta$, the double integral extending over the projection of S upon the $r\theta(xy)$-plane.

To illustrate, consider the area of the paraboloid $x^2 + z^2 = 4y$, intercepted by the cylinder $x^2 = y$ and the plane $y = 3$ (OAB,

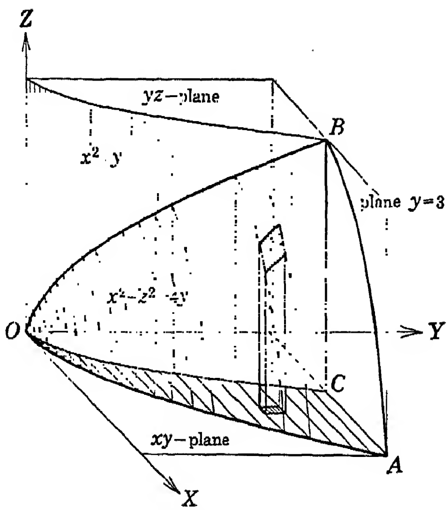

Fig. 188.

the part shown in Fig. 188, is one-fourth of the area in question). The figure shows an element $\Delta x\, \Delta y$ in the xy-plane and the element ΔS of which it is the projection.

At any point on OAB we have

$$z = \sqrt{4y - x^2}, \qquad z_x = \frac{-x}{\sqrt{4y - x^2}} \qquad z_y = \frac{}{\sqrt{4y - x^2}},$$

and hence, by the formula above,

$$= 4\int_0^3 \int_{\sqrt{y}}^{\sqrt{4y}} \sqrt{1 + \frac{x^2 + 4}{4y - x^2}}\, dx\, dy,$$

where the order of integration and the limits are as indicated in

the figure and the coefficient 4 is used because the area OAB is but one-fourth of the desired area.

Exercise 4. Show that the above indicated integration gives the result $112\pi/9$ for the area called for in the illustration.

Problems

1. Verify that the above process gives $4\pi a^2$ as the area of a sphere. HINT: Assume the sphere so placed that its equation is $x^2 + y^2 + z^2 = a^2$.

2. Verify that the area of a right circular cone of radius a and altitude h is $\pi a\sqrt{a^2 + h^2}$. HINT: Assume the cone so placed that its equation is $h^2x^2 + h^2y^2 - a^2z^2 = 0$.

3. Compute the area of

(*a*) The portion of the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ $(a, b, c > 0)$ in the first octant. *Ans.* $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$.

(*b*) The portion of the cylinder $x^2 + z^2 = a^2$, intercepted by the cylinder $x^2 + y^2 = a^2$. *Ans.* $8a^2$.

(*c*) The portion of the plane $x + y + z = 4$, intercepted by the cylinder $x^2 + z^2 = 2$. *Ans.* $2\pi\sqrt{3}$.

(*d*) The cylinder $x^2 - ay + y^2 = 0$, intercepted by the sphere $x^2 + y^2 + z^2 = a^2$. *Ans.* $4a^2$.

(*e*) The sphere $x^2 + y^2 + z^2 = a^2$, intercepted by the cylinder $x^2 - ay + y^2 = 0$. *Ans.* $2(\pi - 2)a^2$.

(*f*) The cylinder $x^2 - ay + y^2 = 0$, intercepted by the cone $x^2 + y^2 = z^2$.

(*g*) The portion of the sphere $x^2 + y^2 + z^2 = 2y$, inside the paraboloid $y = x^2 + z^2$. *Ans.* 2π.

(*h*) The part of the cylinder $z = y^2$, intercepted by the planes $2x + y = 4$, $x = 0$, $y = 0$.

Ans. $\frac{1}{24}[31\sqrt{65} + 1 + 12 \log (8 + \sqrt{65})]$.

(*i*) The sphere $x^2 + y^2 + z^2 = a^2$, intercepted by the cylinder constructed on a single loop of the curve $r = a \cos n\theta$ as base, and elements parallel to OZ. HINT: Use cylindrical coordinates. *Ans.* $\frac{2a^2}{n}(\pi - 2)$.

(*j*) The cylinder $y^{2/3} + z^{2/3} = a^{2/3}$, which is contained in the cylinder $x^{2/3} + y^{2/3} = a^{2/3}$.

4. For a hemispherical surface of radius a, find

(*a*) The moment of inertia about the radius perpendicular to its base. *Ans.* $2a^2M/3$.

(*b*) The attraction upon a unit mass situated at the center of the base.

5. For the surface of a cone of revolution of semivertical angle 45° and altitude h, find

(*a*) The moment of inertia about the central axis. *Ans.* $Mh^2/2$.

(*b*) The attraction upon a unit mass situated at the center of the base.

107. Triple Integrals. Given a region V of space and a function $f(x,y,z)$, defined, single-valued, and continuous at all points of the region, as well as on its boundary, let the region be partitioned, in any manner whatever, into a number of nonoverlapping subregions. If we multiply the volume of each of these subregions by the value of the function computed at some point either within it or on its boundary, and sum these products, we obtain an expression of the form

$$\sum_{i=1}^{n} f(x_i,y_i,z_i)\Delta_i v \tag{143}$$

where (x_i,y_i,z_i) is a point in, or on the boundary of, the subregion of volume $\Delta_i v$. In entire analogy with the case of a function of two variables defined over a plane region (Sec. 103), there obtains, also in this case, the theorem that *the sum* (143) *approaches a limit, as the number of subregions increases beyond bound, while simultaneously the volume of each approaches zero; and that this limit is independent of the manner of partitioning the region V into subregions and of the mode in which the points (x_i,y_i,z_i) are chosen in the corresponding subregions.* This limit is called the *triple integral* of the function $f(x,y,z)$ over the region V.

To compute such a triple integral, we may pass planes parallel to the coordinate planes at intervals Δx, Δy, and Δz, thus dividing the region V into rectangular parallelepipeds of volume $\Delta x\ \Delta y\ \Delta z$ each (except that at the boundary we may have only parts of such parallelepipeds). If we hold, say, x and y constant and integrate with respect to z, we obtain

$$\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)dz \cdot \Delta y\ \Delta x$$

as the sum of our products over a column of the region V, parallel to OZ (see Fig. 189), of height $f_2(x,y) - f_1(x,y)$ and of cross-sectional area $\Delta x\ \Delta y$. [To be exact, this column is not, in general, identical with the column actually intercepted by the surfaces $z = f_1(x,y)$ and $z = f_2(x,y)$, a discrepancy which is immaterial in view of subsequent integrations.] This integral is a function of x and y alone, and hence the result of the first integration can be represented as $\varphi(x,y)\Delta y\ \Delta x$. If we now hold x constant and integrate with respect to y we obtain

$$\int_{y=g_1(x)}^{y=g_2(x)} \varphi(x,y)dy\ \Delta x$$

as the sum of the products over a slice of volume parallel to the yz-plane. The result now is a function of x alone, say $\psi(x)\Delta x$. Integrating with respect to x from the least to the greatest value of x over the entire volume V, say from $x = a$ to $x = b$, we obtain $\int_a^b \psi(x)dx$ as the value of the triple integral.

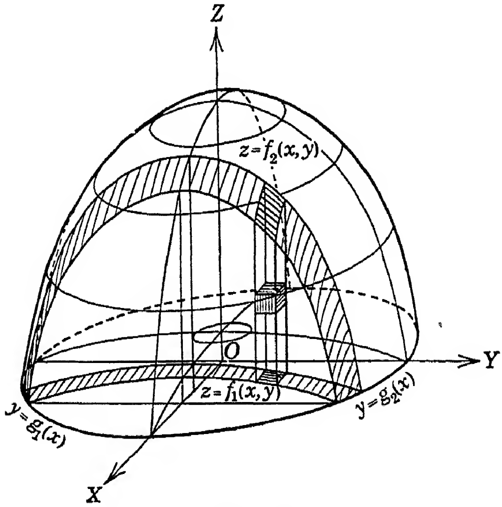

FIG. 189.

In other words, we have evaluated the triple integral as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)dz\ dy\ dx,$$

the notation, as in similar former cases, having the force

$$\int_a^b \left[\int_{g_1(x)}^{g_2(x)} \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)dz \right) dy \right] dx.$$

The student will perceive at once that a possible, and perhaps in some cases more convenient, order of integration may differ from the one just displayed, and that the value of the triple integral may in such cases be represented by

$$\int_c^d \int_{g_1(z)}^{g_2(z)} \int_{f_1(y,z)}^{f_2(y,z)} f(x,y,z)dx\ dy\ dz$$

or

$$\int_e^f \int_{g_1(z)}^{g_2(z)} \int_{f_1(x,z)}^{f_2(x,z)} f(x,y,z)dy\ dx\ dz,$$

etc.

Exercise 1. Show that when cylindrical coordinates (r,θ,z) are employed, the value of the triple integral of a function $f(r,\theta,z)$ over a region V may be represented as

$$\int_{\alpha}^{\beta}\int_{g_1(\theta)}^{g_2(\theta)}\int_{f_1(r,\theta)}^{f_2(r,\theta)} f(r,\theta,z)r\,dz\,dr\,d\theta.$$

If spherical coordinates (ρ,φ,θ) are employed, an element of volume bounded by two concentric spheres differing in radius by

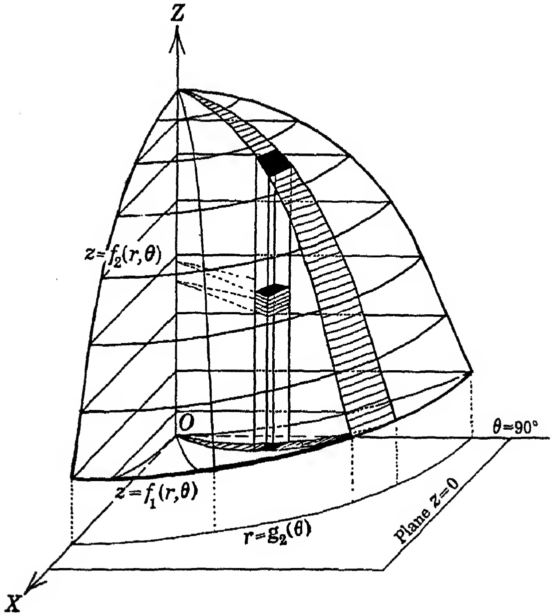

Fig. 190.

$\Delta\rho$, two planes through the z-axis meeting at the angle $\Delta\theta$, and two cones with OZ as axis and O as vertex, with semivertical angles differing by $\Delta\varphi$ is, apart from infinitesimals of higher order, equal to $\rho^2 \sin\varphi\,\Delta\rho\,\Delta\phi\,\Delta\theta$. Indeed, the volume of the element, excepting for such infinitesimals, is equal to $\Delta\rho$ times the area of $CDEF$ (Fig. 191), an area which may be computed as CD times CF. Now $CD = \rho\,\Delta\varphi$ and $CF = AB = OA \cdot \Delta\theta = \rho \sin\varphi\,\Delta\theta$. Hence,

$$\Delta V = CD \cdot CF \cdot \Delta\rho = (\rho\,\Delta\varphi)(\rho \sin\varphi\,\Delta\theta)(\Delta\rho) = \rho^2 \sin\varphi\,\Delta\rho\,\Delta\varphi\,\Delta\theta.$$

If, now, the triple integral of a function $f(\rho,\varphi,\theta)$ is to be computed over a region V and spherical coordinates are employed, the value of the integral may be represented as

$$\int_{\alpha}^{\beta}\int_{g_1(\theta)}^{g_2(\theta)}\int_{f_1(\varphi,\theta)}^{f_2(\varphi,\theta)} f(\rho,\varphi,\theta)\cdot\rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta.$$

Exercise 2. Show that $\iiint dx\,dy\,dz$, $\iiint r\,dz\,dr\,d\theta$, and

$$\iiint \rho^2 \sin\varphi\,d\rho\,d\varphi\,d\theta,$$

computed over a region V of space, all represent the volume of that region.

Exercise 3. Show that the triple integrals of the functions $\delta(x^2 + y^2)$, δr^2, and $\delta\rho^2 \sin^2\varphi$ over a region V, represent the moment of inertia of the mass of V about the z-axis, where δ is the density of a point of V and by the z-axis in spherical coordinates we mean the line $\varphi = 0$.

Exercise 4. Show that the quotients, of the triple integrals of $\delta \cdot x$, $\delta \cdot r\cos\theta$, and $\delta \cdot \rho \sin\varphi\cos\theta$ over a region, by the mass of the region represent the x-coordinate of the center of gravity of V, where δ represents the density at a point of V.

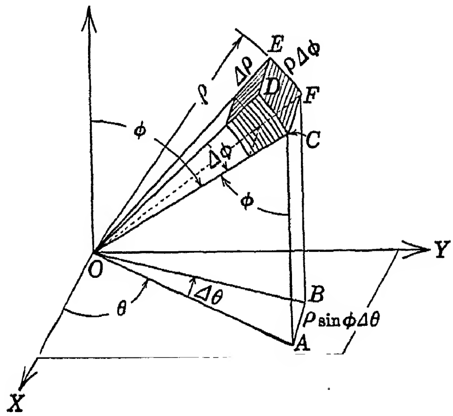

FIG. 191.

Problems

1. Compute each of the triple integrals below. Identify the function integrated and the region over which the integral is computed.

(a) $\displaystyle\int_0^2\int_{-2}^0\int_0^1 (y + z)dx\,dy\,dz.$ *Ans.* 0.

(b) $\displaystyle\int_0^1\int_{x^2}^{2-x}\int_0^{\sqrt{y-x^2}} x\,dz\,dy\,dx.$

(c) $\displaystyle\int_0^{\pi/2}\int_0^{2\sin\theta}\int_0^{\sqrt{4-r^2}} r^2\,dz\,dr\,d\theta.$ *Ans.* $\dfrac{\pi^2}{4}$.

(d) $\displaystyle\int_0^a\int_0^{2\pi}\int_0^{\pi/4} \rho^2\cos\varphi\sin\varphi\,d\varphi\,d\theta\,d\rho.$

2. Compute the triple integral of each function below over the region indicated.

(a) $f(x,y,z) = y\sqrt{x}$; region in the first octant, bounded by the planes $x = 0$, $z = 0$, $y = 1$, and $3x + 2y + z = 6$.

Ans. $64(243\sqrt{2} - 104\sqrt{3})/8505$.

(b) $f(x,y,z) = xz$; region in the first octant, bounded by the coordinate planes, the cylinder $x^2 + z^2 = 25$, and plane $5x + 15y + 14z = 90$.

(c) $f(x,y,z) = \dfrac{y}{x^2 + y^2}$; region bounded by the paraboloid

$$x^2 + y^2 + z = 9,$$

the cylinder $x^2 + y^2 = 3y$, and the xy-plane. HINT: Change to cylindrical coordinates. *Ans.* $81\pi/8$.

(d) $f(x,y,z) = \dfrac{x}{x^2 + y^2}$; the region bounded by the two cones $x^2 + y^2 = z^2$, $x^2 + y^2 = 3z^2$ and the sphere $x^2 + y^2 + z^2 = 4$. HINT: Change to spherical coordinates.

3. For the region in the first octant, bounded by the planes $x = 0$, $z = 0$, $x + y = 1$, $y - 2x = 4$, and the cylinder $x^2 + z^2 = 1$, compute as a triple integral the following:

(a) The volume. *Ans.* $\dfrac{3\pi}{4} + 1$.

(b) The moment of inertia about the y-axis.

Ans. $\dfrac{(15\pi + 24)M}{30\pi + 40}$.

(c) The x-coordinate of the center of gravity. *Ans.* $\dfrac{3\pi + 16}{12\pi + 16}$.

(d) The mass, if the density at any point is proportional to the distance from the xy-plane. *Ans.* $11k/8$.

4. For the region in the first octant, bounded by the coordinate planes, the plane $x + 2y = 6$, and the surface $x^2 + y^2 = 36 - 6z$, compute by means of a triple integral:

(a) The volume.

(b) The moment of inertia about the x-axis.

(c) The y-coordinate of the center of gravity.

(d) The mass, if the density at any point is proportional to the distance from the yz-plane.

5. For the region in the first octant, bounded by the planes $x = 0$, $y = 0$, $z - y = 6$, $x + 2y + 2z = 8$ and the cylinder $x^2 + y^2 = 4$, compute by means of a triple integral:

(a) The volume.

(b) The moment of inertia about the z-axis.

(c) The x-coordinate of the center of gravity.

(d) The moment of inertia about the line $x = 2$, $y = 0$.

6. For the pyramid whose faces are the coordinate planes and the plane $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$, find by means of a triple integral:

(a) The moment of inertia about the edge of length a.

Ans. $\dfrac{M(b^2 + c^2)}{10}$.

(b) The y-coordinate of the center of gravity. *Ans.* $b/4$.

(c) The moment of inertia about a line parallel to the edge of length a, and passing through the midpoint of the opposite edge.

Ans. $M(b^2 + c^2)/10$.

7. For the volume above the xy-plane and inside both the cylinder $x^2 + y^2 - 3x = 0$ and the sphere $x^2 + y^2 + z^2 = 9$, compute, using cylindrical coordinates and triple integration,

(*a*) The z-coordinate of the center of gravity.

(*b*) The moment of inertia about the x-axis.

(*c*) The moment of inertia about the line through (3,0,0), parallel to OZ.

8. For a hollow cylinder of altitude h and radii of base a and b $(a > b)$, using triple integration and cylindrical coordinates, find

(*a*) The moment of inertia about the axis of the cylinder.

(*b*) The moment of inertia about a diameter of either base.

(*c*) The moment of inertia about an outside element.

(*d*) The center of gravity, on the assumption that the density at any point is proportional to its distance from one of the bases.

9. A volume is in the form of a cone of semivertical angle α, capped by a portion of a sphere of radius a whose center is at the vertex of the cone. Using triple integration and cylindrical or spherical coordinates, find

(*a*) The volume. *Ans.* $\dfrac{2\pi a^3(1 - \cos\alpha)}{3}$.

(*b*) The moment of inertia of the volume about the axis of the cone.

Ans. $\dfrac{a^2M(2 - \cos\alpha - \cos^2\alpha)}{5.}$

(*c*) The moment of inertia of the volume about a line through the vertex of the cone perpendicular to the axis.

Ans. $\dfrac{a^2M(4 + \cos\alpha + \cos^2\alpha)}{10}$.

(*d*) The moment of inertia of the volume about a diameter of the base of the cone. *Ans.* $\dfrac{a^2M(8 - 13\cos\alpha + 7\cos^2\alpha)}{20}$.

(*e*) The center of gravity, on the assumption that the density at any point is proportional to the distance from the plane through the vertex of the cone perpendicular to the axis.

Ans. Distance from the vertex equals $\dfrac{8a}{15}\left(\dfrac{1 + \cos\alpha + \cos^2\alpha}{1 + \cos\alpha}\right)$.

10. A lune is cut from a sphere of radius a by two planes through a diameter making an angle 2α with each other. Find

(*a*) The center of gravity of the lune if its volume is homogeneous.

Ans. $\dfrac{3\pi a \sin\alpha}{16\alpha}$ from center.

(*b*) The center of gravity of the lune if the density at any point is proportional to the distance from the edge of the dihedral angle 2α.

Ans. $\dfrac{32a\sin\alpha}{15\pi\alpha}$ from center.

(*c*) The moment of inertia of the lune about the edge when the volume is homogeneous.

(*d*) The moment of inertia of the lune about the edge with the density as in (*b*).

11. A cylindrical hole of radius 8 ft. is cut from a sphere of radius 10 ft., the axis of the hole coinciding with a diameter of the sphere. For the remaining part of the sphere, find

(*a*) The moment of inertia about the axis of the cylinder.

(*b*) The moment of inertia about a diameter of the sphere perpendicular to the axis of the cylinder.

(*c*) The moment of inertia about an element of the cylinder.

(*d*) The center of gravity.

12. Find the attraction of a hemispherical shell of radii a and b $(b > a)$ upon a unit mass placed at the center of its base.

(*a*) If the shell is homogeneous. *Ans.* $\dfrac{3M}{2(b^2 + ab + a^2)}$.

(*b*) If the density at any point of the shell is proportional to its distance from the radius of symmetry. *Ans.* $\dfrac{k\pi(b^2 - a^2)}{3}$.

13. A solid is bounded by the plane $z = 8$ and the surface $x^2 + y^2 = z - 4$. Find its attraction upon a unit mass placed at the origin

(*a*) If the solid is homogeneous.

(*b*) If the density at any point is proportional to the square of the distance from the origin.

14. A hemispherical shell of radii a and b $(b > a)$ is intercepted by a right circular cone of semivertical angle α and with vertex at the center of the shell. Find the attraction of that part of the shell outside the cone upon a unit mass placed at the center

(*a*) If the volume is homogeneous.

(*b*) If the density at any point is proportional to the distance from the base of the shell.

108. Line Integrals. Let a function $P(x,y)$ be defined, single-valued, and continuous over a region R of the xy-plane, and let C be a curve in that region at all points of which y is a single-valued and continuous function of x, say $y = f(x)$, and x is a single-valued function of y, say $x = g(y)$. Let the portion of C between any two of its points (a,b) and (c,d) be divided into n parts of length $\Delta_1 s, \Delta_2 s, \ldots, \Delta_n s$, whose projections upon the coordinate axes are $\Delta_1 x, \Delta_2 x, \ldots, \Delta_n x$ and $\Delta_1 y, \Delta_2 y, \ldots, \Delta_n y$, respectively. Compute the function $P(x,y)$ for some point, (ξ_i,η_i) at random in every subdivision of C and set up the sums

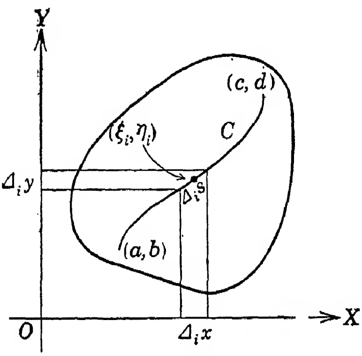

FIG. 192.

$$\sum_{i=1}^{n} P(\xi_i,\eta_i)\Delta_i x, \qquad \sum_{i=1}^{n} P(\xi_i,\eta_i)\Delta_i y.$$

Since, at all points of C, $P(x,y)$ is essentially a function of x alone, *viz.*, $P[x,f(x)]$, single-valued and continuous in view of the

assumptions made on P and f, and likewise a single-valued and continuous function of y alone, *viz.*, $P[g(y),y]$, each of the above sums approaches a limit as $n \to \infty$ and Δx, $\Delta y \to 0$. These limits are called *line integrals* of the function $P(x,y)$ along the curve C, between (a,b) and (c,d), and are designated, respectively, as

$$\int_{(a,b)}^{(c,d)} P(x,y)\,dx \qquad \text{and} \qquad \int_{(a,b)}^{(c,d)} P(x,y)\,dy,$$

taken along C, or as

$$\int_C P(x,y)\,dx \qquad \text{and} \qquad \int_C P(x,y)\,dy,$$

between (a,b) and (c,d). The curve C is spoken of as the *path of integration.*

This notion of a line integral is easily extended to three dimensions, and for a function $P(x,y,z)$, single-valued and continuous in a region R of space, and a curve C in that region, we obtain

$$\int_C P(x,y,z)\,dx, \qquad \int_C P(x,y,z)\,dy, \qquad \int_C P(x,y,z)\,dz$$

between two points (a,b,c) and (d,e,f) of C, as line integrals taken along the path of integration, C.

The computation of a line integral follows directly from its definition. Thus,

$$\int_{(0,0)}^{(4,2)} (x^2 - xy)\,dx \text{ along } y^2 = x$$

reduces to

$$\int_0^4 (x^2 - x^{3/2})\,dx = {}^{128}\!/_{15},$$

while

$$\int_{(0,0)}^{(4,2)} (x^2 - xy)\,dy \text{ along } y^2 = x$$

reduces to

$$\int_0^2 (y^4 - y^3)\,dy = {}^{12}\!/_{5}.$$

Again, to compute $\displaystyle\int_{(2/\sqrt{5},4/\sqrt{5},0)}^{(\sqrt{2/5},\sqrt{8/5},2)} (x^2 + xy - z^2)\,dz$ along the curve $\left.\begin{array}{l} x^2 + y^2 = 4 - z \\ y = 2x \end{array}\right\}$, note that along the path of integration, $y^2 = 4x^2$, $5x^2 = 4 - z$, $xy = 2x^2$. Hence, the integral reduces to

$$\int_0^2 \left(\frac{4 - z}{5} + \frac{2(4 - z)}{5} - z^2\right)dz = \frac{14}{15}.$$

The line integrals of most frequent occurrence in problems are of the form

$$\int_C P(x,y)dx + Q(x,y)dy,$$

understood as $\int_C P(x,y)dx + \int_C Q(x,y)dy$, or of the form

$$\int_C P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz,$$

understood as $\int_C P(x,y,z)dx + \int_C Q(x,y,z)dy + \int_C R(x,y,z)dz.$

As an example, let a particle be displaced along the curve C from (a,b) to (c,d) under the action of a force F whose x- and y-components, say $X(x,y)$ and $Y(x,y)$ are single-valued and continuous functions of x and y in some region containing the curve C. If we resolve the force F into two components, one along the tangent and the other along the normal, hence represented by $F \cdot \cos\beta$ and $F \cdot \sin\beta$, the amount of work done by the force in displacing the particle is

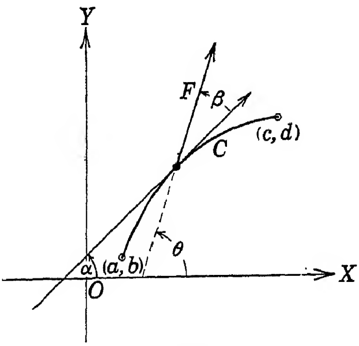

FIG. 193.

$$\lim_{\Delta s\to 0}\sum(F\cdot\cos\beta)\Delta s = \int_{(a,b)}^{(c,d)} F\cos\beta\, ds, \text{ taken along } C,$$

since the direction of the curve at any point is that of its tangent, and the only effective component of the force F in displacing the particle is its component along the tangent. Now, by Fig. 193, $\beta = \theta - \alpha$, whence $\cos\beta = \cos\theta\cos\alpha + \sin\theta\sin\alpha$. Hence,

$$\begin{aligned} F\cos\beta\, ds &= F\cos\theta\cos\alpha\, ds + F\sin\theta\sin\alpha\, ds \\ &= F\cos\theta\, dx + F\sin\theta\, dy \\ &= X(x,y)dx + Y(x,y)dy, \end{aligned}$$

since $\cos\alpha\, ds = dx$, $\sin\alpha\, ds = dy$, $F\cos\theta = X(x,y)$ and $F\sin\theta = Y(x,y)$, and the work done is, thus, represented by

$$\int_{(a,b)}^{(c,d)} X(x,y)dx + Y(x,y)dy, \text{ taken along } C.$$

The student will extend this result to three dimensions by

Exercise 1. Prove that the work done by a force F whose components parallel to the x-, y-, and z-axes, $X(x,y,z)$, $Y(x,y,z)$, and $Z(x,y,z)$ are single-valued and continuous functions of x, y, and z in some region in space, in displacing a particle along a curve C in that region, is given by

$$\int_C X(x,y,z)dx + Y(x,y,z)dy + Z(x,y,z)dz.$$

Exercise 2. Prove that $\int_C P(x,y)ds$ may be put in the form

$$\int_C Q(x,y)dx + R(x,y)dy,$$

where ds is the element of arc of the path of integration. HINT: If α is the inclination of the tangent of C to the x-axis, then $ds = dx \cdot \cos\alpha + dy \cdot \sin\alpha$.

The student will note in passing that the expression for the area bounded by a curve $y = f(x)$, the x-axis and the two ordinates $x = a$, and $x = b$, *viz.*, $\int_a^b f(x)dx$, is actually a line integral of the form $\int_{a,f(a)}^{b,f(b)} P(x,y)dx$, taken along the curve, where $P(x,y) = y$.

Exercise 3. Exhibit the area bounded by the curve $x = g(y)$, the y-axis and two lines $y = c$, $y = d$, as a line integral along that curve.

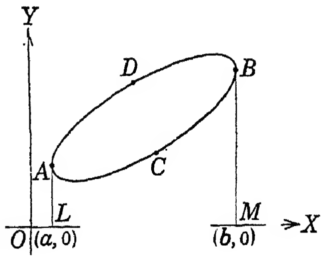

FIG. 194.

As an important application of line integrals, we consider the problem of finding the area enclosed by a curve. We assume, first, that the curve is such that a line parallel to OX meets it in two points at most, thus having $y = y_1(x)$ along the portion ACB and $y = y_2(x)$ along ADB. Then the area enclosed by the curve is equal to the area $LADBM$ − area $LACBM$

$$\begin{aligned} &= \int_{ADB} y_2(x)dx - \int_{ACB} y_1(x)dx \\ &= -\int_{ACB} y_1(x)dx - \int_{BDA} y_2(x)dx = -\int_{ACBDA} y(x)dx \\ &\qquad = -\int_C y\,dx.^* \end{aligned}$$

An alternative expression for the area enclosed by a curve the student is asked to obtain in

* This number will always be positive if the path of integration is so traversed that the enclosed area is to the left of it.

Exercise 4. Prove that the area enclosed by a curve C is represented by the line integral $\int_C x\,dy$. HINT: Assume that the curve is met by a line parallel to OY in at most two points.

On account of the possibility of dual representation just brought out, it is customary to display the area enclosed by a curve C in the form $\frac{1}{2}\int_C x\,dy - y\,dx$.

Exercise 5. Prove, in case a closed curve C is met by some lines parallel to OX and OY in more than two places, that the formulas $-\int_C y\,dx$ and $\int_C x\,dy$ still represent the area if it can be subdivided into a finite number of subareas, each of which is bounded by a curve which is met at most twice by lines parallel to OX and OY. HINT: It is sufficient to consider two subareas, as in Fig. 195. Let C represent the curve $ABDEA$, C_1 represent the curve $ABDA$, and C_2 represent the curve $ADEA$. Note that in traversing C_1 and C_2 each once and each counterclockwise, the line AD is traversed twice, once from D to A and once from A to D.

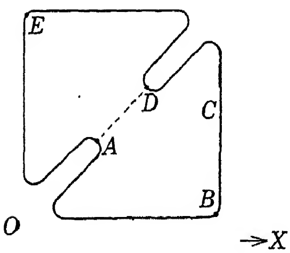

FIG. 195.

By changing the expression $\frac{1}{2}\int_C x\,dy - y\,dx$ from rectangular to polar coordinates, prove

Exercise 6. The area enclosed by a curve C is represented by the line integral $\frac{1}{2}\int_C r^2\,d\theta$.

The student learned in an earlier chapter to express the area OAB bounded by the two lines OA and OB and the curve C as $\frac{1}{2}\int_\alpha^\beta r^2\,d\theta$. He now recognizes it as the line integral $\frac{1}{2}\int r^2\,d\theta$ between $[f(\alpha),\alpha]$ and $[f(\beta),\beta]$ where $r = f(\theta)$ is the equation of C. This will square with the result of Exercise 6 if we note that the boundary of the area consists of the lines $\theta = \alpha$, $\theta = \beta$ and the curve C, that along the two lines $d\theta = 0$, and hence, that the line integral along the entire boundary reduces to the line integral along C.

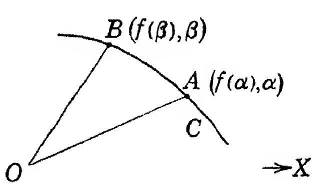

FIG. 196.

One feature of a line integral will have been observed by the student, *viz.*, that its value depends not only upon the inte-

grand function and the limits, but also upon the path of integration. Consider, however, the integral

$$\int_{(a,b)}^{(c,d)} P(x,y)dx + Q(x,y)dy$$

in the case when $P(x,y)dx + Q(x,y)dy$ is an exact differential. (A necessary and sufficient condition for this, the student will recall from Art. 102, is that $P_y = Q_x$.) Let any curve C, passing through (a,b) and (c,d), say $y = f(x)$, be taken as the path of integration. Now, to say that $P(x,y)dx + Q(x,y)dy$ is an exact differential, amounts to saying that a function $\varphi(x,y)$ exists such that

$$d\varphi = P(x,y)dx + Q(x,y)dy.$$

Hence, the line integral

$$\int_{(a,b)}^{(c,d)} P(x,y)dx + Q(x,y)dy, \text{ taken along } C,$$

reduces to

$$\int_{(a,b)}^{(c,d)} d\varphi(x,y), \text{ taken along } C,$$

and this, in turn to

$$\begin{aligned}\int_a^c dg(x) &= g(c) - g(a)\\ &= \varphi[c,f(c)] - \varphi[a,f(a)]\\ &= \varphi(c,d) - \varphi(a,b),\end{aligned}$$

where we have used $g(x)$ to represent $\varphi[x,f(x)]$. The result clearly does not involve the equation $y = f(x)$ which we assume for the path of integration. In other words, *if*

$$P(x,y)dx + Q(x,y)dy$$

is an exact differential, the value of the integral

$$\int_{(a,b)}^{(c,d)} P(x,y)dx + Q(x,y)dy$$

depends upon the functions P and Q and the limits, but not upon the path of integration.

Exercise 7. Show that if $Pdx + Qdy + Rdz$ is an exact differential, the value of $\int_{(a,b,c)}^{(d,e,f)} P\,dx + Q\,dy + R\,dz$ is independent of the path of integration.

Problems

1. Compute $\int_{(0,0)}^{(2,2)} (x^2 + xy)dx + (y^2 - xy)dy$

   (*a*) Along the line $y = x$. *Ans.* $16/3$.

(b) Along the parabola $x^2 = 2y$. *Ans.* $\frac{62}{15}$.

(c) Along the x-axis from the origin to (2,0), then along the line $x = 2$ from (2,0) to (2,2). *Ans.* $\frac{4}{3}$.

2. Compute $\int_{(0,0)}^{(1,-1)} yx^2\,dx + (x + y)dy$

(a) Along the straight line joining the two points.

(b) Along the curve $y = -x^3$.

(c) Along OY to $(0,-1)$, then along the line $y = -1$.

(d) Along the circle $x^2 + y^2 - 2x = 0$.

3. Compute $\int_{(0,2)}^{(3,0)} (x - 2y)dx + xy\,dy$

(a) Along the straight line joining the two points. *Ans.* $-\frac{7}{2}$.

(b) Along the ellipse $x = 3\cos\theta$, $y = 2\sin\theta$. *Ans.* $\frac{1}{2} - 3\pi$.

(c) Along the parabola $2(x - 3)^2 = 9y$. *Ans.* $-\frac{7}{10}$.

4. Compute $\int_{(0,0,0)}^{(1,2,4)} (x + y)dx + (y + z)dy + (z + x)dz$

(a) Along the straight line joining the two points. *Ans.* $\frac{35}{2}$.

(b) Along OX to (1,0,0), along a straight line to (1,2,0), and then along a straight line to (1,2,4). *Ans.* $\frac{29}{2}$.

(c) Along the curve $x^2 - y + z = 3x$. $z = 2y$.

5. Compute $\int (x - y)dx + (y + x)dy$ around the entire circle $x^2 + y^2 = 4$. *Ans.* 8π.

6. Compute $\int_{(1,0,0)}^{(3,4,8)} (xy - z)dx + e^x\,dy + y\,dz$

(a) Along the straight line joining the two points.

(b) Along the curve $x = t + 1$, $y = t^2$, $z = t^3$.

7. Show that the value of each of the following is independent of the path of integration, and compute in any convenient manner.

(a) $\int_{(-1,0)}^{(2,1)} (x^2 + 2y)dx + (2x - y)dy$.

(b) $\int_{(1,3)}^{(3,1)} (e^x + 2y)dx + (2x + \log y)dy$.

(c) $\int_{(2,0)}^{(-1,3)} (2xy - y^2 + y)dx + (x^2 - 2xy + x)dy$.

(d) $\int_{(0,2)}^{(1,0)} \frac{2y}{(xy + 1)^2}dx + \frac{2x}{(xy + 1)^2}dy$.

(e) $\int_{(0,0,0)}^{(1,3,2)} (y - z)dx + (x + z^2)dy + (2yz - x)dz$.

(f) $\int_{(2,-1,0)}^{(1,0,3)} (yz + 2x)dx + xz\,dy + (xy - 2z)dz$.

8. Compute the following areas as line integrals, by using the formulas $\int_C x\,dy$, $-\int_C y\,dx$, $\frac{1}{2}\int_C x\,dy - y\,dx$, or $\frac{1}{2}\int_C r^2\,d\theta$, as may be found convenient:

(a) The area bounded by one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

(b) The area bounded by the straight line $x + y = a$ and the first quadrant arc of the four-cusped hypocycloid $x = a\cos^3\theta$, $y = a\sin^3\theta$.

(*c*) The area in the first quadrant bounded by OX, the ellipse $x = a \cos \theta$, $y = b \sin \theta$, and the line $y = x$.

(*d*) The area in the first quadrant bounded by OX, the circle $x = a \cos \theta$, $y = a \sin \theta$, and the line $x = y\sqrt{3}$.

(*e*) The area outside the cardioid $r = a(1 + \cos \theta)$ and inside the circle $r = 3a \cos \theta$.

9. Compute (*a*) $\int_{(1,0)}^{(2, \log 2)} (x + e^y)\, ds$ along the curve $y = \log x$.

(*b*) $\int_{(2,1)}^{(4,9)} (3x - y) ds$ along the curve $x = t + 1$, $y = t^2$.

(*c*) $\int (x^2 + xy) ds$ around the circle $x^2 + y^2 - 4y = 0$.

10. Show that if $P(x,y)dx + Q(x,y)dy$ is an exact differential, the integral $\int_C P(x,y)dx + Q(x,y)dy$ equals zero if C is a closed curve lying entirely in a region where P and Q are continuous, single-valued functions of x and y. HINT: Choose any two points, (a,b) and (c,d) on the curve, and integrate between the two points over each portion of C joining them.

11. State and prove a three-dimensional proposition similar to that of Prob. 10.

12. Verify the proposition of Prob. 10 for $\int_C \frac{-y\, dx}{x^2 + y^2} + \frac{x\, dy}{x^2 + y^2}$ where C is the boundary of the square whose vertices are $(1,0)$, $(3,0)$ $(3,2)$, and $(1,2)$.

13. Verify that the value of the integral of Prob. 12 over the boundary of the square with vertices at $(-2,-1)$, $(1,-1)$, $(1,2)$, and $(-2,2)$ is 2π. Explain in what way the proposition of Prob. 10 fails to be met in this case.

CHAPTER XIV

INFINITE SERIES

109. Series of Constant Terms. One frequently meets, in mathematics, with *infinite sequences, i.e.*, sequences of numbers in which the number of terms is infinite. Thus, the sequence of positive integers, 1, 2, 3, . . . , is such a one. Usually, some law is given by which the terms of the sequence can be determined. For example, if the nth term of a sequence is $\frac{n}{2n+1}$ ($n = 1,2,3,$ $\cdots$), the sequence itself is $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \cdots$.

Associated with every infinite sequence $u_1, u_2, u_3, \ldots$ is the *infinite series* $u_1 + u_2 + u_3 + \cdots$. A question of importance that arises in connection with an infinite series is this. If we form the sequence $S_1, S_2, S_3, \cdots$, where $S_1 = u_1$, $S_2 = u_1 + u_2$, $S_3 = u_1 + u_2 + u_3, \cdots,$ $S_n = u_1 + u_2 + u_3 + \cdots + u_n$, does $\lim\limits_{n\to\infty} S_n$ exist?

This question presented itself, in fact, in the student's earlier experience, in the case of the sequence $a, ar, ar^2, \ldots$ and of the corresponding series $a + ar + ar^2 + \cdots$ (the geometric series), and the fact encountered in that connection was that $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ does indeed approach a limit when $|r| < 1$, and that in that case $\lim\limits_{n\to\infty} S_n = \frac{a}{1-r}$, while no limit is approached by S_n when $|r| \geqq 1$.

We now lay down the

Definition. An infinite series $u_1 + u_2 + u_3 + \cdots$ *is said to be convergent when* $\lim\limits_{n\to\infty} S_n$ *exists, where*

$$S_n = u_1 + u_2 + u_3 + \cdots + u_n.$$

The value of that limit is then said to be the *sum* of the series.

A series that is not convergent is said to be divergent.

A series may diverge by reason of S_n becoming positively infinite (as is the case with $1 + 2 + 4 + 8 + \cdots + 2^{n-1} + \cdots$),

negatively infinite (as in the series $-1 - 2 - 4 - 8 - \cdots$), infinite but with no definite sign (as in the case of $1 - 2 + 4 - 8 + \cdots$), or remaining finite but failing to approach a limit (as in the series $1 - 1 + 1 - 1 + \cdots$, where $S_n = 0$ for even values of n and $S_n = 1$ for odd values of n).

The problem now arises, given an infinite series, what means to employ to ascertain whether it converges or diverges.* The student's first attempt to solve this problem might be to exhibit S_n as a function of n and to determine from that whether $\lim\limits_{n\to\infty} S_n$ exists. This may be easily done occasionally, but in general the task is prohibitive. An observant student may note, for instance, that the series of which the nth term is

$$\frac{1}{n(n+1)} \quad (n = 1,2,3, \cdots),$$

viz., $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \cdots$, may be displayed as

$$(\tfrac{1}{1} - \tfrac{1}{2}) + (\tfrac{1}{2} - \tfrac{1}{3}) + (\tfrac{1}{3} - \tfrac{1}{4}) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots$$

$\left(\text{since } \dfrac{1}{n(n+1)} = \dfrac{1}{n} - \dfrac{1}{n+1}\right)$, hence $S_n = 1 - \dfrac{1}{n+1}$, and $\lim\limits_{n\to\infty} S_n = 1$. This informs us at once that the series converges and that its sum, as defined above, is 1. In most cases, however, it would be out of the question to obtain an expression for S_n, and we must seek some means of determining from the series itself its property of convergence or nonconvergence. To that end we lay down a number of theorems.

Theorem 1. *A necessary condition for the convergence of the series* $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ *is that* $\lim\limits_{n\to\infty} u_n = 0$.

For, to say that the series converges to S as a limit amounts to saying that given any arbitrary positive constant ϵ, a number m may be found such that whenever $n \geqq m$, we have $|S - S_n| < \epsilon$. We can, therefore, fix an integer p so that whenever $n \geqq p$, we have $|S - S_n| < \epsilon/2$, and hence, for such values of n, we have

* That the question of the convergence or divergence of a series is of significance, will appear abundantly as we go on.

$$|S - S_n| < \frac{\epsilon}{2}$$

and

$$|S - S_{n+1}| < \frac{\epsilon}{2}.$$

Now

$$|u_{n+1}| = |S_{n+1} - S_n| = |S - S_n + S_{n+1} - S| \leqq |S - S_n| + |S_{n+1} - S| = |S - S_n| + |S - S_{n+1}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

To say, now, that given any positive quantity ϵ (however small), a number p may be found large enough (*i.e.*, a place in the series far enough) so that $|u_{n+1}| < \epsilon$ for all values of $n \geqq p$, is only another way of saying that $\lim\limits_{n \to \infty} u_n = 0$.

By means of this theorem, we can at once pronounce a series as divergent if it appears that, for it, $\lim\limits_{n \to \infty} u_n \neq 0$. For example, in the series

$$\frac{1}{5} + \frac{3}{8} + \frac{5}{11} + \frac{7}{14} + \cdots + \frac{2n-1}{3n+2} + \cdots,$$

$\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} \left(\dfrac{2n-1}{3n+2}\right) = \lim\limits_{n \to \infty} \left(\dfrac{2 - \frac{1}{n}}{3 + \frac{2}{n}}\right) = \dfrac{2}{3} \neq 0$, and the series diverges.

The condition of Theorem 1 is not, however, a sufficient condition for convergence of the series. That is, we may have a series for which $\lim\limits_{n \to \infty} u_n = 0$ and which, nevertheless, diverges. The case of the so-called *harmonic* series,

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

is an illustration. Here $\lim\limits_{n \to \infty} u_n = \lim\limits_{n \to \infty} (1/n) = 0$. Let us group its terms as follows:

$$1/1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8) + (1/9 + 1/10 + \cdots + 1/16) + (1/17 + 1/18 + \cdots + 1/32) + \cdots.$$

It is evident that the first pair grouped exceed $2 \cdot \frac{1}{4}$, *i.e.*, $\frac{1}{2}$, the next group of four terms exceeds $4 \cdot \frac{1}{8}$, *i.e.*, $\frac{1}{2}$. the next group of eight terms exceeds $8 \cdot \frac{1}{16}$, *i.e.*, $\frac{1}{2}$, etc. Since each of these groups exceeds $\frac{1}{2}$ and the number of groups is infinite we may readily choose a value for n large enough so that, for $m > n$, S_m will exceed any preassigned positive number. Evidently the series diverges.

Theorem 2. **(Comparison test.)** (*a*) *Given a series of positive terms* $v_1 + v_2 + v_3 + \cdots$, *known to converge, if each term of the series* $u_1 + u_2 + u_3 + \cdots$ *is positive or zero and less than or equal to the corresponding term of the given series* (*i.e.*, $0 \leqq u_i \leqq v_i$ *for every positive integer* i), *then the series* $u_1 + u_2 + u_3 + \cdots$ *converges.*

(*b*) *Given a series of terms* $v_1 + v_2 + v_3 + \cdots$, *positive or zero, known to diverge, if each term of the series* $u_1 + u_2 + u_3 + \cdots$ *is positive and greater than or equal to the corresponding term of the given series* (*i.e.*, $u_i \geqq v_i \geqq 0$ *for every positive integer* i), *then the series* $u_1 + u_2 + u_3 + \cdots$ *diverges.*

Proof of (*a*). Let V_n represent the sum of the first n terms of the given series and U_n the sum of the first n terms of the series to be tested (*viz.*, the series $u_1 + u_2 + u_3 + \cdots$). Since the terms of the latter series are positive or zero, U_n is a function of n that increases as n increases. By hypothesis $U_n \leqq V_n$. If V is the sum of the given series, $V_n < V$, whence $U_n < V$ ($n = 1,2,3, \cdots$). Since U_n increases but always remains less than a finite fixed number V, it approaches a limit which is $\leqq V$ (page 151, Theorem 1) and the series $u_1 + u_2 + u_3 + \cdots$ converges.

Exercise 1. Prove part (*b*) of this theorem.

Note, in passing, that the convergence or divergence of a series is not altered by discarding a finite number of terms or by prefixing a finite number of terms. Hence, in applying the comparison test embodied in Theorem 2, if it appears that $u_i \leqq v_i$ or $u_i \geqq v_i$ ($i = m + 1, m + 2, m + 3, \cdots$), where m is finite, we draw the appropriate conclusion regardless of how the first m terms of the two series compare.

Let us employ Theorem 2 to study the behavior of the series

$$\frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} + \cdots,$$

the *hyperharmonic* series, to use the term introduced by the American mathematician Pierpont.

1. When $k = 1$, this series becomes

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots,$$

the harmonic series which we studied above and found to be divergent.

2. When $k < 1$, each term (after the first) of the hyperharmonic series is greater than the corresponding term of the harmonic series, since $n^k < n$ $(n = 2,3,4, \cdots)$ when $k < 1$ and hence $1/n^k > 1/n$. By part (*b*) of Theorem 2, the hyperharmonic series is, in this case, divergent.

3. When $k > 1$, let us group the terms as follows*:

$$\frac{1}{1^k}+\left(\frac{1}{2^k}+\frac{1}{3^k}\right)+\left(\frac{1}{4^k}+\frac{1}{5^k}+\frac{1}{6^k}+\frac{1}{7^k}\right) \\ +\left(\frac{1}{8^k}+\frac{1}{9^k}+\cdots+\frac{1}{15^k}\right)+\cdots. \quad (144)$$

Now, the series

$$\frac{1}{1^k}+\frac{2}{2^k}+\frac{4}{4^k}+\frac{8}{8^k}+\cdots$$

converges, since it is a geometric series whose ratio is

$$\frac{2}{2^k}=\frac{1}{2^{k-1}},$$

a positive number less than 1 when $k > 1$. We can display its terms as

$$\frac{1}{1^k}+\left(\frac{1}{2^k}+\frac{1}{2^k}\right)+\left(\frac{1}{4^k}+\frac{1}{4^k}+\frac{1}{4^k}+\frac{1}{4^k}\right) \\ +\left(\frac{1}{8^k}+\frac{1}{8^k}+\cdots+\frac{1}{8^k}\right)+\cdots. \quad (145)$$

A comparison of (144) and (145) shows that the group of terms (after the first term) in each parenthesis is less in sum for (144)

* If a series of positive terms is grouped, without changing the order of the terms, the property of convergence or divergence, and the value of the sum of the series, in case it converges, remain undisturbed. For, if the nth term of the original series occurs in the mth group and Σ_m represents the sum of the first m groups, then $\Sigma_{m-1} < S_n \leqq \Sigma_m$.

than it is for (145). By part (*a*) of Theorem 2, the hyperharmonic series is therefore, in this case, convergent.

The hyperharmonic series, thus, diverges when $k \leqq 1$ and converges when $k > 1$.

Exercise 2. Prove that if a series converges (or diverges), it will still converge (or diverge) if every term of it is multiplied by the same constant $\neq 0$.

Exercise 3. Prove that if the series of positive terms $v_1 + v_2 + v_3 + \cdots$ is known to converge and $u_n < kv_n$, for every n, where k is a finite constant, then the series of positive terms $u_1 + u_2 + u_3 + \cdots$ converges. HINT: Use Exercise 2 and Theorem 2(*a*).

Exercise 4. Prove that if the series of positive terms $v_1 + v_2 + v_3 + \cdots$ is known to diverge and $u_n > kv_n$, for every n, where k is a finite positive constant, then the series of positive terms $u_1 + u_2 + u_3 + \cdots$ diverges.

Exercise 5. (*a*) Prove that if the series of positive terms

$$v_1 + v_2 + v_3 + \cdots$$

is known to converge and $\lim\limits_{n \to \infty} (u_n/v_n) = k$, a finite constant, the series of positive terms $u_1 + u_2 + u_3 + \cdots$ converges. HINT: For any positive constant ϵ, there exists a value of n, say m, such that for every $n > m$ we have $u_n/v_n < k + \epsilon = l$, hence $u_n < lv_n$ $(n = m + 1, m + 2, m + 3, \cdots)$.

(*b*) Prove that if the series of positive terms $v_1 + v_2 + v_3 + \cdots$ is known to diverge and $\lim\limits_{n \to \infty} (u_n/v_n) = k$, a finite constant not zero, the series of positive terms $u_1 + u_2 + u_3 + \cdots$ diverges.

Problems

1. Write the first four terms of the series whose nth term is

(*a*) $\dfrac{2n}{n+1}$. (*b*) $3n - 2$ (*c*) $\dfrac{\sqrt{n}}{e^n}$. (*d*) $\dfrac{(-1)^n(n-2)}{n^2+1}$.

2. Write the nth term of each of the following series.

(*a*) $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \frac{7}{8} + \cdots$.

(*b*) $\sqrt{2} + \dfrac{\sqrt{3}}{4} + \dfrac{2}{9} + \dfrac{\sqrt{5}}{16} + \cdots$.

(*c*) $2 - \dfrac{4}{\sqrt{2}} + \dfrac{8}{\sqrt{3}} - \dfrac{16}{2} + \cdots$.

(*d*) $\dfrac{1}{\sqrt{3}} - \dfrac{2}{2} + \dfrac{6}{\sqrt{5}} - \dfrac{24}{\sqrt{6}} + \dfrac{120}{\sqrt{7}} - \cdots$.

3. Show that every infinite arithmetical series diverges, except when $a = d = 0$

4. Prove that the following series diverge:

(a) $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} + \cdots .$

(b) $\frac{1}{2} - \frac{4}{5} + \frac{9}{10} - \frac{16}{17} + \cdots .$

(c) $\frac{4}{5} - \frac{5}{7} + \frac{6}{9} - \frac{7}{11} + \cdots .$

(d) $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots .$

(e) $\frac{|\sec A|}{2} + \frac{|\sec 2A|}{3} + \frac{|\sec 3A|}{4} + \frac{|\sec 4A|}{5} + \cdots .$

(f) $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$

(g) $(\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + (\sqrt{5} - \sqrt{4}) +$

(h) $\frac{1 \cdot 2}{3 \cdot 4} + \frac{2 \cdot 3}{4 \cdot 5} + \frac{3 \cdot 4}{5 \cdot 6} + \frac{4 \cdot 5}{6 \cdot 7} + \cdots .$

5. Prove that the following series converge:

(a) $\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots .$

(b) $\frac{\sin B}{2} + \frac{\sin (B + 30°)}{4} + \frac{\sin (B + 60°)}{8} + \frac{\sin (B + 90°)}{16} +$

(c) $\frac{}{2\sqrt{2} + 1} + \frac{}{3\sqrt{3} + 1} + \frac{}{4\sqrt{4} + 1} + \frac{1}{5\sqrt{5} + 1} +$

(d) $\frac{1}{6} + \frac{1}{11} + \frac{1}{18} + \frac{1}{27} + \cdots .$

(e) $\frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \frac{1}{4 \cdot 7} + \cdots .$

(f) $\frac{3 - 1}{2} + \frac{3 - \frac{1}{2}}{2^2} + \frac{3 - \frac{1}{3}}{2^3} + \frac{3 - \frac{1}{4}}{2^4} + \cdots .$

6. Test the following series for convergence:

(a) $\frac{3}{5} + \frac{5}{8} + \frac{7}{11} + \frac{9}{14} + \cdots .$ *Ans.* Div.

(b) $\frac{2^{-1}}{1^3} + \frac{2^{-2}}{2^3} + \frac{2^{-3}}{3^3} + \frac{2^{-4}}{4^3} + \cdots .$ *Ans.* Conv.

(c) $\frac{2}{1 \cdot 1} - \frac{5}{2 \cdot 3} + \frac{10}{3 \cdot 5} - \frac{17}{4 \cdot 7} + \cdots .$

(d) $\frac{2}{9} + \frac{2}{12} + \frac{2}{15} + \frac{2}{18} + \cdots .$

(e) $\frac{1}{1^1} + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots .$

(f) $-\frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot} - \frac{1}{5 \cdot 6} - \cdots .$

(g) $\frac{\sqrt[3]{2}}{1} + \frac{\sqrt[3]{3}}{2} + \frac{\sqrt[3]{4}}{3} + \frac{\sqrt[3]{5}}{4} + \cdots .$

(h) $(\sqrt[3]{2} - \sqrt[3]{1}) + (\sqrt[3]{3} - \sqrt[3]{2}) + (\sqrt[3]{4} - \sqrt[3]{3}) + (\sqrt[3]{5} - \sqrt[3]{4}) + \cdots .$

110. Further Tests for Convergence. The comparison test, embodied in Theorem 2 and in Exercises 3, 4, and 5 of the foregoing section, has the weakness that it is not always easy or even possible, to make use of an appropriate series whose behavior is known. Further tests relating to the series itself, which is to be tested, are contained in the following theorems.

Theorem 3 (D'Alembert's Ratio Test). *If, in the series of positive terms* $u_1 + u_2 + u_3 + \cdots$, $\lim\limits_{n\to\infty} (u_{n+1}/u_n)$ *exists and equals* r, *then if* $r < 1$ *the series converges, while if* $r > 1$ *the series diverges.* (If $r = 1$, no conclusion can be drawn.)

Proof. If $r < 1$, let us set up a positive number ϵ such that $r + \epsilon = S$ is still less than 1. Corresponding to ϵ, we can find a value of n, say m, large enough so that for every $n > m$ the fraction u_{n+1}/u_n will take on values in the interval $r \pm \epsilon$ and hence be less than S. Then

$$\begin{aligned} u_{m+2} &< Su_{m+1}, \\ u_{m+3} &< Su_{m+2} < S^2u_{m+1}, \\ u_{m+4} &< Su_{m+3} < S^3u_{m+1}, \qquad \cdots . \end{aligned}$$

Applying Exercise 3 of the preceding section to the series $u_{m+2} + u_{m+3} + u_{m+4} + \cdots$ and $S + S^2 + S^3 + \cdots$ (the latter being the geometric series and known to converge, since $S < 1$), the conclusion is established. If $r > 1$, it is easy to show that u_n fails to approach zero, and hence, by Theorem 1 of the preceding section, the series diverges.

The example of the hyperharmonic series, in which $r = 1$, independently of the value of k, shows that a series may converge or diverge when its test ratio r has the value 1.

To illustrate the ratio test consider the series

$$\frac{1}{2^1} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} + \cdots$$

Here

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{n+1}{2^{n+1}}\cdot\frac{2^n}{n}\right) = \lim_{n\to\infty}\left(\frac{n+1}{2n}\right) = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2}\right) = \frac{1}{2}$$

a number less than 1, and hence the series converges.

Again, for the series

$$\frac{2^1}{1^2} + \frac{2^2}{2^2} + \frac{2^3}{3^2} + \cdots + \frac{2^n}{n^2} + \frac{2^{n+1}}{(n+1)^2} + \cdots$$

$$\lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left(\frac{2^{n+1}}{(n+1)^2}\cdot\frac{n^2}{2^n}\right) = \lim_{n\to\infty}\left(\frac{2n^2}{(n+1)^2}\right) = \lim_{n\to\infty}\left(\frac{2}{(1+1/n)^2}\right) = 2,$$

a number greater than 1, and the series diverges.

Theorem 4. (Cauchy's Integral Test.) *Given a function $f(x)$, positive for all $x > a$, constantly decreasing as x increases, then if $\int_a^\infty f(x)dx$ exists, the series*

$$f(a) + f(a+1) + f(a+2) + \cdots$$

converges, otherwise the series diverges.

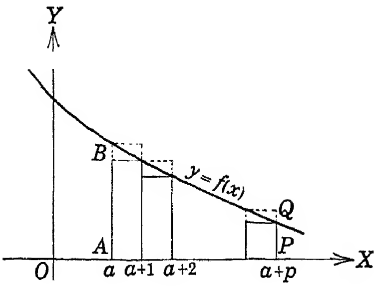

FIG. 197.

Proof. Consider the set of rectangles each of base length 1, and of altitudes

$$f(a+1), f(a+2), \cdots, f(a+p).$$

From the hypothesis on $f(x)$, the sum of the areas of these rectangles is less than the area $ABQP$ under the curve (see Fig. 197), *i.e.*,

$$f(a+1) + f(a+2) + \cdots + f(a+p) < \int_a^{a+p} f(x)dx.$$

Hence

$$\lim_{p\to\infty} [f(a+1) + f(a+2) + \cdots + f(a+p)] \leqq \lim_{p\to\infty} \int_a^{a+p} f(x)dx = \int_a^\infty f(x)dx,$$

and if the latter exists, the conclusion about the convergence of the series is evident.

On the other hand, it is seen, by considering the rectangles with altitudes $f(a), f(a+1), f(a+2), \cdots$ that

$$f(a) + f(a+1) + \cdots + f(a+p-1) > \int_a^{a+p} f(x)dx$$

whence

$$\lim [f(a) + f(a+1) + \cdots + f(a+p-1)] \geqq \lim_{p\to\infty} \int_a^{a+p} f(x)dx$$

If the latter limit does not exist, the integral $\int_a^{a+p} f(x)dx$ becomes infinite, as p becomes infinite (since it increases as p increases and would approach a limit if it remained finite). The series in question, evidently, diverges.

To illustrate Cauchy's integral test, examine, once more, the hyperharmonic series

$$\frac{1}{1^k} + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} + \cdots .$$

The function $f(x) = 1/x^k$ satisfies the hypotheses of Theorem 4 (with $a = 1$), while our series is of the form

$$f(1) + f(2) + f(3) + \cdots .$$

Consider, then, $\int_1^\infty \frac{dx}{x^k}$, *i.e.*, $\lim_{p\to\infty} \int_1^p \frac{dx}{x^k}$.

(1) If $k = 1$, this gives $\lim_{p\to\infty} \Big(\log x\Big)_1^p = \lim_{p\to\infty} (\log p) = \infty$.

(2) If $k \neq 1$, we have $\lim_{p\to\infty} \left[\frac{x^{1-k}}{1-k}\right]_1^p = \lim_{p\to\infty} \left(\frac{p^{1-k} - 1}{1-k}\right)$. This last limit is ∞ when $k < 1$ and equals $\frac{-1}{1-k}$ if $k > 1$. By Theorem 4, then, the series converges for $k > 1$ and diverges for $k \leqq 1$, as has been previously shown.

Exercise 1. Prove, for the series of positive terms $u_1 + u_2 + u_3 + \cdots$, that if $\lim_{n\to\infty} \sqrt[n]{u_n}$ exists and equals l, the series converges if $l < 1$ and diverges if $l > 1$. HINT: If $l < 1$, assume an ϵ such that $l + \epsilon = S < 1$, and for values of n from a certain number m on, $u_{m+1} < S^{m+1}$, $u_{m+2} < S^{m+2}$, etc.

As to series in which positive and negative terms occur, the following should be noted:

Theorem 5. *A series which has both positive and negative terms converges if the series of the absolute values of the terms converges* (or, what is the same, if the series converges when all its terms are made positive. The series is said, in such a case, to converge *absolutely*.)

Proof. Let the first n terms of the series contain p positive terms of sum P_p and q negative terms of sum $-N_q$. Then $S_n = P_p - N_q$, while for the corresponding positive series, $S'_n = P_p + N_q$. By hypothesis, the latter series converges, to some sum S. Now P_p increases as p increases but remains less than S, and likewise N_q increases as q increases but remains less than S. Hence, P_p and N_q both approach limits, say P and N (see Theorem 1, page 151). S_n, then, approaches a limit, and

$$\lim_{n\to\infty} S_n = \lim_{p\to\infty} P_p - \lim_{q\to\infty} N_q = P - N.$$

NOTE: If the number of either positive or negative terms in the series is finite, Theorem 5 is evident at once. Explain why.

A series which has positive and negative terms may, however, converge while the corresponding positive series does not. The series, in such a case, is said to converge *conditionally*. Thus, the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \cdots,$$

we shall presently show, converges, while the corresponding positive series is the now familiar harmonic series, which diverges.

Exercise 2. Prove that a series containing positive and negative terms converges only if the two series, one of the positive terms alone and the other of the negative terms alone, both converge or both diverge, and that in the first case the series converges absolutely, while in the second case it converges conditionally, if it converges at all.

Theorem 6. An alternating *series* (*i.e.*, a series in which the terms are alternately positive and negative) *converges if each term is numerically less than the preceding one and the limit of the* nth *term is zero as* $n \to \infty$.

Proof. Let the series be $u_1 + u_2 + u_3 + u_4 + \cdots$, where the odd-numbered terms are positive. We write S_n, where n is odd, as

$$u_1 + (u_2 + u_3) + (u_4 + u_5) + \cdots + (u_{n-1} + u_n)$$

and as

$$(u_1 + u_2) + (u_3 + u_4) + \cdots + (u_{n-2} + u_{n-1}) + u_n.$$

Since the number in each parenthesis is, by hypothesis, negative in the first case and positive in the second case, the first expression shows that S_n decreases as n increases through odd values, while the second expression shows that S_n, with $n > 2$, is greater than $u_1 + u_2$. Hence, S_n approaches a limit S as n increases through odd values. Also, $S_{n+1} = S_n + u_{n+1}$ and since $u_{n+1} \to 0$, S_{n+1} approaches the same limit as S_n, *i.e.*, S_n approaches the same limit as n becomes infinite through even or odd values. The series, thus, converges.

Note that

$$S - S_n = u_{n+1} + (u_{n+2} + u_{n+3}) + (u_{n+4} + u_{n+5}) + \cdots,$$
$$S_n - S = -u_{n+1} - (u_{n+2} + u_{n+3}) - (u_{n+4} + u_{n+5}) - \cdots,$$

If u_{n+1} is positive the first form shows that $S - S_n < u_{n+1}$, and if u_{n+1} is negative the second form shows that $S_n - S < -u_{n+1}$. Since $S - S_n$ and $S_n - S$ are positive in the respective cases,

we conclude that in every instance $|S - S_n| < |u_{n+1}|$. For example, in the case of the alternating series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$

the sum of the first nine terms differs from the sum of the series by less than the absolute value of the tenth term, *i.e.*, by less than $\frac{1}{10}$.

Exercise 3. Prove that Theorem 3 can be stated as follows. If in the series $u_1 + u_2 + u_3 + \cdots \lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ exists and equals r, then if $|r| < 1$ the series converges and if $|r| > 1$ the series diverges. (If $|r| = 1$, no conclusion can be drawn).

Problems

1. Test each of the following series for convergence.

(a) $1 - \dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{3}} - \dfrac{1}{\sqrt{4}} + \dfrac{1}{\sqrt{5}} - \cdots$. *Ans.* Conv.

(b) $\dfrac{1^3}{1!} + \dfrac{2^3}{2!} + \dfrac{3^3}{3!} + \dfrac{4^3}{4!} + \cdots$. *Ans.* Conv.

(c) $\dfrac{3}{4} + \dfrac{3\cdot 6}{4\cdot 6} + \dfrac{3\cdot 6\cdot 9}{4\cdot 6\cdot 8} + \dfrac{3\cdot 6\cdot 9\cdot 12}{4\cdot 6\cdot 8\cdot 10} + \cdots$. *Ans.* Div.

(d) $(\frac{2}{1})^1 + (\frac{3}{4})^2 + (\frac{4}{9})^3 + (\frac{5}{16})^4 + \cdots$. *Ans.* Conv.

(e) $\dfrac{3}{1} - \dfrac{3^2}{2} + \dfrac{3^3}{3} - \dfrac{3^4}{4} + \cdots$. *Ans.* Div.

(f) $\dfrac{1!}{3} - \dfrac{2!}{3^2} + \dfrac{3!}{3^3} - \dfrac{4!}{3^4} + \cdots$. *Ans.* Div.

(g) $\log(\frac{3}{1}) - \log(\frac{4}{2}) + \log(\frac{5}{3}) - \log(\frac{6}{4}) + \cdots$. *Ans.* Conv.

(h) $\dfrac{1}{1\cdot 3^1} + \dfrac{1}{2\cdot 3^2} + \dfrac{1}{3\cdot 3^3} + \dfrac{1}{4\cdot 3^4} + \cdots$.

(i) $\dfrac{1}{1+\sqrt{1}} + \dfrac{1}{2+\sqrt{2}} + \dfrac{1}{3+\sqrt{3}} + \dfrac{1}{4+\sqrt{4}} + \cdots$.

(j) $\dfrac{2}{\pi^2} + \dfrac{3}{\pi^3} + \dfrac{4}{\pi^4} + \dfrac{5}{\pi^5} + \cdots$

(k) $\dfrac{1}{1\cdot 3} + \dfrac{1}{3\cdot 5} + \dfrac{1}{5\cdot 7} + \dfrac{1}{7\cdot 9} + \cdots$.

(l) $\dfrac{1}{2\sqrt{2^2-1}} + \dfrac{1}{3\sqrt{3^2-1}} + \dfrac{1}{4\sqrt{4^2-1}} + \cdots$.

(m) $\dfrac{1}{1^2+2} + \dfrac{2}{2^2+2} + \dfrac{3}{3^2+2} + \cdots$.

(n) $\dfrac{1}{(\log 2)^2} + \dfrac{1}{(\log 3)^3} + \dfrac{1}{(\log 4)^4} + \cdots$.

(o) $\dfrac{3}{1!\cdot 2} + \dfrac{3^2}{2!\cdot 2^2} + \dfrac{3^3}{3!\cdot 2^3} + \dfrac{3^4}{4!\cdot 2^4} + \cdots$.

(p) $\frac{2}{3} + \sqrt{2}(\frac{2}{3})^2 + \sqrt{3}(\frac{2}{3})^3 + \sqrt{4}(\frac{2}{3})^4 + \cdots$.

(q) $\frac{1}{1^{\log 1}} + \frac{1}{2^{\log 2}} + \frac{1}{3^{\log 3}} + \frac{1}{4^{\log 4}} + \cdots$.

2. Examine the following series for absolute and conditional convergence, with reference to the proposition of Exercise 2.

(a) $1 - \frac{3^2}{2!} + \frac{3^4}{4!} - \frac{3^6}{6!} + \frac{3^8}{8!} - \cdots$. *Ans.* Conv. absolutely.

(b) $1 - \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} - \frac{1}{\sqrt[3]{4}} + \frac{1}{\sqrt[3]{5}} - \cdots$.

Ans. Conv. conditionally.

(c) $1 - \frac{1}{2} + \frac{1}{3^3} - \frac{1}{4} + \frac{1}{5^3} - \frac{1}{6} + \frac{1}{7^3} - \frac{1}{8} + \frac{1}{9^3} - \cdots$.

Ans. Div.

(d) $1 - \frac{1}{2^1} + \frac{1}{3^2} - \frac{1}{4^3} + \frac{1}{5^4} - \frac{1}{6^5} + \cdots$.

(e) $\frac{1}{2(\log 2)^2} - \frac{1}{3(\log 3)^2} + \frac{1}{4(\log 4)^2} - \frac{1}{5(\log 5)^2} + \cdots$.

(f) $\frac{\sin A}{1^2} - \frac{\sin 2A}{2^2} + \frac{\sin 3A}{3^2} - \frac{\sin 4A}{4^2} + \cdots$.

3. (a) Show that the series $\frac{2}{3!} - \frac{3}{4!} + \frac{4}{5!} - \frac{5}{6!} + \cdots$ converges, and that its sum lies between $\frac{5}{24}$ and $\frac{29}{120}$. How many terms of the series, beginning with the first, must be summed to represent the sum of the series correctly in the second decimal place?

(b) Show that the series $1 - \frac{2}{10} + \frac{4}{10^2} - \frac{6}{10^3} + \frac{8}{10^4} - \cdots$ converges, and that its sum lies between 0.84 and 0.834. Obtain its sum correctly in the fifth decimal place.

111. Power Series. Consider the series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

where a_i ($i = 0,1,2, \cdots$) are real constants, positive, negative or zero and x is a variable. Such a series is called a *power series* in x. Its behavior, as to convergence or nonconvergence, will evidently depend on the value assigned to the variable x in it, though it is entirely conceivable—and in fact we shall meet such cases—that a given power series may converge for all real values of x, or diverge for all real values excepting $x = 0$. Evidently, whatever the coefficients, when $x = 0$, $S_n = a_0$ and hence $\lim_{n\to\infty} S_n = a_0$. In other words, every power series converges for $x = 0$.

An important property of power series is embodied in

Theorem 1. *If the power series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ converges for $x = b$, then it converges absolutely for $x = c$, where $|c| < |b|$.*

Proof. By hypothesis, the series $a_0 + a_1b + a_2b^2 + a_3b^3 + \cdots$ converges. Consequently, every term in it is finite, *i.e.*, $|a_ib^i| < N$ $(i = 1,2,3, \cdots)$, where N is some finite positive number. We now write the series

$$|a_0| + |a_1c| + |a_2c^2| + |a_3c^3| + \cdots$$

as

$$|a_0| + |a_1b| \cdot \left|\frac{c}{b}\right| + |a_2b^2| \cdot \left|\frac{c}{b}\right|^2 + |a_3b^3| \cdot \left|\frac{c}{b}\right|^3 + \cdots .$$

The terms in this series, after the first, are all less than the corresponding terms of the series

$$|a_0| + N \cdot \left|\frac{c}{b}\right| + N \cdot \left|\frac{c}{b}\right|^2 + N \cdot \left|\frac{c}{b}\right| + \cdots .$$

Beginning with the second term, the latter is a geometric series in which $r = |c/b|$ and is, by hypothesis, less than 1. It, thus, converges, and hence, with it, the series

$$|a_0| + |a_1c| + |a_2c^2| + |a_3c^3| + \qquad\qquad Q.E.D.$$

An obvious corollary to Theorem 1 is that if a power series diverges for $x = d$ it must diverge for $x = e$, if $|e| > |d|$. For, if it converged for $x = e$ it would converge, absolutely, for $x = d$.

It follows that the range of values of x for which a given power series converges constitutes an interval from $x = -l$ to $x = l$, the value of l, depending, of course, on the coefficients $a_0, a_1, a_2, \ldots$ of the series; and that, furthermore, for every value of x, such that $|x| < l$, the series converges absolutely. Theorem 1 and its corollary give no clue, however, to whether the series converges for $x = l$ or $x = -l$. That must be ascertained by testing the series itself with l and $-l$ substituted for x.

To determine the interval of convergence for a given power series, we may resort, and usually do, to the ratio test, in which we inquire into the $\lim\limits_{n\to\infty} \dfrac{a_{n+1}x^{n+1}}{a_nx^n}$. The student will establish

Theorem 2. *If* $\lim \dfrac{a_{n+1}}{a_n}$ *exists and equals* r, *the series converges absolutely for* $< \left|\dfrac{1}{r}\right|$ *and diverges for* $|x| > \left|\dfrac{1}{r}\right|$ $\left(\text{No conclusion for } |x| = \left|\dfrac{1}{r}\right|.\right)$

The student may find it desirable, sometimes, to employ Cauchy's radical test, embodied in Exercise 1, Sec. 110. Its application to power series is contained in

Exercise 1. Prove that if $\lim_{n\to\infty} \sqrt[n]{|a_n|}$ exists and equals r, the series converges absolutely for $|x| < \frac{1}{r}$ and diverges for $|x| > \frac{1}{r}$. $\left(\text{No conclusion for } |x| = \frac{1}{r}.\right)$

As an illustration, consider the power series

$$\frac{2x}{1\cdot 3} - \frac{4x^2}{2\cdot 9} + \frac{8x^3}{3\cdot 27} - \cdots \pm \frac{2^n x^n}{n\cdot 3^n} \mp \frac{2^{n+1}x^{n+1}}{(n+1)\cdot 3^{n+1}} \pm \cdots.$$

Here

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = -\lim_{n\to\infty}\left(\frac{2^{n+1}\cdot n\cdot 3^n}{(n+1)\cdot 3^{n+1}\cdot 2^n}\right) = -\frac{2}{3}\lim_{n\to\infty}\left(\frac{n}{n+1}\right) = -\frac{2}{3}.$$

By Theorem 2, the series converges absolutely for $|x| < 3/2$ and diverges for $|x| > 3/2$. As to the cases when $x = 3/2$ and $x = -3/2$, we note that these values of x convert the given power series into $1/1 - 1/2 + 1/3 - 1/4 + \cdots$ and

$$-1/1 - 1/2 - 1/3 - 1/4 - \cdots,$$

respectively. The first is readily seen to converge, and the other diverges. We may now display the interval of convergence, if we wish to, as all values of x satisfying $-3/2 < x \leqq 3/2$.

Problems

Find the interval of convergence for each of the following:

1. $\dfrac{x}{2\cdot 5} + \dfrac{x^2}{3\cdot 5^2} + \dfrac{x^3}{4\cdot 5^3} + \cdots.$ *Ans.* $-5 \leqq x < 5.$

2. $\dfrac{x^2}{2^2+1} + \dfrac{x^3}{3^2+1} + \dfrac{x^4}{4^2+1} + \cdots.$ *Ans.* $-1 \leqq x \leqq 1.$

3. $\dfrac{x}{1\cdot 3} + \dfrac{x^3}{3\cdot 3^3} + \dfrac{x^5}{5\cdot 3^5} + \cdots.$ *Ans.* $-3 < x < 3.$

4. $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots.$ *Ans.* All real values of x.

5. $1 + (1!)x + (2!)x^2 + (3!)x^3 + \cdots.$ *Ans.* $x = 0.$

6. $1 - \dfrac{2x^2}{3} + \dfrac{4x^4}{5} - \dfrac{8x^6}{7} + \dfrac{16x^8}{9} \cdots.$

Ans. $-1/\sqrt{2} \leqq x \leqq 1/\sqrt{2}.$

7. $\dfrac{1}{1\cdot 2} - \dfrac{x}{2\cdot 4} + \dfrac{x^2}{4\cdot 6} - \dfrac{x^3}{6\cdot 8} + \cdots$

8. $x + 2^{10}x^2 + 3^{10}x^3 + 4^{10}x^4 + \cdots$.

9. $1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$.

10. $2x - \dfrac{3x^2}{2!} + \dfrac{4x^3}{3!} - \dfrac{5x^4}{4!} + \cdots$.

11. $1 + \dfrac{x^2}{2} + \dfrac{3x^4}{2 \cdot 4} + \dfrac{3 \cdot 5x^6}{2 \cdot 4 \cdot 6} + \dfrac{3 \cdot 5 \cdot 7x^8}{2 \cdot 4 \cdot 6 \cdot 8} + \cdots$.

12. $\dfrac{3^2x}{4+1} - \dfrac{3^3x^2}{4^2+1} + \dfrac{3^4 \cdot x^3}{4^3+1} \quad \dfrac{3^5x^4}{4^4+1} + \cdots$.

13. $\dfrac{x-1}{1} - \dfrac{(x-1)^2}{2} + \dfrac{(x-1)^3}{3} - \dfrac{(x-1)^4}{4} + \cdots$. NOTE: This is a power series in $u = x - 1$. *Ans.* $0 < x \leqq 2$.

14. $\dfrac{x-2}{1!} - \dfrac{(x-2)^3}{3!} + \dfrac{(x-2)^5}{5!} \quad \dfrac{(x-2)^7}{7!} +$

15. $\dfrac{1(x+2)}{2} + \dfrac{3(x+2)^2}{4} + \dfrac{5(x+2)^3}{6} + \dfrac{7(x+2)^4}{8} + \cdots$.

16. $1 + \dfrac{1}{x} + \dfrac{4}{x^2} + \dfrac{9}{x^3} + \dfrac{16}{x^4} + \cdots$. NOTE: This is a power series in $u = 1/x$. *Ans.* $x < -1$ and $x > 1$.

17. $1 - \dfrac{1}{3x} + \dfrac{2^2}{3^2 \cdot x^2} - \dfrac{3^2}{3^3x^3} + \dfrac{4^2}{3^4x^4} - \cdots$.

112. Maclaurin's Series. For every value of x, within its interval of convergence, the power series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

determines a corresponding number, *viz.*, the value of the sum of the series for that value of x. We speak, then, of the power series as representing a function of x, say $P(x)$, [where by $P(b)$ is meant the sum of the series $a_0 + a_1b + a_2b^2 + \cdots$ if $x = b$ is within the interval of convergence] and represent this relation by writing

$$P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots.$$

We state, without proof, the following important properties of the function $P(x)$:

I. *It is a continuous function of x.*

II. *The series obtained by differentiating the successive terms of the given series with respect to x, converges in the same interval as the latter,* and the function represented by it is the derivative of $P(x)$, i.e.,*

$$P'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots.$$

* With the possible exception of the end points.

III. *The series obtained by integrating the successive terms of the given series converges in the same interval as the latter,* and the function represented by it is the integral of* $P(x)$, *i.e.*,

$$\int P(x)dx = c + a_0x + \frac{a_1x^2}{2} + \frac{a_2x^3}{3} + \cdots,$$

where c is an arbitrary constant.

The problem that we shall confront in connection with power series is this. Given a function of x, say, $f(x)$, to set up a power series such that the function represented by it in its interval of convergence shall be precisely the given function.

We note, in the first place, that if the problem has a solution at all, the solution is unique, *i.e.*, if there should exist two series $a_0 + a_1x + a_2x^2 + \cdots$ and $b_0 + b_1x + b_2x^2 + \cdots$ such that each equals $f(x)$ then $a_i = b_i$ $(i = 0,1,2, \cdots)$. For, by II, we have, in that case,

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \cdots,$$
$$f'(x) = b_1 + 2b_2x + 3b_3x^2 + 4b_4x^3 + \cdots.$$

Again, by II

$$f''(x) = 2a_2 + 6a_3x + 12a_4x^2 + \cdots,$$
$$f''(x) = 2b_2 + 6b_3x + 12b_4x^2 + \cdots,$$

and again

$$f'''(x) = 6a_3 + 24a_4x + \cdots,$$
$$f'''(x) = 6b_3 + 24b_4x + \cdots.$$

Now $x = 0$ is certainly within the interval of convergence of each series and of the successive derived series. The sum of any power series for $x = 0$ is its constant term. Thus putting $x = 0$ in the above equalities we obtain

$$a_0 = f(0) \qquad a_1 = f'(0) \qquad a_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$$
$$a_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}$$
$$b_0 = f(0) \qquad b_1 = f'(0) \qquad b_2 = \frac{f''(0)}{2} = \frac{f''(0)}{2!}$$
$$b_3 = \frac{f'''(0)}{6} = \frac{f'''(0)}{3!}.$$

* With the possible exception of the end points.

Continuing our differentiation, and substituting $x = 0$ in the resulting equalities, we shall find

$$a_n = \frac{f^{(n)}(0)}{n!}, \qquad b_n = \frac{f^{(n)}(0)}{n!} \qquad (n = 1,2,3,4,5, \cdots),$$

and the corresponding coefficients of the two power series are identical, as we set out to prove.

The fact that the power series representing a given function is unique is of importance, inasmuch as it tells us, if we succeed in any manner whatever to obtain a power series to represent it, that it is the only series possible. Thus, by division, we obtain $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$, the interval of convergence for the series being $-1 < x < 1$. This, then, is the power series for the function $\frac{1}{1-x}$. Again, by division, we obtain

$$\frac{}{1 + n} = 1 - n + n^2 - n^3 + \quad \cdot \ (-1 < n < 1).$$ By III

$$\int_0^x \frac{dn}{1+n} = \int_0^x dn - \int_0^x n\, dn + \int_0^x n^2\, dn - \int_0^x n^3\, dn + \cdots$$
$$(-1 < x < 1)$$

Hence, $\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \ (-1 < x \leqq 1)$, and this, then, is the power series for the function $\log(1 + x)$.

Note again that in our argument above we ascertained the values of the coefficients $a_0, a_1, a_2, \ldots, a_n, \ldots$ as $f(0)$, $\frac{f'(0)}{1!}, \frac{f''(0)}{2!}, \ldots, \frac{f^{(n)}(0)}{n!}, \cdots$ and we are prepared to state

Theorem 1. *If $f(x)$ can be represented by a power series in x, then*

$$f(x) = f(0) + \frac{f'(0)}{1!} \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3$$
$$+ \qquad + \frac{f^{(n)}(0)}{n!} x^n + \cdots.$$

The series on the right of the last equality is called a *Maclaurin series* for the function $f(x)$, after the mathematician Maclaurin (1698–1746), and the content of Theorem 1 is usually spoken of as *Maclaurin's formula.*

To illustrate, consider again the function $f(x) = \frac{1}{1-x}$. For it,

$$f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = \frac{2}{(1-x)^3}, \quad f'''(x) = \frac{6}{(1-x)^4}, \quad \cdots, \quad f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

Hence, $f(0) = 1, f'(0) = 1 = 1!, f''(0) = 2 = 2!, f'''(0) = 6 = 3!, \cdots, f^{(n)}(0) = n!$, and by Maclaurin's formula we have

$$\frac{1}{1-x} = 1 + \frac{1}{1!}x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \cdots + \frac{n!}{n!}x^n + \cdots$$
$$= 1 + x + x^2 + x^3 + \cdots + x^n + \cdots, \text{ as above.}$$

Again, for the function $\log(1 + x)$,

$$f(x) = \log(1+x), \quad f'(x) = \frac{1}{1+x}, \quad f''(x) = \frac{-1}{(1+x)^2},$$
$$f'''(x) = \frac{2}{(1+x)^3}, \quad f^{\text{IV}}(x) = \frac{-6}{(1+x)^4}, \quad \cdots .$$

Hence $f(0) = 0, f'(0) = 1, f''(0) = -1 = -1!, f'''(0) = 2 = 2!, f^{\text{IV}}(0) = -6 = -3!, \cdots$, and Maclaurin's formula yields

$$\log(1+x) = 0 + \frac{1}{1!}x - \frac{1!}{2!}x^2 + \frac{2!}{3!}x^3 - \frac{3!}{4!}x^4 + \cdots$$
$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$$

as above.

As still another illustration, consider $f(x) = \sin x$. Here $f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x, f^{\text{IV}}(x) = \sin x, f^{\text{V}}(x) = \cos x, \cdots$, and $f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1, f^{\text{IV}}(0) = 0, f^{\text{V}}(0) = 1, \cdots$ and by Maclaurin's formula,

$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 - \cdots,$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

This series, the student will find, converges for all values of x.

A question of importance now is this. We know, from its mode of derivation, that Maclaurin's series represents the func-

tion $f(x)$ from which it was derived, for $x = 0$; also that the successive derived series represent the successive derivatives of $f(x)$, at $x = 0$. But what assurance have we that the series represents $f(x)$ for *all* values of x in its interval of convergence, *i.e.*, that the sum of the series, say for $x = a$, where a is any value within the interval of convergence, is $f(a)$? Indeed, there is nothing in the argument employed above to give us that assurance, though, as a matter of fact, in the case of the functions that the student will meet in elementary calculus this will indeed be the case, *i.e.*, that the function determined by the Maclaurin series will, for all values of x in its interval of convergence, represent the function from which it was derived.

We may be able, in some cases, to check up on the question raised above in this manner. In Sec. 62 we developed Taylor's formula with the remainder, which stated that

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$
$$+ \frac{f^{(n)}(0)}{n!} \cdot x^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1} \qquad (|\xi| < |x|)$$

provided, of course, that $f(0)$ and all the derivatives involved exist and are finite. Now if we can ascertain that property for every value of x in the interval of convergence, and if we may, furthermore, by taking n large enough, make the last term above (the remainder) as small as we please, we can conclude that the sum of the first n terms in the Maclaurin series indeed approaches $f(x)$ as a limit, *i.e.*, the Maclaurin series actually converges to $f(x)$. For example, in the case of the function $\sin x$, just treated, we know that $|f^{(n)}(x)| = |\cos x|$ or $|\sin x|$, and hence $|f^{(n+1)}(\xi)| = |\cos \xi|$ or $|\sin \xi|$, and the remainder $\dfrac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ has the absolute value $\dfrac{|\cos \xi|}{(n+1)!}$ or $\dfrac{|\sin \xi|}{(n+1)!}$. For any finite value of x, this indeed approaches zero as n becomes infinite, and the series found for $\sin x$ represents that function for every finite value of x.

Suppose, now, we wish to compute $\sin 8°$, by means of the series obtained above for $\sin x$, to five decimal places. Substituting for x the value, in radian measure, of $8°$, *viz.*, 0.139626, we obtain

$$\sin 8^\circ = \sin (0.139626) = 0.139626 - \frac{(0.139626)^3}{3!} + \frac{(0.139626)^5}{5!} - \cdots.$$

This is an alternating series in which the terms diminish in numerical value and approach zero as a limit. By the property of such a series pointed out in Sec. 110, the sum of the series differs from the sum of the first two terms by less than the absolute value of the third term. That term, it is easy to verify, is less than 0.000001 in absolute value. The sum of the first two terms is $0.139626 - 0.000454 = 0.139172$, and the value, to five decimal places, of $\sin 8^\circ$, is, then, 0.13917.

Problems

1. Obtain, by Maclaurin's formula, the following expansions, and verify in each case the interval of convergence indicated.

(a) $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$ (all values of x)

(b) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$ (all values of x)

(c) $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots$ $(-1 < x < 1)$

(d) $\log (1 - x) = -x - \dfrac{x^2}{2} - \dfrac{x^3}{3} - \dfrac{x^4}{4} - \cdots$ $(-1 \leqq x < 1)$

(e) $(1 + x)^m = 1 + mx + \dfrac{m(m-1)}{2!}x^2 + \dfrac{m(m-1)(m-2)}{3!}x^3 + \cdots$ $(-1 < x < 1)$,

when m is not a positive integer.

(f) $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$ (all values of x)

(g) $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$ (all values of x)

2. (a) Obtain the power series of 1(a) by making use of the fact that $D_x(\sin x) = \cos x$ and employing the series for $\sin x$ displayed in the text.

(b) Obtain the series in 1(c), by making use of the series displayed in the text, for $\dfrac{1}{1-x}$.

(c) Obtain the series in 1(d) by making use of the series displayed in the text for $\log (1 + x)$.

3. (a) Obtain, by any method, the expansion

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad (-1 < x < 1),$$

and from it, by integration, the result

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad (-1 \leqq x \leqq 1).$$

(*b*) By the result of 3(*a*) compute π to four decimal places, from the formula $\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$.

4. From the series in 1(*e*), by setting $m = -\frac{1}{2}$ and replacing x by $-x^2$, obtain:

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \cdots \qquad (-1 < x < 1),$$

and, from that, by integration

$$\sin^{-1} x = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \cdots \qquad (-1 < x < 1).$$

5. Obtain the expansion

$$(a+y)^m = a^m + ma^{m-1}y + \frac{m(m-1)}{2!}a^{m-2}y^2 + \frac{m(m-1)(m-2)}{3!}a^{m-3}y^3 + \cdots \qquad (|y| < |a|),$$

(*a*) by using Maclaurin's formula; (*b*) from the series in 1(*e*); by setting $x = y/a$.

6. From the series in 1(*d*), or otherwise, obtain

$$\log(a-y) = \log a - \frac{y}{a} - \frac{y^2}{2a^2} - \frac{y^3}{3a^3} - \cdots \qquad (-a \leqq y < a),\ (a > 0)$$

7. From the series in 1(*b*), or otherwise, obtain (*a*) the expansion for e^{-x}; (*b*) the expansion for e^{-x^2}.

8. Show that the expansion, by Maclaurin's formula, of the polynomial $a + bx + cx^2 + dx^3 + \cdots + kx^m$ reproduces the polynomial.

9. If we designate by S_n the sum of the terms in the Maclaurin expansion of $f(x) = \log(1+x)$ as far as $\frac{f^{(n)}(0)x^n}{n!}$, show that, for values of x within the interval of convergence of the series, $|\log(1+x) - S_n| < \frac{x^{n+1}}{n+1}$ when $x > 0$ and $|\log(1+x) - S_n| < \frac{|x^{n+1}|}{(n+1)(1+x)^{n+1}}$ when $x < 0$.

10. Compute to five decimal places, by employing an appropriate Maclaurin series (*a*) $\sin 6°\,30'$; (*b*) $\cos 4°\,20'$; (*c*) $e^{-0.2}$; (*d*) $(0.8)^{1/5}$; (*e*) $\log(1.2)$; (*f*) $\cosh(0.3)$; (*g*) $\tan^{-1}(\frac{1}{4})$.

11. By using $\pi/6 = \sin^{-1}(\frac{1}{2})$, compute π to four decimal places from the Maclaurin series for $\sin^{-1} x$.

12. By employing a Maclaurin series for the integrand and integrating term by term, compute to four decimal places

(*a*) $\int_0^1 \cos x^2\, dx$;

(*b*) $\int_0^{0.5} e^{-x^2}\, dx$;

(*c*) $\int_{0.3}^1 \frac{\sin x}{x}\, dx$;

(*d*) $\int_0^{0.36} \sqrt{x}\tan^{-1} x\, dx$;

(*e*) $\int_0^{0.2} \log(1+\sqrt{x})\, dx$.

13. Show, for the error committed in taking $\int_0^{1/2}\left(1 - \frac{x^2}{6}\right)dx$ as an approximation to $\int_0^{1/2}\frac{\sin x}{x}dx$, that its absolute value is less than $\frac{1}{2^5 \cdot 120}$. HINT: Set $\frac{\sin x}{x} - 1 + \frac{x^2}{6} = R(x)$. By the mean value theorem for integrals, $\int_0^{1/2} R(x)dx = \frac{1}{2}R(\xi) \qquad (0 < \xi < \frac{1}{2})$.

14. For the catenary, the curve assumed by a hanging chain, and whose equation is $y = a \cdot \cosh \frac{x}{a}$, show that the length from the lowest point to $\left(b,\ a \cosh \frac{b}{a}\right)$ is $a \sinh \frac{b}{a}$. Show that an approximation to that length is $b + \frac{b^3}{6a^2} + \frac{b^5}{120a^4}$. Show that an approximation to the sag in the chain is $\frac{b^2}{2a} + \frac{b^4}{24a^3}$.

113. Taylor's Series. It is evident that in order that a function may be developed into a Maclaurin series, it must be defined, along with all its derivatives, at $x = 0$. Such is not the case, for example, with the function $\log x$, and no Maclaurin expansion, *i.e.*, no expansion into a power series in x, is possible for it. We may, however, in such a case, be able to expand the function into a power series in $x - a$, where a is some constant, *i.e.*, into a series of the form

$$a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots.$$

It is clear that all we have learned about a power series in x may at once be applied to this form of a series, *viz.*, that there exists for it an interval of convergence $(-l < x - a < l)$ (with possible equality signs adjoined to either "less than" sign), that within the interval of convergence the series defines a function of x, that the series obtained by differentiating its successive terms defines, within the same interval of convergence, the derivative of that function, etc.

Again, if a given function, say $f(x)$, may be represented by such a series, *i.e.*,

$$f(x) = a_0 + a_1(x - a) + a_2(x - a)^2 + \cdots + a_n(x - a)^n + \cdots,$$

we may show, precisely as in the former case, by considering the additional equalities

$$
\begin{aligned}
f'(x) &= a_1 + 2a_2(x - a) + 3a_3(x - a)^2 + 4a_4(x - a)^3 + \cdots, \\
f''(x) &= 2a_2 + 6a_3(x - a) + 12a_4(x - a)^2 + \cdots, \\
f'''(x) &= 6a_3 + 24a_4(x - a) + \cdots, \\
f^{\mathrm{IV}}(x) &= 24a_4 + \cdots
\end{aligned}
$$

and setting $x = a$ in all of these (since each series surely converges for $x = a$) that $a_0 = f(a)$, $a_1 = f'(a)$, $a_2 = \dfrac{f''(a)}{2!}$, $a_3 = \dfrac{f'''(a)}{3!}$, $\cdots$, $a_n = \dfrac{f^{(n)}(a)}{n!}$, $\cdots$, *i.e.*, that the series is unique, and that the resulting relation is

$$
f(x) = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \cdots. \tag{146}
$$

This equality is known as *Taylor's formula*, after Brook Taylor (1685–1731), and the series on the right is called a *Taylor series*.

It is evident that a Maclaurin series is a Taylor series with the constant a set equal to zero.

By setting $x = a + y$, we may write the Taylor formula as

$$
f(a + y) = f(a) + \frac{f'(a)}{1!}y + \frac{f''(a)}{2!}y^2 + \frac{f'''(a)}{3!}y^3 + \cdots + \frac{f^n(a)}{n!}y^n + \cdots. \tag{147}
$$

By appealing to Taylor's formula with the remainder (Exercise 5, Section 62), which may be displayed as

$$
f(a + y) = f(a) + \frac{f'(a)}{1!}y + \frac{f''(a)}{2!}y^2 + \cdots + \frac{f^{(n)}(a)}{n!}y^n + \frac{f^{(n+1)}(a + \theta y)}{(n + 1)!}y^{n+1} \qquad (0 < \theta < 1)
$$

we may be able, in some cases, to ascertain that the last term above, the remainder, may be made to approach zero by taking n large enough, for all values of y within the interval of convergence of the Taylor series, and in such cases we have the assurance, precisely as was found for the Maclaurin series, that the series represents the function from which it was developed, for all values of the variable within the interval of convergence. Generally,

again in this case, the functions met with in elementary calculus will be represented by their Taylor expansions for the entire range of the interval of convergence of the latter.

Let us illustrate Taylor's formula by developing the function $\log x$ about $x = 1$, *i.e.*, by expanding it into a power series in $x - 1$. Here $f(x) = \log x$, $f'(x) = 1/x$. $f''(x) = -1/x^2$, $f'''(x) = 2/x^3$, $f^{\text{IV}}(x) = -6/x^4, \cdots$, hence $f(1) = 0$, $f'(1) = 1$, $f''(1) = -1$, $f'''(1) = 2 = 2!$, $f^{\text{IV}}(1) = -6 = -3!, \cdots$, and Taylor's formula yields

$$\log x = 0 + \frac{1}{1!}(x-1) - \frac{1}{2!}(x-2)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \cdots$$
$$= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots.$$

The interval of convergence of the series is found to be $(-1 < x - 1 \leqq 1)$, or $(0 < x \leqq 2)$.

Exercise 1. Given a function $f(x)$, set $f(x + a) = g(x)$, hence

$$f'(x+a) = g'(x),$$

$f''(x + a) = g''(x), \cdots$, and $f(a) = g(0)$, $f'(a) = g'(0)$, $f''(a) = g''(0)$, $\cdots$. *Show that the Maclaurin expansion for $g(x)$ leads to Taylor's* formula in the second form displayed above.

As an illustration of the use of a Taylor series, consider the problem of computing the value of cos 47°. We might be able to use Maclaurin's series, *viz.*,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$$

inasmuch as that series converges for all values of x. However, with the radian measure of 47°, *viz.*, 0.82030 substituted in it, the series would converge quite slowly, *i.e.*, the terms would not diminish rapidly enough, and a greater number of them than is desired would have to be summed before we could get an approximation, say, to five decimal places to the value sought. Let us, instead, set up a Taylor expansion for $\cos x$, with a set equal to $\pi/4$. Now, $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$, $f'''(x) = \sin x, \cdots$, hence $f(a) = \sqrt{2}/2$, $f'(a) = -\sqrt{2}/2$,

$f''(a) = -\sqrt{2}/2$, $f'''(a) = \sqrt{2}/2$, $\cdots$ and, thus, by Taylor's formula,

$$\cos x = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\left(x - \frac{\pi}{4}\right) - \frac{\sqrt{2}}{2 \cdot 2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{\sqrt{2}}{2 \cdot 3!}\left(x - \frac{\pi}{4}\right)^3 + \cdots$$
$$= \frac{\sqrt{2}}{2}\left[1 - \left(x - \frac{\pi}{4}\right) - \frac{1}{2!}\left(x - \frac{\pi}{4}\right)^2 + \frac{1}{3!}\left(x - \frac{\pi}{4}\right)^3 + \cdots\right].$$

We now put $x = 47°$, and hence, $x - a = 2° = 0.034907$ (in radian measure), and obtain

$$\cos 47° = \frac{\sqrt{2}}{2}\left(1 - 0.034907 - \frac{(0.034907)^2}{2} + \frac{(0.034907)^3}{6} + \cdots\right).$$

The first four terms in the parenthesis come to

$$1 - 0.034907 - 0.000609 + 0.000007 = 0.964491,$$

and the product of that by $\sqrt{2}/2$ gives 0.68200 as the value of $\cos 47°$ correct to five decimal places.

Exercise 2. State a necessary condition that $f(x)$ and its successive derivatives must meet, in order that $f(x)$ may be developed into a power series in $x - a$.

Problems

1. (*a*) By using form (147) of Taylor's series, obtain the expansion

$$\log (a + y) = \log a + \frac{y}{a} - \frac{y^2}{2a^2} + \frac{y^3}{3a^3} + \cdots,$$

and from that obtain the expansion of Prob. 6, of the preceding section. State the interval of convergence for the series representing $\log (a + y)$.

(*b*) By using form (147) of Taylor's series, obtain the expansion for $(a + y)^m$ and check with the result of Prob. 5 of the preceding section.

(*c*) By using form (147) of Taylor's series, obtain the expansion for $\cosh (a + x)$, and from that, by differentiation, the expansion for $\sinh (a + x)$. State the interval of convergence for each series.

2. Using Taylor's formula, expand

(*a*) e^x in powers of $x - 2$; (*c*) $\sqrt[3]{x}$ in powers of $x - 1$;

(*b*) $x^2 + 3x - 6$ in powers of $x + 1$; (*d*) $\sin x$ in powers of $x - \frac{\pi}{6}$.

3. Compute the following to five decimal places, by employing an appropriate Taylor series:

(a) $\cos 42° 30'$. (c) $\sin 58° 40'$. (e) $\sqrt{8.6}$.
(b) $\sin 34°$. (d) $\sqrt[3]{8.4}$. (f) $e^{3.1}$.

4. By use of the series for $\log (a + y)$ given in Prob. 1(a), compute, to five decimal places,

(a) $\log_{10} 10.8$. HINT: Note that $\log_{10} m = \log_e m \cdot \log_{10} e = M \cdot \log m$, where $M = \log_{10} e = 0.434294$, to six decimal places. Set $a = 10$, and note that $M \cdot \log_e 10 = \log_{10} 10 = 1$.

(b) $\log_{10} 8.8$ (c) $\log_{10} 105$. (d) $\log_{10} 980$

5. Show that in computing $\int_a^b f(x)dx$, if we replace $f(x)$ in the integrand by the first four terms of its Taylor expansion in powers of $(x - a)$, the result is the prismoidal formula.

6. (a) Show that the expansion of a polynomial in x of any degree into a power series in $x - a$ is a polynomial of the same degree in $(x - a)$.

(b) By (a) show that the prismoidal formula gives the exact value of the integral $\int_a^b f(x)dx$, if $f(x)$ is a polynomial of degree three or less.

114. Operations with Power Series. Let two functions, $f(x)$ and $g(x)$, be represented in power series as

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$
$$g(x) = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots,$$

then

I. *For values of x that belong to both intervals of convergence, the function $h(x) = f(x) + g(x)$ is represented by a power series as*

$$h(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots$$

Proof. If x_0 is such a value, then $\lim_{n\to\infty} S_n(x_0) = f(x_0)$ and $\lim_{n\to\infty} S'_n(x_0) = g(x_0)$, where S_n is the sum of the first n terms of the first series and S'_n, of the second. If we denote by S''_n the sum of the first n terms of the third series, we have

$$S''_n(x_0) = S_n(x_0) + S'_n(x_0)$$

and

$$\lim_{n\to\infty} S''_n(x_0) = \lim_{n\to\infty} S_n(x_0) + \lim_{n\to\infty} S'_n(x_0) = f(x_0) + g(x_0) = h(x_0),$$

which proves the proposition.

We state without proof II and III.

II. *For values of x that belong to both intervals of convergence, the function $\varphi(x) = f(x) \cdot g(x)$ is represented by a power series as*

$$\varphi(x) = a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2 + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \cdots$$

III. *If y and z are defined by the power series*

$$y = b_0 + b_1x + b_2x^2 + b_3x^3 + \cdots \qquad |x| < l_1$$
$$z = a_0 + a_1y + a_2y^2 + a_3y^3 + \cdots \qquad |y| < l_2$$

then, under certain restrictions, one of which is that $|b_0| < l_2$, z can be represented as a power series in x by substituting from the first series into the second and collecting like powers of x.

The three properties given above clearly apply to two series in $x - a$ as well.

Exercise 1. If $f(x)$ and $g(x)$ are represented as in Property I above, a necessary condition that $\psi(x) = \dfrac{f(x)}{g(x)}$ be represented by a power series is that $b_0 \neq 0$. Prove. HINT. Every power series in x is convergent for $x = 0$.

Exercise 2. If $\psi(x)$ of Exercise 1 is represented as

$$\psi(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots,$$

show, in view of the uniqueness of the power series for a given function, that

$$c_0 = \frac{a_0}{b_0}, \qquad c_1 = \frac{a_1 - b_1c_0}{b_0}, \qquad c_2 = \frac{a_2 - b_2c_0 - b_1c_1}{b_0},$$
$$c_3 = \frac{a_3 - b_3c_0 - b_2c_1 - b_1c_2}{b_0}, \ \cdots,$$
$$c_n = \frac{a_n - b_nc_0 - b_{n-1}c_1 - b_{n-2}c_2 - \cdots - b_1c_{n-1}}{b_0}.$$

Exercise 3. Show by I that

$$\log\left(\frac{1 + x}{1 - x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \qquad (-1 < x < 1).$$

Exercise 4. By setting $\dfrac{1 + x}{1 - x} = \dfrac{n + 1}{n}$ and using the result of Exercise 3, show that

$$\log(n + 1) = \log n + 2\left[\frac{1}{2n + 1} + \frac{1}{3(2n + 1)^3} + \frac{1}{5(2n + 1)^5} + \cdots\right] \qquad (n > 0).$$

Problems

1. Using the Maclaurin series for $\sin x$ and $\cos x$, obtain the Maclaurin series for $\tan x = \dfrac{\sin x}{\cos x}$. *Ans.* $x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \cdots$.

NOTE: While the series for $\sin x$ and $\cos x$ converge for all values of x, the series for the quotient converges only for $|x| < \pi/2$. The student is

not asked to prove that. However, the fact that $\tan x$ becomes discontinuous for $|x| = \pi/2$ may serve as a clue.

2. Show, in the light of Exercise 1, or otherwise, that no power series in x is possible for representing $\text{ctn}\, x$.

3. Obtain the Maclaurin series for $\sec x$.

$$\textit{Ans.}\quad 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots, \left(|x| < \frac{\pi}{2}\right).$$

4. Obtain the following expansions, continuing each by one more term, and state the interval of convergence in each.

$$(a)\ e^x \sin x = x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots$$

$$(b)\ e^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} - \cdots$$

$$(c)\ \sqrt{1+x}\log(1+x) = x - \frac{x^3}{24} + \frac{x^4}{24} - \cdots$$

5. Obtain the Maclaurin expansions to the 8th powers of x, inclusive, for
(*a*) $\sin^2 x$. (*b*) $\cos^2 x$. (*c*) $\tan^2 x$.

6. Obtain the Maclaurin expansions for

(*a*) $\dfrac{\sin x}{e^x}$. (*b*) $\dfrac{e^x}{\cos x}$. (*c*) $\dfrac{1}{2 + \log(1 - x)}$.

7. (*a*) Obtain the Maclaurin expansion for $e^{\sin x}$. HINT: Set

$$y = \sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots,$$

and

$$z = e^{\sin x} = e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{6} + \frac{y^4}{24} + \cdots$$

and apply III. $\quad$ *Ans.* $\quad 1 + x + \dfrac{x^2}{2!} - \dfrac{3x^4}{4!} - \dfrac{8x^5}{5!} - \dfrac{3x^6}{6!} + \cdots.$

(*b*) Obtain the Maclaurin expansion for $e^{\cos x}$.

$$\textit{Ans.}\quad e\left(1 - \frac{x^2}{2!} + \frac{4x^4}{4!} - \frac{31x^6}{6!} + \cdots\right).$$

(*c*) Obtain the Maclaurin expansion for $\log(1 + \tan x)$, giving the first four terms.

8. By use of the series of Exercise 4, compute to five decimal places successively, (*a*) $\log 2$, (*b*) $\log 3$, and (*c*) $\log 4$.

9. By use of the series of Exercise 3, compute to five decimal places (*a*) $\log \frac{5}{2}$. (*b*) $\log \frac{19}{21}$.

10. Assuming that the series for e^y has meaning when y is complex,

(*a*) Show that $e^{ix} = \cos x + i \sin x$, $(i^2 = -1)$.

(*b*) Show, from (*a*), or otherwise, that $e^{-ix} = \cos x - i \sin x$.

(*c*) Show, from (*a*) and (*b*), that $\sin(iy) = i \sinh y$, and $\cos(iy) = \cosh y$.

(The student may take these formulas as the definition of the functions *sine* and *cosine* when the argument is imaginary.)

(*d*) Derive, from (*a*), DeMoivre's theorem:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

CHAPTER XV

DIFFERENTIAL EQUATIONS

115. Definitions. A relation between two variables may give rise to an equation that involves, besides the variables themselves, their derivatives (or differentials). Such an equation is called a *differential equation.* For example, if a curve has the property that at each of its points the square of its slope, less twice the ordinate, equals the abscissa, we should express the relation between x and y, the coordinates of any of its points, by the equation

$$\left(\frac{dy}{dx}\right)^2 - 2y = x.$$

This is a differential equation, and is of the first order, in view of the following classification:

A differential equation is said to be of order n if the highest ordered derivative involved in it is of order n. Thus, the equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + y = 0$$

is of the second order, while

$$\frac{d^3y}{dx^3} - \left(\frac{dy}{dx}\right)^2 + x = 0$$

is of the third order.

We also classify differential equations as to *degree* by defining *the degree of an equation* as *its degree in the highest ordered derivative entering it,* after the equation has been made rational and integral in all of its derivatives. For the three differential equations written so far, the highest ordered derivatives enter to degree two, one, and one, respectively. They are, thus, of the second, first, and first degrees. Again, the equation

$$\frac{d^2y}{dx^2} = \sqrt{1 - 2\left(\frac{dy}{dx}\right)^2},$$

when cleared of radicals, is $(d^2y/dx^2)^2 + 2(dy/dx)^2 - 1 = 0$, and is of the second degree (as well as of the second order).

If partial derivatives enter an equation, the equation is called a *partial* differential equation, otherwise it is an *ordinary* differential equation. Thus, all the equations shown so far are ordinary, while the equation $z_x^3 = z_y + x$ is partial. We shall treat only ordinary differential equations in this book.

In the foregoing pages, the student has met repeatedly a particularly simple form of differential equation, of the first order and degree, *viz.*, $dy/dx = f(x)$. He has found a *solution*, in the sense that, by integrating, he has found a function $y = g(x)$, such that by substituting $g(x)$ for y and $g'(x)$ for dy/dx in the differential equation, the latter was satisfied for all values of x for which $f(x)$ and $g(x)$ are both defined. In a similar way, any differential equation

$$f(x,y,y',y'', \cdot \cdot \cdot) = 0 \tag{A}$$

is said to have

$$g(x,y) = 0 \tag{B}$$

as a *solution* if every value of (x,y) satisfying (B), and the corresponding values of $y',y'', \ldots$ determined by (B), constitute a set of values $(x,y,y',y'', \ldots)$ that satisfies (A). How to proceed about finding such solutions, at least for certain types of differential equations that frequently arise in practice, will be treated in the succeeding pages. Let us now illustrate the notion of a solution of a differential equation by the following examples.

Example I. Show that $y = 3x^2 + x$ is a solution of the equation

$$x^2y'' - 2xy' + 2y = 0.$$

Proof. If $y = 3x^2 + x$, $y' = 6x + 1$ and $y'' = 6$. Substituting these, the left-hand member of the differential equation becomes

$$x^2 \cdot 6 - 2x(6x + 1) + 2(3x^2 + x) = 6x^2 - 12x^2 - 2x + 6x^2 + 2x = 0.$$

The differential equation is, thus, satisfied by the relation $y = 3x^2 + x$, and the latter is a solution.

Example II. Show that $xy + \log (y/x) = C$, where C is any constant, is a solution of the equation $y'(x^2y + x) + (xy^2 - y) = 0$.

Proof. Upon differentiating the given relation we obtain

$$x\,dy + y\,dx + \frac{dy}{y} - \frac{dx}{x} = 0,$$

which may be written as

$$\left(x + \frac{1}{y}\right)dy + \left(y - \frac{1}{x}\right)dx = 0$$

or, finally, as

$$\frac{dy}{dx} = \frac{y - xy^2}{x^2y + x}$$

The left-hand member of the differential equation, with this expression for dy/dx substituted, becomes

$$\frac{y - xy^2}{x^2y + x}(x^2y + x) + (xy^2 - y) = y - xy^2 + xy^2 - y = 0$$

The solution is verified.

Problems

1. Name the order and degree of each of the following:

(a) $xy' + y = 2x$.

(b) $4y'' - 3y' + 2y = x^2$.

(c) $y'' - 3y'^2 + 2y = x$.

(d) $(y''')^2 + x^2(y'')^3 - 3x = 0$.

2. In each of the following cases, verify that the given relation is a solution of the given differential equation.

(a) $y = 3x - x^2$, $x^2y'' - 2xy' + 2y = 0$.

(b) $y = e^{3x} - 2e^{2x}$, $y'' - 5y' + 6y = 0$.

(c) $y = x - 1 + Ce^{-x}$ (C any constant), $y' + y = x$.

(d) $y^2 = 2Cx + C^2$ (C any constant), $y \cdot y'^2 + 2xy' = y$.

(e) $y = e^{-x}(C_1 \cos 3x + C_2 \sin 3x)$ (C_1, C_2 any constants), $y'' + 2y' + 10y = 0$.

(f) $x^2 - y^2 = Ce^{-x}$ (C any constant), $x^2 - y^2 + 2x = 2yy'$.

(g) $y = C_1e^x + C_2xe^x + C_3e^{-2x}$ (C_1, C_2, C_3 any constants), $y''' - 3y' + 2y = 0$.

116. Primitives. Let us start with the equation $y = Cx + x^2$, where C is an arbitrary constant. Graphically, this represents an infinity of parabolas, and we wish to find the differential equation satisfied at every point of each. We proceed as follows. Differentiating the given equation, we obtain $y' = C + 2x$. We now eliminate C between this and the original equation, getting, first $C = y' - 2x$ and, hence, $y = (y' - 2x)x + x^2$, or $xy' - x^2 - y = 0$. This is the differential equation sought, and from the mode of its derivation it is clear that $y = Cx + x^2$ is a solution of it. We call the given relation the *primitive* of the differential equation that we found, in accordance with the custom of calling the relation from which a differential equation is derived, the primitive of that equation.

Again, let us start with the equation $y = C_1x^2 + C_2x$, where C_1 and C_2 are each arbitrary constants, as the primitive of a proposed differential equation, and find that equation. Graphically, we are seeking the differential equation satisfied at every

point of each of the doubly infinite system of parabolas represented by the given primitive. To that end, we differentiate twice, obtaining $y' = 2C_1x + C_2$, $y'' = 2C_1$, and eliminate C_1 and C_2 from these and the given primitive. We get $C_2 = y' - 2C_1x = y' - y''x$, and hence, $y = \frac{y''x^2}{2} + (y' - y''x)x$, or $2y = y''x^2 + 2y'x - 2y''x^2$, that is, $x^2y'' - 2xy' + 2y = 0$. Again, the given primitive is, of necessity, a solution of the differential equation obtained.

The significant thing to note is that a primitive containing one arbitrary constant gave rise to a differential equation of the first order, while a primitive with two arbitrary constants led to an equation of the second order. It is easy to see that a primitive containing three arbitrary constants would need to be differentiated three times in order to obtain four equations, counting the given one, from which to eliminate the three constants, and the resulting differential equation would be of order three. In general, we may conclude that *the order of the resulting differential equation is equal to the number of essentially distinct* arbitrary constants in the primitive.*†

We note further that the primitive is, in each case, a solution of the corresponding differential equation. Now, we call a solution of a differential equation the *general solution* if the number of essentially distinct arbitrary constants involved in it equals the order of the equation. By that definition, the primitive of a differential equation is its general solution.

It is clear that the equation obtained, by assigning any particular values to the constants in the primitive, will be a solution

* $f(x,y,C_1,C_2, \ldots, C_n)$ is said to involve n essentially distinct arbitrary constants if it is not possible to replace it by a function $g(x,y,k_1,k_2, \ldots, k_m)$ $(m < n)$ in such a way that for every particular set $(C_1,C_2, \ldots, C_n)$ there exists a particular set $(k_1,k_2, \ldots, k_m)$ so that

$$f(x,y,C_1,C_2, \cdots, C_n) = g(x,y,k_1,k_2, \cdots, k_m),$$

for every value of x and y. Thus, the function $y - C_1e^{x+C_2}$, which appears to involve two arbitrary constants, has essentially only one arbitrary constant, since it may be replaced by the function $y - k_1e^x$, where $k_1 = C_1e^{C_2}$.

† Exceptions to this statement arise in certain cases, due to the nature of the function $f(x,y,c_1,c_2, \cdots, c_n)$, where $f(x,y,c_1,c_2, \cdots, c_n) = 0$ is the primitive. Such exceptions cannot be entered into in an elementary work on differential equations.

of the corresponding differential equation. Such a solution is called a *particular solution*. The curves represented by the various particular solutions of the differential equation are called *integral curves* of that equation. Thus, in the first example above, $y = 3x + x^2$ is a particular solution of the equation

$$xy' - x^2 - y = 0,$$

and in the second example $y = 2x^2 - x$ is a particular solution of the equation $x^2y'' - 2xy' + 2y = 0$. The two parabolas represented by these equations would, in turn, be integral curves of the corresponding differential equations.

Problems

1. Form in each case the differential equation whose general solution (primitive) is given. (The C's are understood to be arbitrary constants.)

(*a*) $y = C_1x + C_2$. *Ans.* $y'' = 0$.

(*b*) $y = Cx^2 + x$. *Ans.* $xy' - 2y + x = 0$.

(*c*) $Cy = x^2 - x$.

(*d*) $y = C_1e^x + C_2x$.

(*e*) $y = C_1x^2 + C_2x + C_3$.

(*f*) $C_1x^2 + C_2x = C_3y^2$.

2. Form the differential equation of all parabolas with vertex at the origin and focus on OX. HINT: The equation of this system is $y^2 = Cx$. *Ans.* $2xy' = y$.

3. Form the differential equation of all circles of radius 3. HINT: The equation of this system is $(x - c_1)^2 + (y - c_2)^2 = 9$.

4. Form the differential equation of all parabolas whose axis is the y-axis.

5. State the equation of the tangent drawn at (2,1) to that integral curve of the equation $y' = x + 2y^2$ which passes through (2,1). *Ans.* $y = 4x - 7$.

6. Find the curvature, at the point (2,3) of that integral curve of the equation $xy' + 2y = x$ which passes through that point.

117. Equations of First Order and Degree. We turn now to the main problem of this chapter, that of devising means for solving differential equations of certain types, and we begin by treating the equation of the first order and the first degree. The general form of such an equation is $M(x,y) + N(x,y)y' = 0$, or, in the more symmetrical form,

$$M(x,y)dx + N(x,y)dy = 0. \tag{148}$$

I. *Variables Separable.* It may happen that (148) is of the form

$$F_1(x) \cdot F_2(y)dx + F_3(x) \cdot F_4(y)dy = 0.$$

We may, then, rewrite the equation as

$$\frac{F_1(x)}{F_3(x)}dx + \frac{F_4(y)}{F_2(y)}dy = 0,$$

or

$$g_1(x)dx + g_2(y)dy = 0,$$

and the solution is obtained directly by integrating as

$$\int g_1(x)dx + \int g_2(y)dy = C,$$

where C is an arbitrary constant, *i.e.*,

$$h_1(x) + h_2(y) = C.$$

We say, in such a case, that the *variables are separable* in the equation.

For example, the equation $y \sin x\, dx + (1 + y) \cos^2 x\, dy = 0$ is of that type. We rewrite it as

$$\frac{\sin x}{\cos^2 x}dx + \frac{1 + y}{y}dy = 0.$$

Hence,

$$\int \frac{\sin x}{\cos^2 x}dx + \int \frac{1 + y}{y}dy = C,$$

and the solution is $\dfrac{1}{\cos x} + \log y + y = C.$

II. *Exact Equation.* If the left-hand member of (148) is an exact differential, *i.e.*, if a function $f(x,y)$ exists such that $M(x,y)dx + N(x,y)dy = df(x,y)$, the equation is equivalent to $df(x,y) = 0$, and the solution is given directly by

$$f(x,y) = C.$$

An equation of this kind is said to be *exact*. In Sec. 102 the student has learned that the test for the form $M\,dx + N\,dy$ being an exact differential is the equality $M_y = N_x$, as well as how to proceed about finding the function $f(x,y)$ when that equality is satisfied.

Example. Given the equation

$$(y^2 + y \cos x - 2x \cos y)dx + (2xy + \sin x + x^2 \sin y)dy = 0,$$

we note that $M_y = 2y + \cos x + 2x \sin y = N_x$, and hence that the equation is exact. For the function $f(x,y)$, we have

$$f(x,y) = \int (y^2 + y \cos x - 2x \cos y)dx = y^2x + y \sin x - x^2 \cos y + g(y),$$

where $g(y)$ is a function of y to be determined. By differentiating,

$$f_y(x,y) = 2xy + \sin x + x^2 \sin y + g'(y) = N(x,y) = 2xy + \sin x + x^2 \sin y,$$

making $g'(y) = 0$ and $g(y) = C$. Except for an arbitrary constant, we thus have

$$f(x,y) = xy^2 + y \sin x - x^2 \cos y,$$

and the solution of the equation is

$$xy^2 + y \sin x - x^2 \cos y = C.$$

It may happen that the equation is not exact, but may be made so by multiplying it by an appropriate factor. Such a multiplier is said to be an *integrating factor*. To illustrate, consider the equation

$$(x^2 + 1 - y)dx + x\,dy = 0.$$

Here $M_y = -1 \neq N_x$, and the equation is not exact. Let us group the terms as $x\,dy - y\,dx + (x^2 + 1)dx = 0$. The fact that $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$ at once suggests $\frac{1}{x^2}$ as an integrating factor. Multiplying through by it, we obtain

$$\frac{x\,dy - y\,dx}{x^2} + \frac{x^2 + 1}{x^2}dx = 0,$$

or

$$d\left(\frac{y}{x}\right) + \frac{x^2 + 1}{x^2}dx = 0.$$

The second term on the left is obviously the differential of a function of x, in fact it is $d\left(x - \frac{1}{x}\right)$. The equation is now of the form

$$d\left(\frac{y}{x} + x - \frac{1}{x}\right) = 0,$$

and the solution is

$$\frac{y}{x} + x - \frac{1}{x} = C,$$

or

$$y + x^2 - 1 = Cx.$$

As a second example, consider the equation

$$(x^2 + y^2 + 2xy)dx - 2x^2\,dy = 0.$$

For it, $M_y = 2y + 2x \neq N_x$. Let us now group the terms as

$$(x^2 + y^2)dx - 2x(x\,dy - y\,dx = 0.$$

We recall that $d \tan^{-1} \dfrac{y}{x} = \dfrac{x\,dy - y\,dx}{x^2\left(1 + \dfrac{y^2}{x^2}\right)} = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$. The factor $\dfrac{1}{x(x^2 + y^2)}$ will thus reduce the second term on the left in the equation to $-d\left(2 \tan^{-1} \dfrac{y}{x}\right)$; it will also reduce the first term to $\dfrac{dx}{x}$, or $d(\log x)$. We thus identify $\dfrac{1}{x(x^2 + y^2)}$ as an appropriate integrating factor, and by means of it the equation becomes

$$\frac{dx}{x} - \frac{2(x\,dy - y\,dx)}{x^2 + y^2} = 0,$$

or

$$d(\log x) - d\left(2 \tan^{-1} \frac{y}{x}\right) = 0,$$

and the solution is

$$\log x - 2 \tan^{-1} \frac{y}{x} = C.$$

The treatment of integrating factors is extensive in the theory of differential equations. We shall confine ourselves merely to cases where an integrating factor may be recognized by inspection.

Exercise 1. Show that $1/x^2$ is an integrating factor of the equation $[f(x) - y]dx + x\,dy = 0$.

Exercise 2. Show that $1/y^2$ is an integrating factor of the equation $y\,dx - [x + f(y)]dy = 0$.

Problems

1. The variables are separable in each of the following equations. Find the general solution in each case.

(a) $(2 - x)y^2\,dx + x\,dy = 0$. *Ans.* $2y \log x - xy - 1 = Cy$.

(b) $e^y\,dx + x(1 + e^y)dy = 0$. *Ans.* $\log x - e^{-y} + y = C$.

(c) $\tan y(1 + \sqrt{1 - x^2})dx + \sqrt{1 - x^2}(\sin y + 1)dy = 0$.

(d) $y' \cdot x(1 + x^2) \sec^2 y = 2x^2 + x + 2$.

Ans. $\tan^{-1} x + \log x^2 = \tan y + C$.

(e) $(xy^2 + x)dx + (x^2 + 1)(y - 1)dy = 0$.

2. Show that each of the following equations is exact, and solve:

(a) $(y^2 - 2)dx + (2xy + 1)dy = 0$. *Ans.* $xy^2 + y - 2x = C$.

(b) $(\sin y + e^x \cos y)dx + (x \cos y - e^x \sin y)dy = 0$.

Ans. $x \sin y + e^x \cos y = C$.

(c) $(2xy^3 - 4xy + 3x^2)dx + (3x^2y^2 - 2x^2 - 1)dy = 0.$
Ans. $x^2y^3 - 2x^2y + x^3 - y = C.$

(d) $\sin^{-1} y \cdot dx + \frac{x - \sqrt{1 - [illegible]}\,\sin y}{[illegible]}dy = 0.$

(e) $(y \sec^2 x + \cos x \cdot \log y)dx + \left(\tan x + \frac{\sin x}{y}\right)dy = 0.$

3. Solve each of the following by employing an integrating factor:

(a) $(x^3 - y)dx + x\,dy = 0.$ *Ans.* $2y + x^3 = Cx.$

(b) $y^2\,dx - (xy - x^2 - y^2)dy = 0.$ *Ans.* $\log y + \tan^{-1}\left(\frac{x}{y}\right) = C.$

(c) $-y\,dx + (x + y - 1)dy = 0.$ *Ans.* $y(\log y + C) = x - 1.$

(d) $(y + x^2y^2e^x)dx + x\,dy = 0.$

(e) $(x^2 + y^2 - x)dx - y\,dy = 0.$

III. *Homogeneous Equations.* Elsewhere in this book (page 348) we have defined a function $f(x,y)$ to be homogeneous and of degree n if it has the property that $f(tx,ty) = t^n \cdot f(x,y)$. Such is the case, for example, with $f(x,y) = x^2 \cos \frac{y}{x} + y^2 - xye^{x/y}$, for which

$$f(tx,ty) = t^2y^2 \cos\left(\frac{ty}{tx}\right) + t^2y^2 - t^2xye^{\frac{tx}{ty}} = t^2 \cdot f(x,y)$$

—a function homogeneous and of degree 2.

We now define the equation $M\,dx + N\,dy = 0$ as *homogeneous* if M and N are homogeneous functions of x and y, of the same degree.

To solve such an equation, we introduce a new variable v, defined as $v = y/x$, $(y = vx)$. With this substitution of vx for y, the equation becomes

$$M(x,vx)dx + N(x,vx) \cdot (v\,dx + x\,dv) = 0,$$

or

$$\frac{v\,dx + x\,dv}{dx} = -\frac{M(x,vx)}{N(x,vx)}.$$

By the hypothesis on the functions M and N, the right-hand member of the above is equivalent to

$$-\frac{x^n \cdot M(1,v)}{x^n \cdot N(1,v)} = -\frac{M(1,v)}{N(1,v)},$$

clearly a function of v alone, say $f(v)$. Now,

$$\frac{v\,dx + x\,dv}{dx} = f(v)$$

reduces to

$$v\,dx + x\,dv = f(v)dx,$$

or

$$dx(v - f(v)) + x\,dv = 0,$$

and the variables are separable, giving

$$\frac{dx}{x} + \frac{dv}{v - f(v)} = 0,$$

and thence the solution.

To illustrate, consider the equation

$$(x^3 + y^3)dx - xy^2\,dy = 0.$$

Here M and N are, evidently, both homogeneous and of degree 3. Thus, the equation is homogeneous. We now put $y = vx$ (hence $dy = v\,dx + x\,dv$), and the equation takes the form

$$(x^3 + v^3x^3)dx - v^2x^3(v\,dx + x\,dv) = 0.$$

From this,

$$(1 + v^3)dx - v^2(v\,dx + x\,dv) = 0,$$
$$(1 + v^3 - v^3)dx - v^2x\,dv = 0,$$

and

$$\frac{dx}{x} - v^2\,dv = 0,$$

with the result,

$$\log x - \frac{v^3}{3} = C.$$

Replacing, now, v by y/x, we have the solution as

$$\log x - \frac{y^3}{3x^3} = C.$$

Exercise 1. For the function $f(v)$, which appears in the discussion above, show that if $f(v) = v$, the solution of the homogeneous equation is $y = Cx$.

Exercise 2. Show that the substitution $x = uy$ will also separate the variables in the homogeneous equation.

Exercise 3. Show that the equation

$$(ax + by + c)dx + (lx + my + n)dy = 0$$

may be made homogeneous in the variables r and s by setting

$$ax + by + c = r, \qquad lx + my + n = s.$$

HINT: The equations

$$ax + by + c = r,$$
$$lx + my + n = s,$$

lead to

$$\begin{aligned} a\,dx + b\,dy &= dr, \\ l\,dx + m\,dy &= ds, \end{aligned}$$

and these give

$$dx = \frac{m\,dr - b\,ds}{am - bl},$$
$$dy = \frac{a\,ds - l\,dr}{am - bl}.$$

These substituted, the given equation reduces to

$$(mr - ls)dr + (as - br)ds = 0,$$

a homogeneous equation.

We illustrate Exercise 3 by treating the equation

$$(2x + y - 1)dx + (x - 3y + 2)dy = 0.$$

Set $\begin{Bmatrix} r = 2x + y - 1 \\ s = x - 3y + 2 \end{Bmatrix}$, hence $\begin{Bmatrix} dr = 2\,dx + dy \\ ds = dx - 3\,dy \end{Bmatrix}$ and, solving for dx and dy, $\begin{Bmatrix} dx = \frac{3}{7}\,dr + \frac{1}{7}\,ds \\ dy = \frac{1}{7}\,dr - \frac{2}{7}\,ds \end{Bmatrix}$. The equation now takes the form $\dfrac{r(3\,dr + ds)}{7} + \dfrac{s(dr - 2\,ds)}{7} = 0$, or

$$(3r + s)dr + (r - 2s)ds = 0.$$

We now employ the method learned for the homogeneous equation and set $s = vr$ $(ds = v\,dr + r\,dv)$, obtaining

$$\begin{aligned} (3r + vr)dr + (r - 2vr)(v\,dr + r\,dv) &= 0, \\ (3 + v)dr + (1 - 2v)(v\,dr + r\,dv) &= 0, \\ (3 + v + v - 2v^2)dr + (1 - 2v)r\,dv &= 0, \\ \frac{dr}{r} + \frac{1 - 2v}{3 + 2v - 2v^2}dv &= 0. \end{aligned}$$

Integrating this, we have

$$\log r + \tfrac{1}{2}\log(3 + 2v - 2v^2) = \log C,$$

or

$$r^2(3 + 2v - 2v^2) = C.$$

Exercise 4. In the last equation above, replace v by s/r, that is, by $\dfrac{x - 3y + 2}{2x + y - 1}$ and r by $2x + y - 1$ and show that the solution reduces to

$$2x^2 + 2xy - 3y^2 - 2x + 4y = C.$$

Exercise 5. Show that the method of Exercise 3 fails if $am - bl = 0$. Note that in that case $a/l = b/m$. If we call the common ratio k and set $lx + my = z$, the equation becomes $(kz + c)dx + (z + n)dy = 0$. Now, $y = \dfrac{z - lx}{m}$ and $dy = \dfrac{dz - l\,dx}{m}$. Substitute this for dy and show that the variables are now separable.

Problems

1. Solve the following homogeneous equations:

(a) $(2x - y)dx + (x + y)dy = 0$.

Ans. $\log (2x^2 + y^2) + \sqrt{2} \tan^{-1} \dfrac{y}{x\sqrt{2}} = C$.

(b) $x\,dy - y\,dx = -\sqrt{x^2 - y^2}\,dx$. *Ans.* $\log x + \sin^{-1} \dfrac{y}{x} = C$.

(c) $y' = \dfrac{y}{x} + \sin \dfrac{y}{x}$. *Ans.* $x = C \tan \dfrac{y}{2x}$.

(d) $y^2\,dx + (xy - x^2)dy = 0$.

(e) $y\,dx - (x + ye^{x/y})dy = 0$,

2. Solve by the method of Exercise 3.

(a) $(x + y - 1)dx + (2x - y + 1)dy = 0$.

(b) $(3x - y)dx + (x - y + 2)dy = 0$.

3. Solve by the method of Exercise 5:

(a) $(2x + 2y - 1)dx + (x + y + 3)dy = 0$.

Ans. $2x + y + 7 \log (x + y - 4) = C$.

(b) $(3x - 3y)dx + (x - y + 1)dy = 0$

IV. *Linear Equations.* An equation of the type

$$y' + y \cdot f(x) = g(x)$$

is called a *linear* equation. (Linearity here implies that the equation is linear in the dependent variable and its derivative.)

To solve this equation we employ the integrating factor $e^{\int f(x)dx}$. Multiplied by it, the equation becomes

$$y' \cdot e^{\int f(x)dx} + y \cdot f(x) \cdot e^{\int f(x)dx} = g(x) \cdot e^{\int f(x)dx}$$

or

$$\frac{d}{dx}(y \cdot e^{\int f(x)dx}) = g(x) \cdot e^{\int f(x)dx},$$

and the equation is promptly solved as

$$y \cdot e^{\int f(x)dx} = \int g(x) \cdot e^{\int f(x)dx}\,dx + C,$$

or

$$y = e^{-\int f(x)dx}[\int g(x) \cdot e^{\int f(x)dx}\,dx + C].$$

Example. To solve $y' + \frac{y}{2x} = x + 1$, we note that $f(x) = 1/2x$ and employ $e^{\int \frac{dx}{2x}}$ as an integrating factor. This reduces to

$$e^{\frac{1}{2}\log x} = (e^{\log x})^{\frac{1}{2}} = \sqrt{x}.$$

Multiplying the equation by $\sqrt{x}$, we obtain

$$y'\sqrt{x} + \frac{y\sqrt{x}}{2x} = \sqrt{x}(x+1),$$

$$\frac{d}{dx}(y\sqrt{x}) = \sqrt{x}(x+1),$$

whence

$$y\sqrt{x} = \int \sqrt{x}(x+1)dx = \tfrac{2}{5}x^2\sqrt{x} + \tfrac{2}{3}x\sqrt{x} + C,$$

or

$$y = \frac{2x^2}{5} + \frac{2x}{3} + \frac{C}{\sqrt{x}}.$$

Exercise 1. Show that the linear equation $y' + y \cdot f(x) = 0$ has for its solution $y = Ce^{-\int f(x)dx}$

Exercise 2. Show that the nonlinear equation

$$y' + y \cdot f(x) = y^n \cdot g(x) \qquad (n \neq 0,\ n \neq 1)$$

may be made linear by introducing a new variable z defined by $z = y^{1-n}$.

HINT: Divide the given equation by y^n, obtaining

$$y' \cdot y^{-n} + y^{1-n} \cdot f(x) = g(x)$$

By $z = y^{1-n}$, $z' = (1-n)y^{-n} \cdot y'$. Show that the equation becomes $z' + (1-n) \cdot z \cdot f(x) = (1-n) \cdot g(x)$, a linear equation.

NOTE: The equation $y' + y \cdot f(x) = y^n \cdot g(x)$, treated in Exercise 2, is known as the *Bernoulli equation*, after James Bernoulli (1654–1705), a member of a celebrated family of mathematicians.

Exercise 3. Discuss the equation of Exercise 2 for the excepted cases, $n = 0$ and $n = 1$.

Exercise 4. Show that if $y = u(x)$ and $y = v(x)$ are any two particular solutions of the equation of Exercise 1, then $y = k_1u(x) + k_2v(x)$, where k_1 and k_2 are any two constants, is also a solution of it.

Problems

1. Solve the following linear equations.

(a) $y' + y(1 - x^2) = 0$. *Ans.* $y = Ce^{\frac{x^3}{3} - x}$.

(b) $y' + y \sin x = xe^{\cos x}$. *Ans.* $y = e^{\cos x}\left(\frac{x^2}{2} + C\right)$.

(c) $dy + \frac{3y\,dx}{2x - 1} = \frac{x\,dx}{\sqrt{2x - 1}}$.

Ans. $y(2x-1)^{\frac{3}{2}} = \frac{2x^3}{3} - \frac{x^2}{2} + C$.

(d) $(y - \sin^{-1} x)dx +$ $0.$

(e) $\dfrac{dx}{dy} + 2x \cdot \csc 2y = \sec^2$

(f) $\dfrac{dx}{dy} + 4xy$ $e^{-2y^2} \cdot \cos y.$

2. Solve the following equations by the method of Exercise 2.

(a) $\dfrac{dy}{dx} + \dfrac{xy}{1 - x^2} = \dfrac{2xy^2}{1 - x^2}.$ *Ans.* $y = \dfrac{}{2\sqrt{1 - x^2} + C}$

(b) $x\,dy + y\,dx = y^3 \cdot x^2 \cdot \log x \cdot dx.$

Ans. $y^2 = \dfrac{1}{x^2[C - (\log x)^2]}.$

V. *Miscellaneous Problems.* Below is appended a list of problems that will provide a review of the methods that the student has learned so far for solving differential equations and illustrate the use of differential equations in applications to geometry and mechanics.

1. Solve each of the following:

(a) $(x^2 + y^2)dx + 3xy\,dy = 0.$ *Ans.* $x^2(x^2 + 4y^2)^3 = C.$

(b) $(e^y + y \sec^2 x - y)dx + (xe^y + \tan x - x)dy = 0.$

Ans. $xe^y + y \tan x - xy = C.$

(c) $(3x - 2)dy + 3y\,dx = \sin x(3x - 2)dx.$

(d) $(x + 2xy)dx + (y - 2xy)dy = 0.$

(e) $y\,dx + (x + xy^2)dy = 0.$ *Ans.* $xy = Ce^{-y^2/2}.$

(f) $(y - x\sqrt{x^2 - 1} \cdot e^{\csc^{-1} x})dx + x\sqrt{x^2 - 1}\,dy = 0.$

Ans. $ye^{\sec^{-1} x} = xe^{\frac{\pi}{2}} + C.$

(g) $(x + 4y - 2)dx + (4x - y)dy = 0.$

(h) $ye^{\cos x}\,dx + \csc x \cdot \log y \cdot dy = 0.$

(i) $\left(y^2 + \dfrac{y}{x}\right)dx + (2xy + \log x - 1)dy = 0.$

(j) $(x - y + x^2 + y^2)dx + (x + y)dy = 0.$

Ans. $\log (x^2 + y^2) + 2 \tan^{-1} \dfrac{y}{x} + 2x = C.$

(k) $y' + \dfrac{2y}{x} = 2xy^2\sqrt{y}.$ *Ans.* $y^{3/2}(Cx^3 + 3x^2) = 1.$

(l) $\left(x - y \sin \dfrac{y}{x}\right)dx + x \sin \dfrac{y}{x}dy = 0.$

Ans. y $x \cos^{-1} (\log x + C).$

(m) $(3x + y - 1)dx + (6x + 2y)dy = 0.$

(n) $e^{\sin y} \cos y\dfrac{dy}{dx} + \dfrac{3x^2}{x^3 - 1} \cdot e^{\sin y} = \log x.$ HINT: Set $z = e^{\sin y}.$

(o) $x\,dy(2x^2y^2 - 1 + y^2) + y\,dx(2x^2y^2 + 1 + x^2) = 0.$ HINT: $\log \left(\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{xy}.$

2. Solve each of the following and determine the equation of the integral curve specified.

(*a*) $y' + y = x/y$. Integral curve to pass through $(0,1)$.

Ans. $y^2 = x - \frac{1}{2} + \frac{3}{2}e^{-2x}$.

(*b*) $(2xy + y \sec^2 xy)dx + (x^2 + x \sec^2 xy)dy = 0$. Integral curve to pass through $(\pi/4,1)$.

(*c*) $\dfrac{dx}{dy} + \dfrac{x}{y} = \sin y$. Integral curve to have slope 1 at the point where it meets the y-axis.

3. Find the equation of the system of curves for which the slope at any point equals the product of the coordinates.

4. If the slope at any point of a curve exceeds the abscissa of the point by 2, show that the curve is a parabola. Find its equation if it is to pass through $(0,1)$.

5. Find the equation of the system of curves for which the x-intercept of any normal is twice the ordinate of the point of contact of the corresponding tangent.

6. Find the equation of the system of curves for which, at any point (r,θ), $\theta = \psi$, where ψ is the angle between the radius vector and the tangent (see Sec. 70). *Ans.* $r = C \sin \theta$.

7. Find the equation of the system of orthogonal trajectories of the parabolas $x^2 = Cy$.

NOTE: *An orthogonal trajectory of a given system of curves is a curve that cuts every curve of the system at right angles.*

HINT: Form the differential equation of the given system of parabolas, which is $y' = 2y/x$. The differential equation of the trajectories is, then, $y' = -x/2y$.

8. Find the equation of the system of orthogonal trajectories

(*a*) Of the circles $x^2 + y^2 = c^2$. *Ans.* $y = Cx$.

(*b*) Of the hypocycloids $x^{2/3} + y^{2/3} = c^{2/3}$. *Ans.* $x^{4/3} - y^{4/3} = C$.

9. The rate at which a body cools is proportional to the difference in temperature between it and the surrounding atmosphere. If in air at 50° the body cools from 70° to 60° in 20 min., what will be its temperature 20 min. later? *Ans.* 55°.

10. A particle falls from rest in a resisting medium, the retardation being $g/100$ times the square of the velocity at any instant, the acceleration of the particle, due to gravity, being the constant g. Show that as time goes on, the velocity of the particle will approach a fixed limit. HINT: The equation of the motion is $dv/dt = g - (g/100)v^2$.

11. A particle falls from rest in a resisting medium, the retardation being proportional to the velocity. Show that the velocity approaches a limit as time goes on.

118. First Order, Not of the First Degree. These, by definition, are equations in which y' enters to degree two or higher.*

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* It is also possible that y' will not enter to any degree. For example, y' enters exponentially in the equation $y = xy' + e^{y'}$, and we cannot speak at all of the degree of this equation.

Several methods of attack may present themselves in connection with such equations.

I. *Equations Solvable for p.* If we represent the quantity y' by the symbol p, the equation is of the form $f(x,y,p) = 0$, and it may be possible to factor the left-hand member into factors of the form $p - g(x,y)$. Each such factor, set equal to zero, will furnish a first-degree equation, whose solution will clearly be a solution of the given equation. Thus, in the case of

$$p^3 - p^2(xy + x - y) + pxy(x - y) = 0$$

we may conveniently display it as

$$p(p - xy)[p - (x - y)] = 0.$$

These factors, set in turn equal to zero, provide three first-degree equations, $p = 0$, $p = xy$, $p = x - y$. The solutions of the first two are obtained immediately as $y = C$ and $y = Ce^{x^2/2}$. The third, when put into the form $(dy/dx) + y = x$, is seen to be linear, and its solution is readily obtained as $y = Ce^{-x} + x - 1$. All the solutions of the given equation are contained, then, in the relations

$$y = C, \qquad y = Ce^{x^2/2}, \qquad y = Ce^{-x} + x - 1.$$

We may leave these equations displayed in this manner or write the one equation

$$(y - C)(y - Ce^{x^2/2})(y - Ce^{-x} - x + 1) = 0,$$

to represent them all.

II. *Equations Solvable for x.* The equation $f(x,y,p) = 0$ may sometimes be conveniently solved for x, yielding the form $x = g(p,y)$. In such a case it is to advantage to differentiate with respect to y, and obtain $\dfrac{1}{p} = g_p(p,y)\dfrac{dp}{dy} + g_y(p,y)\left(\text{since } \dfrac{dx}{dy} = \dfrac{1}{p}\right)$. This differential equation is seen to involve p and y only, and to be of the first order and first degree. If we succeed in obtaining its solution as $\varphi(p,y,C) = 0$, then this and the given equation, taken simultaneously, represent the solution sought as $\left\{\begin{matrix}\varphi(p,y,C) = 0\\ f(x,y,p) = 0\end{matrix}\right\}$, with p as parameter. If desired, and if convenient, the parameter, p, may be eliminated and the solution displayed in the form $F(x,y,C) = 0$.

To illustrate, let us treat the equation

$$p^2 - 2y + x = 0.$$

Solving for x, we have

$$x = 2y - p^2.$$

Differentiating with respect to y, we get

$$\frac{1}{p} = 2 - 2p\frac{dp}{dy}.$$

The variables are separable, so we write it as

$$\frac{2p\,dp}{2 - \dfrac{1}{p}} = dy$$

and this integrates to

$$\frac{p^2}{2} + \frac{p}{2} + \frac{1}{4}\log(2p - 1) = y + C.$$

The solution sought is, then, represented by the equations

$$\left.\begin{cases} 2p^2 + 2p + \log(2p - 1) = 4y + C \\ p^2 - 2y + x = 0 \end{cases}\right\},$$

with p as the parameter.

Exercise 1. Establish the reason for differentiating the equation

$$x = g(p,y)$$

with respect to y, by showing that differentiation with respect to x would lead to an equation involving p, y, and x.

III. *Equations Solvable for y.* Again, it may be convenient to solve the equation $f(x,y,p) = 0$ for y, to obtain $y = h(x,p)$. In this case, differentiation with respect to x will lead to

$$p = h_p(p,x)\frac{dp}{dx} + h_x(p,x),$$

an equation of the first order and degree and involving only p and x. If this can be solved, its solution $\varphi(p,x,C) = 0$, coupled with the given equation, will represent the solution of the latter as $\left.\begin{cases} \varphi(p,x,C) = 0 \\ f(x,y,p) = 0 \end{cases}\right\}$, with p as the parameter.

To illustrate, consider the same equation as above,

$$p^2 - 2y + x = 0.$$

Solving it for y, we have

$$2y = p^2 + x,$$

and differentiation with respect to x gives

$$2p = 2p\frac{dp}{dx} + 1.$$

We separate the variables and write

$$\frac{2p\,dp}{2p - 1} = dx.$$

This integrates to

$$p + \tfrac{1}{2}\log(2p - 1) = x + C.$$

The solution is now represented parametrically, with p as parameter, by

$$\begin{Bmatrix} 2p + \log(2p - 1) = 2x + C \\ p^2 - 2y + x = 0 \end{Bmatrix}.$$

This form of the solution is slightly different from the one obtained in II, but it is demonstrable immediately that they are equivalent.

Exercise 2. Show that differentiation of the equation $y = h(p,x)$ with respect to y will, in general, lead to an equation involving p, x, and y.

An especially interesting form of an equation of the first order, solved for y, is

$$y = px + f(p),$$

known as the *Clairaut equation* after the French astronomer Clairaut (1713–1765). Differentiated with respect to x, this gives

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx},$$

or

$$\frac{dp}{dx}[x + f'(p)] = 0.$$

The solutions of this, *viz.*, $p = C$ and $x + f'(p) = 0$, coupled, in turn, with the given equation, yield

$$\begin{Bmatrix} p = C \\ y = px + f(p) \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} x = -f'(p) \\ y = px + f(p) \end{Bmatrix}$$

as the solutions of the Clairaut equation. If, between the equations of the first pair, we eliminate the parameter p, we obtain

$$y = Cx + f(C),$$

the general solution. (The form is striking inasmuch as it is directly obtained from the differential equation by changing p to C.)

The second pair of equations contains no arbitrary constant and might, therefore, be supposed to represent a particular solution. However, a particular solution must be derivable from the general solution by assigning a fixed value to the arbitrary constant, *i.e.*, it must represent one of the integral curves of the differential equation. That is not the case here. In fact, the general solution represents a one-parameter family of straight lines, while the solution in question represents, as we shall show, the envelope of that family. We call such a solution, representing the envelope of the family of integral curves of the differential equation, a *singular* solution. It appears, then, that the Clairaut equation always possesses a singular solution unless $f'(p)$ is a constant, in which case the integral curves are a family of straight lines through a point.

As an example, consider the equation

$$y = px + p^2 - 2.$$

Differentiate with respect to x, to obtain

$$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx},$$

or

$$\frac{dp}{dx}(x + 2p) = 0.$$

The solutions of the equations are, thus,

$$\begin{Bmatrix} p = C \\ y = px + p^2 - 2 \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} x = -2p \\ y = px + p^2 - 2 \end{Bmatrix}$$

or, upon eliminating p in each case,

$$y = Cx + C^2 - 2 \quad \text{and} \quad y = -\frac{x^2}{4} - 2.$$

The first is the general, the second the singular, solution.

Exercise 3. By the method of Sec. 99, find the envelope of the family of straight lines forming the above general solution, and show that the result is precisely the singular solution.

Exercise 4. By substituting in the given Clairaut equation, show that the singular solution $y = -\frac{x^2}{4} - 2$ actually satisfies the differential equation, $y = px + p^2 - 2$.

Exercise 5. Show that the solution $\begin{cases} x = -f'(p) \\ y = px + f(p) \end{cases}$ of the Clairaut equation, $y = px + f(p)$, represents the envelope of the family of lines of the general solution $y = Cx + f(C)$, or a point.

Exercise 6. Show that if the integral curves of the equation $f(x,y,p) = 0$ have an envelope the equation of that envelope is a solution of the equation. HINT: Consider the value of p at a point (x,y) of the envelope.

Problems

1. Integrate the following by first solving for p:

(a) $p^2 + p(y - \sin x) - y \sin x = 0$.

Ans. $(y + \cos x - C)(y + Ce^{-x}) = 0$.

(b) $p^3 + p^2(y \tan x - xe^y) - pxye^y \tan x = 0$.

(c) $\left(p + \frac{y}{2x}\right)^2 = x$.

Ans. $(2y\sqrt{x} + x^2 + C)(2y\sqrt{x} - x^2 + C) = 0$.

2. Integrate the following by first solving for x:

(a) $p^2 + p - x = 0$. *Ans.* $\begin{cases} 6y = 4p^3 + 3p^2 + C \\ x = p^2 + p. \end{cases}$

(b) $(p - e^{y-x})(p - 1) = 0$.

Ans. $\begin{cases} y = C + \log (p - 1) \\ = C + \log \left(\frac{p-1}{p}\right); \text{ also } y = x + C. \end{cases}$

(c) $x^2(1 + p^2) = 1$.

3. Integrate the following by first solving for y:

(a) $p^2 + y - 2x = 0$.

Ans. $\begin{cases} x = -2p - 4 \log (p - 2) + C \\ y = -4p - 8 \log (p - 2) - p^2 + 2C. \end{cases}$

(b) $px^2 + p - y = 0$.

(c) $y = p - \sqrt{a^2 + p^2}$.

4. Solve the following Clairaut equations, showing the general and the singular solutions.

(a) $y = px - \log p - 1$. *Ans.* Sing. sol.: $y = \log x$.

(b) $y = px + p - 2p^4$. *Ans.* Sing. sol.: $y = \frac{3}{8}(x + 1)^{4/3}$.

(c) $y = \sqrt{p}(x\sqrt{p} + 1)$. *Ans.* Sing. sol.: $2\sqrt{-x}\sqrt{y} = 1$.

5. Solve the following by any convenient method:

(a) $xp^2y^2 - p(x^2 + y^3) + xy = 0$.

(b) $xy + px = 2p$.

(c) $p^2 - 2x + p = 2px$.

(d) $y - \sqrt{1 - p} = px$. *Ans.* (d) Sing. sol.: $2\sqrt{x}\sqrt{y} = \sqrt{4x^2 + 1}$.

(e) $p^2 \cos^2 y + px \cos y = \sin y + 1$. HINT: Set $z = \sin y$.

(f) $yp^2 + 2xp + y = 0$.

(g) $p^2 - 2px + 2y = 0$.

(h) $p = \sin^{-1} (p^2 + x)$.

6. (a) Show that differentiation with respect to x reduces the equation $y = xf(p) + g(p)$ to a linear equation of the form $\frac{dx}{dp} + xF(p) = G(p)$.

(b) Solve the equation $y = xp^2 + p$.

(c) Solve the equation $y = -xp^3 + p^2$.

7. The general solution of a differential equation is $Cx + C^2y = 1$. Find the singular solution as the equation of the envelope of the integral curves.

8. The general solution of a differential equation is $x \cos C + y \sin C = 3$. Find the singular solution as in Prob. 7.

9. Find the equation of the curve such that the area bounded by any arc of it, the two end ordinates of the arc and the x-axis, be proportional to the length of that arc.

Ans. $x = k \log [y + \sqrt{y^2 - k^2}] + C$, also $y = C$.

10. Find the equation of the curve such that the sum of the x- and y-intercepts of its tangents shall be a constant.

119. Linear, with Constant Coefficients. A differential equation is called *linear* if the dependent variable and all its derivatives enter in it linearly. The general form of such an equation is, then,

$$y^{(n)} + f_1(x) \cdot y^{(n-1)} + f_2(x) \cdot y^{(n-2)} + \cdots + f_{n-1}(x) \cdot y' + f_n(x) \cdot y = g(x). \quad (149)$$

We shall restrict ourselves to the case where all the coefficients $f_i(x)$ are constants,* *i.e.*, to equations of form

$$y^{(n)} + a_1y^{(n-1)} + a_2y^{(n-2)} + \cdots + a_{n-1}y' + a_ny = g(x), \quad (150)$$

where the a_i are all real constants. An effective way to study (150) is to introduce the symbol D, defined by

$$\begin{aligned} (D + a)u &= \frac{du}{dx} + au, \qquad (a \text{ constant}) \\ D^m u &= \frac{d^m u}{dx^m}, \qquad (m = 1,2,3, \cdots) \\ (bD^m)u &= b \cdot D^m u \qquad (b \text{ constant}) \end{aligned} \quad (151)$$

From either of the first two, the symbol Du is defined as

$$Du = \frac{du}{dx} \quad (152)$$

We shall read $(D + a)u$ as "$D + a$, operating on u," likewise, Du as "D, operating on u."

* For the case $n = 1$, *i.e.*, the linear equation of the first order, we studied the equation in the form (149), in IV, Sec. 117.

A significant fact about the symbol $D + a$ is that in many respects it behaves as if it were a number that multiplies u, though actually it is not a number but a symbol of an operation that is performed on u—an *operator*, as it is commonly called.

To make that clear, consider the following equalities which we should expect to be true if $D + a$, and also D, were numerical factors.

$$\begin{aligned} D(u + v) &= Du + Dv, \\ D(cu) &= cDu, \qquad (c \text{ constant}) \\ D(D^m u) &= D^{m+1}u, \\ (D + b)[(D + a)u] &= [D^2 + (a + b)D + ab]u \\ &= (D + a)[(D + b)u]. \end{aligned}$$

Now, in view of (151) and (152),

$$D(u + v) = \frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx} = Du + Dv;$$

also

$$\begin{aligned} D(cu) &= \frac{d}{dx}(cu) = c\frac{du}{dx} = cDu; \\ D(D^m u) &= \frac{d}{dx}(D^m u) = \frac{d}{dx}\left(\frac{d^m u}{du^m}\right) = \frac{d^{m+1}u}{dx^{m+1}} = D^{m+1}u; \\ (D + b)[(D + a)u] &= \frac{d}{dx}[(D + a)u] + b[(D + a)u] \\ &= \frac{d}{dx}\left(\frac{du}{dx} + au\right) + b\left(\frac{du}{dx} + au\right) \\ &= \frac{d^2u}{dx^2} + a\frac{du}{dx} + b\frac{du}{dx} + abu \\ &= D^2u + (a + b)Du + abu \\ &= [D^2 + (a + b)D + ab]u, \end{aligned}$$

the last expression following if we agree to write

$$a_1D^m u + a_2D^n u + a_3u = (a_1D^m + a_2D^n + a_3)u,$$

and the like.

Exercise 1. By use of (151) and (152) verify that $(D + a)[(D + b)u]$ has the same value as the one displayed above for $(D + b)[(D + a)u]$.

We have seen, thus, that all of the equalities tested turned out to be true, with the symbols D and $D + a$ having the meaning given them in (151) and (152).

The four equalities verified enable us to write (150) as

$$[D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n]y = g(x) \quad (153)$$

or, in brief, as

$$[P(D)]y = g(x). \quad (154)$$

We shall also need, in our work, a definition for the symbol $\left[\frac{1}{D-a}\right]u$. We define it by saying that if $\left[\frac{1}{D-a}\right]u = v$, then $(D - a)v = u$. The latter, put in the form $\frac{dv}{dx} - av = u$ is a first order linear differential equation which gives at once (see IV, Sec. 117) $v = e^{\int a\,dx}[\int ue^{-\int a\,dx}\,dx + C]$. In other words,

$$\left[\frac{1}{D-a}\right]u = e^{ax}[\int u \cdot e^{-ax}\,dx + C]. \quad (155)$$

Note further that if $(D - a)v = 0$, we have directly,

$$\frac{dv}{dx} - av = 0, \qquad \frac{dv}{v} - a\,dx = 0, \qquad \log v - ax = \log C,$$

$$\text{and} \qquad v = Ce^{ax},$$

which result may be put in the form

$$\left[\frac{1}{D-a}\right]0 = Ce^{ax}. \quad (156)$$

We are now in a position to attack the solution of (150). To simplify the discussion, we consider, first,

I. *Homogeneous Equations.* We shall call (150) *homogeneous* when the right-hand member, that is, $g(x)$, is identically zero. To fix the ideas, let us first consider a second order homogeneous equation

$$y'' + a_1y' + a_2y = 0,$$

or, in the form (153)

$$(D^2 + a_1D + a_2)y = 0.$$

We solve (as though D were a variable) the equation

$$D^2 + a_1D + a_2 = 0,$$

an equation we shall call the *characteristic* equation of the given differential equation. Let its roots be found to be $D = r_1$ and

$D = r_2$, supposed, for the present, to be two distinct real numbers. Then, by a well-known theorem of algebra,

$$D^2 + a_1D + a_2 = (D - r_1)(D - r_2),$$

and our differential equation assumes the form

$$(D - r_1)(D - r_2)y = 0.$$

If, for the moment, we put $(D - r_2)y = u$, we have

$$(D - r_1)u = 0,$$

and, by (156),

$$u = C_1e^{r_1x}.$$

By $(D - r_2)y = u$, we have $y = \dfrac{1}{D - r_2}u$, and this, by (155), yields

$$\begin{aligned} y &= e^{r_2x}[\textstyle\int u \cdot e^{-r_2x}\,dx + C_2] = e^{r_2x}[\textstyle\int C_1e^{(r_1-r_2)x}\,dx + C_2] \\ &= e^{r_2x}\left[\frac{C_1}{r_1 - r_2}e^{(r_1-r_2)x} + C_2\right] = C_1e^{r_1x} + C_2e^{r_2x}, \end{aligned}$$

where we have replaced $\dfrac{C_1}{r_1 - r_2}$ by C_1. This is the general solution, since it contains two arbitrary constants.

Exercise 2. Show that the equation $(D^3 + a_1D^2 + a_2D + a_3)y = 0$ has as its general solution $y = C_1e^{r_1x} + C_2e^{r_2x} + C_3e^{r_3x}$, if $D = r_1$, $D = r_2$, $D = r_3$ are three distinct real solutions of the characteristic equation

$$D^3 + a_1D^2 + a_2D + a_3 = 0.$$

HINT: Write the equation as $(D - r_1)(D - r_2)(D - r_3)y = 0$. Set $(D - r_3)y = u$, then by the above, $u = C_1e^{r_1x} + C_2e^{r_2x}$. Now find y as $\dfrac{1}{D - r_3}u$.

The argument may be extended to yield the result embodied in

Theorem 1. *If the roots of the characteristic equation, $P(D) = 0$, are n distinct real numbers, $r_1, r_2, \ldots, r_n$, then the general solution of the corresponding homogeneous equation, $[P(D)]y = 0$, is $y = C_1e^{r_1x} + C_2e^{r_2x} + \cdots + C_ne^{r_nx}$.*

To illustrate, we solve the equation

$$y''' - 2y'' - 8y' = 0.$$

In the form (153), it is

$$(D^3 - 2D^2 - 8D)y = 0.$$

Its characteristic equation,

$$D^3 - 2D^2 - 8D = 0,$$

has the (real and distinct) roots $D = 0$, $D = 4$, and $D = -2$. Hence, by Theorem 1, the general solution is

$$y = C_1 + C_2e^{4x} + C_3e^{-2x}.$$

Turning again to Theorem 1, we note from it that the general solution of the equation $(D - r_1)(D - r_2) \cdots (D - r_n)y = 0$ (r_i all real and distinct) is the sum of the general solutions of the equations $(D - r_1)y = 0$, $(D - r_2)y = 0$, $\cdots$ $(D - r_n)y = 0$, for $(D - r_i)y = 0$ gives, by (156)

$$y = C_ie^{r_ix} \qquad (i = 1,2, \cdots, n).$$

Let, now, two roots of the characteristic equation, say r_1 and r_2, be alike. Then the differential equation is

$$(D - r_1)^2(D - r_3)(D - r_4) \cdots (D - r_n)y = 0,$$

and its general solution is the sum of the general solutions of $(D - r_1)^2y = 0$ and of $(D - r_3)(D - r_4) \cdots (D - r_n)y = 0$, the latter being

$$y = C_3e^{r_3x} + C_4e^{r_4x} + \cdots + C_ne^{r_nx}.$$

To solve $(D - r_1)^2y = 0$, or $(D - r_1)[(D - r_1)y] = 0$, we put $(D - r_1)y = u$, and $(D - r_1)u = 0$ gives $u = C_1e^{r_1x}$. Hence,

$$\begin{aligned} y = \frac{1}{D - r_1}u &= e^{r_1x}\left[\int u \cdot e^{-r_1x}\,dx + C_2\right] \\ &= e^{r_1x}[\textstyle\int(C_1e^{r_1x})e^{-r_1x}\,dx + C_2] \\ &= e^{r_1x}[\textstyle\int C_1\,dx + C_2] \\ &= C_1xe^{r_1x} + C_2e^{r_1x}. \end{aligned}$$

The general solution of the differential equation is now

$$y = C_1xe^{r_1x} + C_2e^{r_1x} + C_3e^{r_3x} + C_4e^{r_4x} + \cdots + C_ne^{r_nx}.$$

The case when three roots of the characteristic equation are alike may be treated in like manner by the student as

Exercise 3. Show that if three roots, r_1, r_2, r_3, of the characteristic equation, are alike, the general solution of the corresponding homogeneous equation is

$$y = C_1x^2e^{r_1x} + C_2xe^{r_1x} + C_3e^{r_1x} + C_4e^{r_4x} + \cdots + C_ne^{r_nx}.$$

By extending the argument, we arrive at

Theorem 2. *If m roots, $r_1, r_2, \ldots, r_m$, of the characteristic equation $P(D) = 0$, are alike, then that part of the general solution of the corresponding homogeneous linear differential equation, $[P(D)]y = 0$, due to the root r_1, is*

$$y = C_1x^{m-1}e^{r_1x} + C_2x^{m-2}e^{r_1x} + \cdots + C_{m-1}xe^{r_1x} + C_me^{r_1x}.$$

To illustrate, we solve the equation

$$(D^5 - 5D^4 + 9D^3 - 7D^2 + 2D)y = 0,$$

or

$$D(D - 2)(D - 1)^3y = 0.$$

The roots of its characteristic equation are $D = 1$, $D = 1$, $D = 1$, $D = 2$, and $D = 0$. Hence, by Theorem 2, its general solution is

$$y = C_1x^2e^x + C_2xe^x + C_3e^x + C_4e^{2x} + C_5.$$

As another example, we treat the equation

$$(D^5 - 3D^3 + 2D^2)y = 0.$$

This may be written as

$$D^2(D - 1)^2(D + 2)y = 0,$$

showing that its characteristic equation has the roots $D = 0$, $0; 1, 1; -2$. Its general solution is, then,

$$y = C_1x + C_2 + C_3xe^x + C_4e^x + C_5e^{-2x}.$$

We turn our attention, lastly, to the case when two of the roots of the characteristic equation are complex and equal to $a + bi$ and $a - bi$, (a and b real). Should we ignore the fact that these roots are imaginary we would have, as that part of the general solution corresponding to them,

$$y = C_1e^{(a+bi)x} + C_2e^{(a-bi)x} = e^{ax}(C_1e^{bix} + C_2e^{-bix}).$$

Taking a clue from the fact that this has the real factor e^{ax}, let us assume that the equation

$$(D^2 - 2aD + a^2 + b^2)y = 0, \tag{157}$$

whose characteristic equation has the roots $a + bi$ and $a - bi$, has solutions of the form $y = e^{ax} \cdot u(x)$ and determine the function $u(x)$ by substitution in (157). Differentiating, we have

$$y' = ae^{ax} \cdot u + e^{ax} \cdot u',$$
$$y'' = a^2e^{ax} \cdot u + 2ae^{ax} \cdot u' + e^{ax} \cdot u''.$$

Substituting these in (157), we arrive at the result

$$u'' = -b^2u,$$

as the equation that $u(x)$ must satisfy. The functions $\sin bx$ and $\cos bx$ at once suggest themselves as such values of u, and in fact $u = C_1 \cos bx + C_2 \sin bx$, with C_1 and C_2 as arbitrary constants, is directly seen to satisfy. Hence, the general solution of (157) is $y = e^{ax}(C_1 \cos bx + C_2 \sin bx)$. We, thus, arrive at

Theorem 3. *The part of the general solution of the homogeneous linear differential equation,* $[P(D)]y = 0$, *corresponding to the pair,* $a \pm bi$, *of complex roots of the characteristic equation, is* $y = e^{ax}(C_1 \cos bx + C_2 \sin bx)$.

Thus, in the case of

$$(D^2 - 6D + 13)(D - 1)(D + 2)y = 0,$$

for which the roots of the characteristic equation are $D = 1$, $D = -2$, $D = 3 \pm 2i$, the general solution is

$$y = C_1e^x + C_2e^{-2x} + e^{3x}(C_3 \cos 2x + C_4 \sin 2x).$$

Exercise 4. Show by (155) that $\dfrac{1}{D - a}(u + v) = \dfrac{1}{D - a}u + \dfrac{1}{D - a}v.$

Exercise 5. Show that if $y = u(x)$ is a solution of the homogeneous linear equation, then $y = cu(x)$, where c is any constant, is also a solution.

Exercise 6. Show that if $y = u(x)$ and $y = v(x)$ are two solutions of the homogeneous linear equation, then $y = u(x) + v(x)$ is also a solution.

Problems

1. Solve each of the following:

(a) $y'' + y' = 0$. *Ans.* $y = C_1 + C_2e^{-x}$.

(b) $y'' - y' - 6y = 0$. *Ans.* $y = C_1e^{3x} + C_2e^{-2x}$.

(c) $y''' + 4y'' + 4y' = 0$. *Ans.* $y = C_1 + C_2xe^{-2x} + C_3e^{-2x}$.

(d) $y''' + 4y' = 0$. *Ans.* $y = C_1 + C_2 \cdot \cos 2x + C_3 \cdot \sin 2x$.

(e) $\dfrac{d^4y}{dx^4} - 6\dfrac{d^3y}{dx^3} + 9\dfrac{d^2y}{dx^2} = 0$.

(f) $y''' + (a - b)y'' - aby' = 0$.

(g) $y''' - 3y'' + 4y' - 2y = 0$.

(h) $(8D^4 - 12D^3 + 6D^2 - 2D)y = 0$

(i) $\dfrac{d^4y}{dx^4} - 6\dfrac{d^3y}{dx^3} + 12\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} = 0$.

(j) $\dfrac{d^4y}{dx^4} + 9\dfrac{d^2y}{dx^2} = 0$.

2. Write the differential equation whose general solution is

(a) $y = C_1e^{2x} + C_2 + C_3x$.

(b) $y = C_1 + C_2 \cos 3x + C_3 \sin 3x + x(C_4 \cos 3x + C_5 \sin 3x)$.

3. Find that solution of the equation

(a) $y'' + 3y' - 10y = 0$ for which $y = 1$, and $y' = 9$ when $x = 0$.

Ans. $y = 2e^{2x} - e^{-5x}$.

(b) $y''' + y' = 0$ for which $y = -1$, $y' = -1$, and $y'' = 2$ when $x = \pi$.

4. Solve the equation $x^2y'' - 2xy' - 4y = 0$. HINT: Reduce this to an equation with constant coefficients by introducing a new variable t, defined as $t = \log x$ (and hence

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\cdot\frac{dy}{dt},\quad \frac{d^2y}{dx^2} = \frac{1}{x}\cdot\frac{d^2y}{dt^2}\cdot\frac{dt}{dx} - \frac{1}{x^2}\cdot\frac{dy}{dt} = \frac{1}{x^2}\cdot\frac{d^2y}{dt^2} - \frac{1}{x^2}\cdot\frac{dy}{dt}\Big).$$

Ans. $y = C_1x^4 + \dfrac{C_2}{x}$.

5. Solve the following by the method of Prob. 4.

(a) $x^2y'' + 3xy' - 8y = 0$. *Ans.* $y = C_1x^2 + C_2x^{-4}$.

(b) $x^3y''' - 2xy' = 0$. *Ans.* $y = C_1 + C_2 \log x + C_3x^3$.

6. Show that the substitution of Prob. 4 will reduce every linear equation of the form

$$x^ny^{(n)} + a_1x^{n-1}y^{(n-1)} + a_2x^{n-2}y^{(n-2)} + \cdots + a_{n-1}xy' + a_ny = g(x)$$

(the so-called *Cauchy equation*) to one with constant coefficients.

II. *Nonhomogeneous Equations.* When the right-hand member of the linear equation is not zero, we have recourse to the following

Theorem 4. *If $y = y_1(x)$ is a solution of a given nonhomogeneous linear differential equation, and $y = y_2(x)$ is the general solution of the corresponding homogeneous equation, then $y = y_1(x) + y_2(x)$ is the general solution of the given equation.*

It is customary, in this connection, to refer to the equality $y = y_1(x)$ as *a particular integral,* and to the function $y_2(x)$ as the *complementary function.*

Theorem 4 is easily established by setting $y_3 = y_1 + y_2$ and noting that $y_3' = y_1' + y_2'$, $y_3'' = y_1'' + y_2''$, etc. The result obtained by substituting y_3 in the left-hand member of the differential equation is, then, equal to the sum of the two results obtained by substituting y_1 and y_2 in turn. The first result, by hypothesis, is $g(x)$, the right-hand member of the equation; the second is zero. The substitution of y_3 in the left-hand member, then, reduces the latter to $g(x)$, hence $y = y_3(x)$ is a solution. It is also the general solution, since it contains n constants.

Query. In what way would the presence of a nonlinear term in the equation, such as $y''^2 \sin x$, destroy the above proof?

Let us consider now the nonhomogeneous equation

$$y'' - y' - 2y = \sin x.$$

The complementary function (the solution of $y'' - y' - 2y = 0$) is readily found to be $y = C_1e^{2x} + C_2e^{-x}$.

To find a particular integral of the given equation, we write

$$(D + 1)(D - 2)y = \sin x.$$

Setting, for the moment, $(D - 2)y = u$, this reads

$$(D + 1)u = \sin x$$

and by (155)

$$u = \frac{1}{D + 1} \sin x = e^{-x} \int \sin x \cdot e^x \, dx.$$
$$= \frac{e^{-x}e^x(\sin x - \cos x)}{2} = \frac{\sin x - \cos x}{2}.$$

(We ignore the constants of integration in this and the following integrations, because we are finding a *particular* integral.) Now,

$$y = \frac{1}{D - 2}u = e^{2x} \int \frac{\sin x - \cos x}{2} \cdot e^{-2x} \, dx$$
$$= \frac{e^{2x}}{2}\left[\frac{e^{-2x}(-2 \sin x - \cos x)}{5} - \frac{e^{-2x}(\sin x - 2 \cos x)}{5}\right]$$
$$= \frac{\cos x - 3 \sin x}{10}.$$

The general solution is now at hand, by Theorem 4, as

$$y = C_1e^{2x} + C_2e^{-x} + \frac{\cos x - 3 \sin x}{10}.$$

The integration just gone through with in finding the particular integral might have been simplified if we had written

$$y = \frac{1}{(D + 1)(D - 2)} \sin x$$

and resolved the operator $\dfrac{1}{(D + 1)(D - 2)}$ into *partial fractions.* (The legitimacy of that step will be verified presently.) The student will easily establish that

$$\frac{1}{(D + 1)(D - 2)} = \frac{-\frac{1}{3}}{D + 1} + \frac{\frac{1}{3}}{D - 2}.$$

Hence,

$$\begin{aligned} y &= \frac{1}{(D+1)(D-2)} \sin x = -\frac{1}{3} \cdot \frac{1}{D+1} \sin x + \frac{1}{3} \frac{1}{D-2} \sin x \\ &= -\tfrac{1}{3} e^{-x} \textstyle\int e^x \sin x \, dx + \tfrac{1}{3} e^{2x} \textstyle\int e^{-2x} \sin x \, dx \\ &= \frac{1}{3}\left[-\frac{\sin x - \cos x}{2} + \frac{-2 \sin x - \cos x}{5} \right] \\ &= \frac{\cos x - 3 \sin x}{10}, \end{aligned}$$

as above.

We now justify the resolution of the operator into partial fractions by considering the identity

$$\frac{1}{(D-a)(D-b)} = \frac{1}{a-b}\left[\frac{1}{D-a} - \frac{1}{D-b}\right].$$

We wish to show that

$$\frac{1}{(D-a)(D-b)} u = \frac{1}{a-b}\left[\frac{1}{D-a} u - \frac{1}{D-b} u\right],$$

(where u is any function of x) or, what is the same, that

$$\frac{1}{a-b}[(D-a)(D-b)]\left[\frac{1}{D-a} u - \frac{1}{D-b} u\right] = u, \quad (158)$$

or, again, that

$$\frac{1}{a-b}[(D-a)(D-b)]\left[e^{ax} \int u e^{-ax} \, dx - e^{bx} \int u e^{-bx} \, dx\right] = u.$$

The verification will be left to the student as

Exercise 7. Verify the identity (158).

The foregoing argument can be extended to operators of a more complicated type, such as $\dfrac{1}{(D-a)(D-b)(D-c)}$ or $\dfrac{1}{(D-a)^2(D-b)}$, etc., and the legitimacy of resolving them into partial fractions can thus be established for all cases.

We shall now consider another method for finding the particular integral—that of *undetermined coefficients*. Let us return to the equation just treated, *viz.*,

$$y'' - y' - 2y = \sin x.$$

In seeking a particular integral for this equation, we are seeking a function of x such that a certain linear combination of that function and its first and second derivatives shall be identically equal to $\sin x$. Now, $\sin x$ and $\cos x$ are functions whose successive derivatives, apart from sign, are alternately $\sin x$ and $\cos x$, and this suggests that the function we are attempting to find might well be one of the form $A \sin x + B \cos x$, with proper choices of the coefficients A and B. Acting on that clue, let us assume as our particular integral

$$y = A \sin x + B \cos x.$$

Hence, we have the derivatives

$$\begin{aligned} y' &= -B \sin x + A \cos x, \\ y'' &= -A \sin x - B \cos x \end{aligned}$$

To have our equation satisfied by this y, we must have

$$(-A \sin x - B \cos x) - (-B \sin x + A \cos x) \\ 2(A \sin x + B \cos x) = \sin x,$$

or

$$(-A + B - 2A) \sin x + (-B - A - 2B) \cos x = \sin x.$$

For this to be an identity, we must have

$$\begin{cases} B - 3A \quad = 1 \\ -3B - A = 0 \end{cases}$$

from which we obtain $A = -\frac{3}{10}$, $B = \frac{1}{10}$, and hence the particular integral as $y = -\frac{3}{10} \sin x + \frac{1}{10} \cos x$, the same as obtained before.

This example will pave the way for the student's appreciation of the following notes.

NOTE (*a*): *If the function $g(x)$ in the right-hand member of the equation $[P(D)]y = g(x)$ is of the form $a \cos mx + b \sin mx$, assume the particular integral as $y = A \cos mx + B \sin mx$ [except that if $P(D)$ has the factor $(D^2 + m^2)^r$, assume $y = x^r(A \sin mx + B \cos mx)$.]*

NOTE (*b*). *If $g(x)$ is of the form ae^{mx}, assume the particular integral as $y = Ae^{mx}$ [except that if $P(D)$ has the factor $(D - m)^r$, assume $y = Ax^re^{mx}$].*

NOTE (*c*). *If $g(x)$ is of the form $ax^m + bx^{m-1} + \cdots + kx + l$, a polynomial of degree m, assume the particular integral as*

$y = Ax^m + Bx^{m-1} + \cdots + Kx + L$ [*except that if* $P(D)$ *has the factor* D^r, *assume* $y = x^r(Ax^m + Bx^{m-1} + \cdots + Kx + L)$].

In all cases, the assumed coefficients $A, B, \ldots$ are to be determined by substituting the assumed y and its derivatives into the equation and imposing the condition that the latter become an identity. We shall make no proof concerning the exceptions quoted for the three notes above, other than to remark that they are concerned with the cases where some of the terms of the assumed particular integral are of a type appearing in the complementary function. Thus, *e.g.*, in Note (*b*), if $(D - m)$ is a factor of $P(D)$, the complementary function will contain a term $C_i e^{mx}$. Let the student reflect to see why it would not do to assume, in that case, a particular integral as Ae^{mx}. In every case, among the exceptions quoted, the exponent r appearing on a function of D is understood to be the largest exponent possible which will leave that power a factor of $P(D)$.

Let us illustrate the method of undetermined coefficients again by finding a particular integral for the equation

$$y'' + 2y' - 3y = e^{-3x} - \cos x + x^2 - x,$$

or

$$(D - 1)(D + 3)y = e^{-3x} - \cos x + x^2 - x.$$

We assume it as

$$y = Axe^{-3x} + B \sin x + C \cos x + Dx^2 + Ex + F,$$

the term Axe^{-3x} following the exception in Note (*b*), the next two terms by Note (*a*) and the last three by reason of Note (*c*). We then have

$$\begin{aligned} y' &= -3Axe^{-3x} + Ae^{-3x} + B \cos x - C \sin x + 2Dx + E, \\ y'' &= 9Axe^{-3x} - 6Ae^{-3x} - B \sin x - C \cos x + 2D. \end{aligned}$$

The completion of the illustration is left to the student as

Exercise 8. Substitute the above expressions for y and its derivatives into the given differential equation, collect the terms and find values of the assumed coefficients which will produce an identity.

Ans. $A = -\frac{1}{4}$, $B = -\frac{1}{10}$, $C = \frac{1}{5}$, $D = -\frac{1}{3}$, $E = -\frac{1}{9}$, $F = -\frac{8}{27}$.

The particular integral sought is, thus

$$y = -\tfrac{1}{4}xe^{-3x} - \tfrac{1}{10} \sin x + \tfrac{1}{5} \cos x - \tfrac{1}{3}x^2 - \tfrac{1}{9}x - \tfrac{8}{27}.$$

Problems

1. Solve the following by the method of operators:

(*a*) $y'' + y' = e^x$. *Ans.* $y = C_1 + C_2e^{-x} + \frac{1}{2}e^x$.

(*b*) $y'' - 5y' + 6y = \cos x + 1$.

(*c*) $y''' + 3y'' = \sin x - 1$.

(*d*) $y''' - y' = x + e^x$.

Ans. $y = C_1 + C_2e^x + C_3e^{-x} - \dfrac{x^2}{2} + \dfrac{xe^x}{2}$.

2. Solve the following by the method of undetermined coefficients:

(*a*) $y'' - 2y' - 8y = x^2 - 1$.

Ans. $y = C_1e^{4x} + C_2e^{-2x} - \dfrac{x^2}{8} + \dfrac{x}{16} + \dfrac{5}{64}$.

(*b*) $y''' - 4y'' + 4y' = \sin x + e^{2x}$.

Ans. $y = C_1 + C_2e^{2x} + C_3xe^{2x} + \dfrac{4}{25}\sin x - \dfrac{3}{25}\cos x + \dfrac{x^2e^{2x}}{4}$.

(*c*) $y''' - y'' - 20y = \cosh x - 2\sinh x$.

(*d*) $y''' + y'' - 6y' = x^3 - x$.

(*e*) $\dfrac{d^4y}{dx^4} + 3\dfrac{d^2y}{dx^2} - 4y = \cos 2x + e^{3x}$.

3. Solve the following by any method.

(*a*) $y''' = x + \log x$.

(*b*) $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} = e^{2x} + x^2 - 1$.

Ans. $y = C_1 + C_2x + C_3xe^{-x} + C_4e^{-x} + \dfrac{x^4}{12} - \dfrac{2x^3}{3} + \dfrac{5x^2}{2} + \dfrac{e^{2x}}{36}$.

(*c*) $\dfrac{d^4y}{dx^4} - a^4y = \cos ax + e^{ax} + 1$.

(*d*) $3y''' - 4y'' + y' = x + \sin x - 1$.

(*e*) $\dfrac{d^4y}{dx^4} + 9\dfrac{d^2y}{dx^2} = x + \cos 3x - \sin 2x$.

(*f*) $\dfrac{d^5y}{dx^5} - 2\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} = x + e^{2x}$.

(*g*) $3y''' + y'' - 2y' = e^{-2x} + x^2 - 3x + 2$.

4. (*a*) Solve $y''' - 8y'' + 16y' = xe^x$. HINT: If done by undetermined coefficients, assume particular integral $y = e^x \cdot z$.

Ans. $y = C_1 + C_2e^{4x} + C_3xe^{4x} + \dfrac{xe^x}{9} - \dfrac{e^x}{27}$.

(*b*) Solve $y''' + 3y'' = e^x \sin x$.

Ans. $y = C_1 + C_2x + C_3e^{-3x} - \dfrac{e^x}{34}(\sin x + 4\cos x)$.

5. Solve the following Cauchy equations (see Prob. 6 under I, this section):

(*a*) $x^2y'' + 4xy' + 2y = (\log x)^2 + x^2$.

Ans. $y = \dfrac{C_1}{x^2} + \dfrac{C_2}{x} + \dfrac{x^2}{12} + \dfrac{7}{4} + \dfrac{(\log x)^2}{2} - \dfrac{3\log x}{2}$.

(*b*) $x^3y''' + x^2y'' - 3xy' = \dfrac{1}{x} + \log x$.

6. (*a*) Find that integral curve of the equation $y'' - 3y' - 10y = 0$ which touches the line $y = x$ at the origin. *Ans.* $y = \dfrac{e^{5x} - e^{-2x}}{7}$.

(*b*) Find that particular integral of the equation $y''' - 3y' = x^2$ for which, at $x = 0$, $y = 0$, $y' = -5\%$, and $y'' = 0$.

7. A particle moving in a straight line is said to be in simple harmonic motion when its acceleration at any instant is directed toward a fixed point on the line and is proportional to the distance from that point. Show that the equation of the motion is $d^2x/dt^2 = -k^2x$, where x represents the distance of the particle from the fixed point. Find x in terms of t, if when $t = 0$, $x = 2k$ and the velocity of the particle is $\frac{1}{2}k$.

Ans. $x = 2k \cdot \cos kt + \frac{1}{2} \sin kt$.

8. A particle moving on a straight line is attracted to a fixed point on that line, as in simple harmonic motion, and is subject to a resistance proportional to the velocity. Show that the equation of its motion is

$$\frac{d^2x}{dt^2} = -k^2x - l^2\frac{dx}{dt}.$$

Solve for x. If we set $l^2 = 2\mu$, distinguish among the cases $\mu^2 - k^2 > 0$, $\mu^2 - k^2 = 0$ and $\mu^2 - k^2 < 0$. In the last case show that the motion is oscillatory, with the period equal to $2\pi/\sqrt{k^2 - \mu^2}$, and that the amplitude approaches zero as time goes on.

NOTE: The motion, in the case $\mu^2 - k^2 < 0$, is called *damped harmonic motion.*

9. The acceleration of a particle moving under the action of gravity is directed towards the center of the earth and is inversely proportional to the square of the distance from that center. Find the velocity with which the particle reaches the surface of the earth, if it starts from rest at an infinite distance. *Ans.* $-\sqrt{2gR}$.

HINT: The equation of the motion is $d^2s/dt^2 = -k^2/s^2$. This is a non-linear equation of the form $d^2s/dt^2 = f(s)$. To solve such, we multiply by $2\frac{ds}{dt}$ obtaining $\frac{d}{dt}\left(\frac{ds}{dt}\right)^2 = 2f(s)\frac{ds}{dt}$ and $\left(\frac{ds}{dt}\right)^2 = \int 2f(s)ds = g(s) + C_1$. Hence

$$\frac{ds}{\sqrt{g(s) + C_1}} = dt,$$

and the variables are separated.

10. (*a*) The plates of a charged condenser, of capacity C farads, are connected in series with a resistance of R ohms and an inductance of L henries. The potential difference between the plates, of v volts, is represented by the equation

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = 0.$$

Find v in terms of t, distinguishing the cases $R^2C - 4L > 0$, $R^2C - 4L = 0$, $R^2C - 4L < 0$.

(*b*) If in (*a*) $R^2C - 4L > 0$, determine v in terms of t, given that when $t = 0$, $v = v_0$ and $dv/dt = 0$.

11. Solve the equation $d^2y/dx^2 = 2y$ by the method of the hint to Prob. 9, given that when $x = 0$, $y = 1$ and $dy/dx = 0$. Check your result by solving it also as a linear equation.

120. Simultaneous Linear Equations. We treat now pairs of linear equations, involving one independent variable, say t, and two dependent variables, say x and y, confining ourselves, as heretofore, to equations with constant coefficients. A general form of such pairs is

$$\begin{cases} P_1(D)x + Q_1(D)y = R_1(t) \\ P_2(D)x + Q_2(D)y = R_2(t), \end{cases} \tag{159}$$

where $P_1(D)$, $P_2(D)$, $Q_1(D)$ and $Q_2(D)$ are polynomials in the operator D, the symbol D denoting differentiation, now, with respect to t.

As in the case of algebraic equations, the procedure is to eliminate one of the variables x,y and reduce the system to a linear equation in the other one, which may then be treated by the methods of the preceding section. One of the dependent variables solved for, it is quite a simple matter to obtain the value of the second.

To illustrate, we consider the pair of equations

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} - 2x - 2y = e^{2t}$$

$$\frac{dx}{dt} + 2\frac{dy}{dt} - y = 0.$$

We write them in the form (159) as

$$\begin{cases} (D^2 - 2)x + (D - 2)y = e^{2t} \\ Dx + (2D - 1)y = 0 \end{cases}$$

To eliminate, say, y, we operate on both members of the first equation by $(2D - 1)$, and of the second by $(D - 2)$, obtaining

$$(2D - 1)(D^2 - 2)x + (2D - 1)(D - 2)y = (2D - 1)e^{2t} = 4e^{2t} - e^{2t} = 3e^{2t},$$

$$(D - 2)Dx + (D - 2)(2D - 1)y = (D - 2)0 = 0.$$

By subtraction, now, we have

$$[(2D - 1)(D^2 - 2) - (D - 2)D]x = 3e^{2t},$$

and this reduces to

$$(D^3 - D^2 - D + 1)x =$$

or

$$(D - 1)^2(D + 1)x = \tfrac{3}{2}e^{2t}.$$

This equation is left to the student to solve as

Exercise 1. Solve the last equation displayed above.

Ans. $x = C_1e^t + C_2te^t + C_3e^{-t} + \frac{1}{2}e^{2t}$.

We may now obtain y either by eliminating x from the given pair of equations, or by substituting the value of x just found into either one of them. By the second plan, let us substitute, say, into the second equation, obtaining

$$(2D - 1)y = -Dx = -C_1e^t - C_2te^t - C_2e^t + C_3e^{-t} - e^{2t},$$

or

$$(D - \tfrac{1}{2})y = -\tfrac{1}{2}[(C_1 + C_2)e^t + C_2te^t - C_3e^{-t} + e^{2t}],$$

whence, by (155) of Sec. 119 (page 475)

$$\begin{aligned} y &= -\tfrac{1}{2}e^{\frac{t}{2}}\left\{C_4 + \int[(C_1 + C_2)e^{\frac{t}{2}} + C_2te^{\frac{t}{2}} - C_3e^{\frac{-3t}{2}} + e^{\frac{3t}{2}}]dt\right. \\ &= -\frac{1}{2}e^{\frac{t}{2}}\left[C_4 + (2C_1 + 2C_2)e^{\frac{t}{2}} + 4C_2e^{\frac{t}{2}}\left(\frac{t}{2} - 1\right) + \frac{2C_3}{3}e^{\frac{-3t}{2}} + \frac{2}{3}e^{\frac{3t}{2}}\right. \\ &= -\tfrac{1}{2}C_4e^{\frac{t}{2}} - (C_1 - C_2)e^t - C_2te^t - \tfrac{1}{3}C_3e^{-t} - \tfrac{1}{3}e^{2t}. \end{aligned}$$

The number of arbitrary constants that appear in our solution is four. By a theorem in differential equations, *the number of arbitrary constants in the solution of a system cannot exceed the sum of the orders of the equations of the system,* which is three in this case. Evidently, there is some relation among the constants we have introduced. To find that relation, we substitute the values that we have found for x and y into the first equation. Operating on x by $D^2 - 2$, we obtain

$$(D^2 - 2)x = (2C_2 - C_1)e^t - C_2te^t - C_3e^{-t} + e^{2t}.$$

Operating on y by $D - 2$, we get

$$(D - 2)y = \tfrac{3}{4}C_4e^{\frac{t}{2}} + (C_1 - 2C_2)e^t + C_2te^t + C_3e^{-t}.$$

The sum of these must be e^{2t}, identically, *i.e.*,

$$\tfrac{3}{4}C_4e^{\frac{t}{2}} + e^{2t} = e^{2t}.$$

Hence, $C_4 = 0$, and the solution of the given system is

$$x = C_1e^t + C_2te^t + C_3e^{-t} + \tfrac{1}{2}e^{2t},$$
$$y = (C_2 - C_1)e^t - C_2te^t - \frac{C_3}{3}e^{-t} - \frac{1}{3}e^{2t}.$$

As an alternative method for obtaining y let us eliminate x from the pair of equations, just as x was found. To that end we phrase the following two exercises.

Exercise 2. Operate on the two members of the first equation of the given pair by D, and of the second by $D^2 - 2$ and eliminate x by subtraction.
Ans. $(D - 1)^2(D + 1)y = -e^{2t}$.

Exercise 3. Find the general solution to the equation obtained in Exercise 2. *Ans.* $y = C_4e^t + C_5te^t + C_6e^{-t} - \frac{1}{3}e^{2t}$.

The constants in the answer to Exercise 3 have been so labeled because at this stage we have no means of knowing how they are related to the constants C_1, C_2, C_3 that appear in the expression for x. To find the relation of the constants, we substitute the values of x and y, as now found, say, into the second of the given equations. Now

$$Dx = (C_1 + C_2)e^t + C_2te^t - C_3e^t + e^{2t},$$
$$(2D - 1)y = (C_4 + 2C_5)e^t + C_5te^t - 3C_6e^{-t} - e^{2t}.$$

The sum must be identically zero, hence

$$\begin{Bmatrix} C_4 + 2C_5 = -C_1 - C_2 \\ C_5 = -C_2 \\ -3C_6 = C_3 \end{Bmatrix}, \qquad \text{so that} \qquad \begin{Bmatrix} C_4 = C_2 - C_1 \\ C_5 = -C_2 \\ C_6 = -\frac{1}{3}C_3 \end{Bmatrix}.$$

and so we have

$$y = (C_2 - C_1)e^t - C_2te^t - \frac{C_3}{3}e^{-t} - \frac{1}{3}e^{2t},$$

the same result as by the other method.

Problems

1. Solve the following pairs of equations.

(a) $\begin{Bmatrix} \dfrac{dx}{dt} - 2\dfrac{dy}{dt} = 0 \\ x + 2\dfrac{dy}{dt} = t^2 \end{Bmatrix}$. *Ans.* $\begin{cases} x = 2C_1e^{-t} + t^2 - 2t + 2, \\ y = C_1e^{-t} + \frac{1}{2}t^2 - t. \end{cases}$

(b) $\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + 3x + y = 0 \\ 2x + \dfrac{dy}{dt} = e^{-t} \end{cases}$

Ans. $\begin{cases} x = C_1e^t + C_2e^{-2t}, \\ y = -2C_1e^t + C_2e^{-2t} - e^{-t}. \end{cases}$

(c) $\begin{cases} \dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} + 2y = \sin t + 1, \\ \dfrac{dx}{dt} + \dfrac{dy}{dt} + x + y + \cos t = 0 \end{cases}.$

Ans. $\begin{cases} x = C_1e^{-t} + C_2te^{-t} + C_3e^{2t} - \dfrac{3}{5}\sin t - \dfrac{3}{10}\cos t - \dfrac{1}{2} \\ y = (3C_2 - C_1)e^{-t} - C_2te^{-t} - C_3e^{2t} + \dfrac{1}{10}\sin t - \dfrac{1}{5}\cos t + \dfrac{1}{2}. \end{cases}$

(d) $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} - 2t - e^{-t} \quad \dfrac{dx}{dt} + \dfrac{dy}{dt} + 4x + 4y - \sin t + t = 0.$

(e) $\begin{cases} \dfrac{dx}{dt} + \dfrac{dy}{dt} + x = 0, \\ \dfrac{dx}{dt} + 2\dfrac{dy}{dt} = t + 1. \end{cases}$

(f) $\begin{cases} \dfrac{dx}{2t + t^2 + y} = dt, \\ 3 - t + x \quad dt. \end{cases}$

2. Extend the principles of a pair of equations to a system of three equations and solve

(a) $\begin{cases} \dfrac{dx}{dt} + y = 3t, \\ 2\dfrac{dy}{dt} - z = 0, \\ 2\dfrac{dx}{dt} + \dfrac{dz}{dt} = 0. \end{cases}$ Ar $\begin{cases} x = -C_1e^t + C_2e^{-t} + C_3, \\ y = C_1e^t + Ce^{-t} + 3t, \\ z = 2C_1e^t - 2C_2e^{-t} + 6. \end{cases}$

(b) $\begin{cases} \dfrac{d^2x}{dt^2} + y \quad e^{2t}, \\ + z = t, \\ -x - 2y + \dfrac{dz}{dt} = 1. \end{cases}$

3. Solve the system $dx/dt - 2x - y = dy/dt - x - 2y = 0$ by assuming $x = Ae^{mt}$, $y = Be^{mt}$.

KEY: By substituting the assumed values, obtain the equations

$$\begin{cases} A(2 - m) + B = 0 \\ A + (2 - m)B = 0 \end{cases}.$$

Elimination of m leads to $A^2 = B^2$. When $A = B$ we find $m = 3$, and when $A = -B$, $m = 1$. Hence, we have the solutions

$$\begin{cases} x = Ae^{3t} \\ y = Ae^{3t} \end{cases} \qquad \begin{cases} x = Be^t \\ y = -Be^t \end{cases}.$$

The student should verify that $\begin{cases} x = Ae^{3t} + Be^t \\ y = Ae^{3t} - Be^t \end{cases}$ is a solution of the given pair of equations, with A and B arbitrary constants. Since the general solution could not involve more than two arbitrary constants, this is the solution.

4. By the method of Prob. 3, solve

$$(a)\ \begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = 2x + y \end{cases} \qquad Ans.\ \begin{cases} x = Ae^{2t} - Be^{-t}, \\ y = 2Ae^{2t} + Be^{-t}. \end{cases}$$

$$(b)\ \begin{cases} \dfrac{dx}{dt} = 4x - 2y, \\ \dfrac{dy}{dt} = 3x - y. \end{cases}$$

121. Integration in Series. It is often possible, in default of other methods for solving a given differential equation, to obtain its solution in the form of a power series in the independent variable x, as an expression for the dependent variable y. Such a series (or rather such series, for the requisite number of arbitrary constants demanded by the order of the equation will actually turn up in the solution) will represent, within their intervals of convergence, the infinity of functions satisfying the equation.

We arrive at the solution in series by assuming y defined as

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \cdots, \tag{160}$$

and hence

$$y' = A_1 + 2A_2x + 3A_3x^2 + 4A_4x^3 + 5A_5x^4 + \cdots, \tag{161}$$

$$y'' = 2A_2 + 6A_3x + 12A_4x^2 + 20A_5x^3 + 30A_6x^4 + \cdots, \tag{162}$$

etc., substituting these expressions for y and its derivatives in the given equation and determining the coefficients A_0, A_1, A_2, . . . , so that it is satisfied identically.

We illustrate by an example. Let the equation be

$$y'(1 + xy) + y = 0.$$

By (160) we have

$$1 + xy = 1 + A_0x + A_1x^2 + A_2x^3 + A_3x^4 + \cdots.$$

The product $y'(1 + xy)$ is found as a series by multiplying (161) and the series last written according to the method of Sec. 114 (page 450), by which we obtain

$$y'(1 + xy) \qquad A_1 + (A_1A_0 + 2A_2)x + (A_1^2 + 2A_2A_0 + 3A_3)x^2 + (3A_1A_2 + 3A_3A_0 + 4A_4)x^3 + \cdots .$$

On adding this and the series for y, the equation assumes the form

$$0 = A_1 + A_0 + (A_1A_0 + 2A_2 + A_1)x + (A_1^2 + 2A_2A_0 + 3A_3 + A_2)x^2 + (3A_1A_2 + 3A_3A_0 + 4A_4 + A_3)x^3 + \cdots .$$

For this to be satisfied identically, every coefficient must be zero, and thus the set of equalities

$$\begin{aligned} &A_1 + A_0 = 0, \\ &A_1A_0 + 2A_2 + A_1 = 0, \\ &A_1^2 + 2A_2A_0 + 3A_3 + A_2 = 0, \\ &3A_1A_2 + 3A_3A_0 + 4A_4 + A_3 = 0, \text{ etc.} \end{aligned}$$

Hence

$$\begin{aligned} A_1 &= -A_0, \\ A_2 &= \frac{-A_1(1 + A_0)}{2} = \frac{A_0(1 + A_0)}{2}, \\ A_3 &= -\frac{1}{3}(A_1^2 + 2A_0A_2 + A_2) = -\frac{A_0(2A_0^2 + 5A_0 + 1)}{6}, \\ A_4 &\quad -\frac{1}{4}(3A_1A_2 + 3A_3A_0 + A_3) = \\ &\frac{A_0(6A_0^3 + 26A_0^2 + 17A_0 + 1)}{24}, \text{ etc.} \end{aligned}$$

In this manner, all the subsequent coefficients are found in terms of A_0, (or A, if we prefer to call it) which is the arbitrary constant in the solution. The latter, thus, takes the form

$$= A\left[1 - x + \frac{1 + A}{2}x^2 \quad 2A^2 + 5A + 1\, x^3 + \frac{6A^3 + 26A^2 + 17A + 1}{24}x^4 + \cdots\right].$$

For linear differential equations, the effective method is to assume the solution as

$$y = A_0x^m + A_1x^{m+s} + A_2x^{m+2s} + A_3x^{m+3s} + \cdots, \tag{163}$$

the values of m and s, as well as the coefficients, to be determined, again as above, by substituting y and its derivatives into the equation.

As an example, we treat the equation

$$y'' - \left(x + \frac{2}{x}\right)y' + \frac{2y}{x} = 0.$$

To simplify the substitution of (163), we consider first $y = x^m$ and its derivatives $y' = mx^{m-1}$, $y'' = (m^2 - m)x^{m-2}$. These substituted, the left-hand member of the equation becomes

$$(m^2 - m)x^{m-2} - \left(x + \frac{2}{x}\right)mx^{m-1} + \frac{2x^m}{x^2}$$

$$= (m^2 - 3m + 2)x^{m-2} - mx^m$$
$$= (m - 1)(m - 2)x^{m-2} - mx^m.$$

Evidently, for $y = x^{m+s}$ we shall obtain the same result, with m changed to $m + s$, *i.e.*, $(m + s - 1)(m + s - 2)x^{m+s-2} - (m + s)x^{m+s}$; for $y = x^{m+2s}$, $(m + 2s - 1)(m + 2s - 2)x^{m+2s-2} - (m + 2s)x^{m+2s}$, etc. Under (163), then, the left-hand member of the equation becomes

$$\begin{aligned} &A_0[(m - 1)(m - 2)x^{m-2} - mx^m] \\ &\quad + A_1[(m + s - 1)(m + s - 2)x^{m+s-2} - (m + s)x^{m+s}] \\ &\quad + A_2[(m + 2s - 1)(m + 2s - 2)x^{m+2s-2} - (m + 2s)x^{m+2s}] \\ &\quad + A_3[(m + 3s - 1)(m + 3s - 2)x^{m+3s-2} - (m + 3s)x^{m+3s}] \\ &\quad + \cdots \end{aligned}$$

and this must be made identically zero.

We see at once that the choice $s = 2$ will make the powers of x alike for the second term in each bracket and the first term in the following bracket. These terms, furthermore, will cancel away if we set

$$\begin{aligned} -A_0m + A_1(m + s - 1)(m + s - 2) &= 0, \\ -A_1(m + s) + A_2(m + 2s - 1)(m + 2s - 2) &= 0, \\ -A_2(m + 2s) + A_3(m + 3s - 1)(m + 3s - 2) &= 0, \text{ etc.} \end{aligned}$$

To insure the vanishing of the first term in the first bracket (and that will complete the vanishing of the entire left-hand member), we must have $(m - 1)(m - 2) = 0$, making $m = 1$ or $m = 2$.

Now, for $m = 1$ (and $s = 2$) the last set of equalities becomes

$$\begin{aligned} &-A_0 + A_1(2 \cdot 1) = 0, & & A_1 = \frac{1}{1 \cdot 2}A_0, \\ &-A_1 \cdot 3 + A_2(4 \cdot 3) = 0, & \text{giving} \quad & A_2 = \frac{A_1}{4} = \frac{1}{1 \cdot 2 \cdot 4}A_0, \\ &-A_2 \cdot 5 + A_3(6 \cdot 5) = 0, & & A_3 = \frac{A_2}{6} = \frac{1}{1 \cdot 2 \cdot 4 \cdot 6}A_0, \\ &\text{etc.} & & \text{etc.} \end{aligned}$$

and hence, as a solution of the equation we have (setting $m = 1$, and $s = 2$ in (163) and using the set of A's found)

$$y_1 = x + \frac{1}{2}x^3 + \frac{1}{2 \cdot 4}x^5 + \frac{1}{2 \cdot 4 \cdot 6}x^7 + \cdots + \frac{1}{2 \cdot 4 \cdot 6 \cdots (2k)}x^{2k+1} + \cdots$$

where we have arbitrarily set $A_0 = 1$.

For $m = 2$ (and again $s = 2$) the equalities become

$$\begin{aligned} &-A_0 \cdot 2 + A_1(3 \cdot 2) = 0, & & A_1 = \frac{A_0}{3}, \\ &-A_1 \cdot 4 + A_2(5 \cdot 4) = 0, & \text{giving} \quad & A_2 = \frac{A_1}{5} = \frac{A_0}{3 \cdot 5}, \\ &-A_2 \cdot 6 + A_3(7 \cdot 6) = 0, & & A_3 = \frac{A_2}{7} = \frac{A_0}{3 \cdot 5 \cdot 7} \\ &\text{etc.} & & \text{etc.} \end{aligned}$$

and hence, as a solution [with $m = 2$ and $s = 2$ in (163)]

$$y_2 = x^2 + \frac{1}{3}x^4 + \frac{1}{3 \cdot 5}x^6 + \frac{1}{3 \cdot 5 \cdot 7}x^8 + \cdots + \frac{1}{3 \cdot 5 \cdot 7 \cdots (2k+1)}x^{2k+2} + \cdots$$

A_0 being set equal to 1.

By the property of linear homogeneous equations,

$$y = C_1y_1 + C_2y_2$$

where C_1 and C_2 are arbitrary constants, is a solution, and, in fact, the general solution, since the equation is of the second order.

To aid the student in grasping the method, we submit, as a second illustration, the equation

$$x^2y'' + \left(3x + \frac{1}{x^2}\right)y' - \frac{y}{x^3} = 0.$$

Here, $y = x^m$ reduces the left-hand member to

$$x^2(m^2 - m)x^{m-2} + \left(3x + \frac{1}{x^2}\right)mx^{m-1} - x^{m-3}$$
$$= m(m + 2)x^m + (m - 1)x^{m-3}.$$

Then $y = x^{m+s}$ will reduce it to

$$(m + s)(m + s + 2)x^{m+s} + (m + s - 1)x^{m+s-3},$$

$y = x^{m+2s}$, to

$$(m + 2s)(m + 2s + 2)x^{m+2s} + (m + 2s - 1)x^{m+2s-3},$$

and the substitution of (163) will render it

$$\begin{aligned}
&A_0[m(m + 2)x^m + (m - 1)x^{m-3}] \\
&\quad + A_1[(m + s)(m + s + 2)x^{m+s} + (m + s - 1)x^{m+s-3}] \\
&\quad + A_2[(m + 2s)(m + 2s + 2)x^{m+2s} + (m + 2s - 1)x^{m+2s-3}] \\
&\quad + A_3[(m + 3s)(m + 3s + 2)x^{m+3s} + (m + 3s - 1)x^{m+3s-3}] \\
&\qquad + \cdots .
\end{aligned}$$

Again here, we can make the powers of x alike in the second term of each bracket and the first term of the following bracket by setting $s = -3$, and we can make these pairs of terms cancel away by setting

$$\begin{aligned}
A_0(m - 1) + A_1(m + s)(m + s + 2) &= 0, \\
A_1(m + s - 1) + A_2(m + 2s)(m + 2s + 2) &= 0, \\
A_2(m + 2s - 1) + A_3(m + 3s)(m + 3s + 2) &= 0, \text{ etc.}
\end{aligned}$$

The first term in the first bracket will vanish if, and only if, $m = 0$ or $m = -2$.

For $m = 0$ (and $s = -3$), the above equalities become

$$\begin{aligned}
&A_0(-1) + A_1(-3)(-1) = 0, & & A_1 = \frac{A_0}{3}, \\
&A_1(-4) + A_2(-6)(-4) = 0, & \text{giving}\quad & A_2 = \frac{A_1}{6} = \frac{A_0}{3 \cdot 6}, \\
&A_2(-7) + A_3(-9)(-7) = 0, & & A_3 = \frac{A_2}{9} = \frac{A_0}{3 \cdot 6 \cdot 9} \\
&\text{etc.} & & \text{etc.}
\end{aligned}$$

and hence a solution is (since $m = 0$ and $s = -3$)

$$y_1 = 1 + \frac{1}{3}x^{-3} + \frac{1}{3 \cdot 6}x^{-} + \frac{1}{3 \cdot 6 \cdot 9}x^{-9} + \cdots + \frac{1}{3 \cdot 6 \cdot 9 \cdots (3k)}x^{-3k} + \cdots,$$

where we set A_0 arbitrarily equal to 1.

For $m = -2$ (and again $s = -3$), the equalities are

$$\left.\begin{aligned} &A_0(-3) + A_1(-5)(-3) = 0, \\ &A_1(-6) + A_2(-8)(-6) = 0, \\ &A_2(-9) + A_3(-11)(-9) = 0, \\ &\text{etc.} \end{aligned}\right\} \text{giving} \left\{\begin{aligned} &A_1 = \frac{A_0}{5}, \\ &A_2 = \frac{A_1}{8} = \frac{A_0}{5 \cdot 8}, \\ &A_3 = \frac{A_2}{11} = \frac{A_0}{5 \cdot 8 \cdot 11} \\ &\text{etc.} \end{aligned}\right.$$

and as a second solution we have, setting $A_0 = 1$,

$$y_2 = x^{-2} + \frac{1}{5}x^{-5} + \frac{1}{5 \cdot 8}x^{-8} + \frac{1}{5 \cdot 8 \cdot 11}x^{-} + \cdots + \frac{1}{5 \cdot 8 \cdot 11 \cdots (3k + 2)}x^{-3k-2} + \cdots.$$

The general solution of the equation is $y = C_1y_1 + C_2y_2$.

Note that the solutions in this example are in the form of power series in $1/x$.

Exercise 1. Show in what way the method of assuming a solution (163) depends on the linearity of the differential equation.

Problems

1. Solve the following nonlinear equations by assuming a series (160) of the text as solution.

(a) $y' + y^2 - 3x = 0$, (c) $y'(1 + y) + y - 2x = 0$,

(b) $y'(1 + xy) - y^2 = 0$, (d) $y'' + y^2 = 2x$.

Ans. (a) $y = A - A^2x + \frac{3 + 2A^3}{2}x^2 - (A + A^4)x^3 + \frac{5A^2 + 4A^5}{4}x^4 + \cdots$,

(d) $y = A + Bx - \frac{A^2}{2}x^2 + \frac{1 - AB}{3}x^3 + \frac{A^3 - B^2}{12}x^4 + \cdots$.

2. The following equations are integrable in finite form by methods previously learned. Solve each also by assuming a series (160) of the text as solution, and compare the two results in each case.

(a) $xy' = x^4 - x^2 + y$. (b) $y' - 2x^2y = 2x^2$.

3. Solve the following linear equations by series.

(a) $y'' + y'\left(x - \dfrac{2}{x}\right) = 0.$

(b) $x^2y'' + y'(x^4 + x) + y(x^3 - 4) = 0.$

(c) $xy'' + y'(x^3 - 3) + y\left(\dfrac{3}{x} - x^2\right) = 0.$

(d) $x^4y'' + xy'(1 + 2x^2) - 2y = 0.$

(e) $x^5y'' + y'(x + 3x^4) + 3y(1 - x^3) = 0.$

(f) $x^4y'' + y'(6x^3 + x) + y(4x^2 + 2) = 0.$

(g) $x^2y'' - 2xy' - 10y = 0.$ (This is a Cauchy equation, also solvable by the method of Prob. 6, page 480.)

Ans. (a) $y = C_1 + C_2\left[x^3 - \dfrac{3}{2 \cdot 5}x^5 + \dfrac{3}{2 \cdot 4 \cdot 7}x^7 - \dfrac{3}{2 \cdot 4 \cdot 6 \cdot 9}x^9 + \cdots\right].$

$$(b)\ y = C_1\left[x^2 - \frac{x^5}{7} + \frac{x^8}{7 \cdot 10} - \frac{x^{11}}{7 \cdot 10 \cdot 13} + \cdots\right] + C_2\left[x^{-2} - \frac{x}{3} + \frac{x^4}{3 \cdot 6} - \frac{x^7}{3 \cdot 6 \cdot 9} + \cdots\right].$$

$$(d)\ y = C_1\left[1 + x^{-2} + \frac{x^{-4}}{3} + \frac{x^{-6}}{3 \cdot 5} + \frac{x^{-8}}{3 \cdot 5 \cdot 7} + \cdots\right] + C_2\left[x^{-1} + \frac{x^{-3}}{2} + \frac{x^{-5}}{2 \cdot 4} + \frac{x^{-7}}{2 \cdot 4 \cdot 6} + \cdots\right].$$

$$(g)\ y = C_1x^5 + \frac{C_2}{x^2}.$$

4. Obtain the general solution of the Legendre equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$$

as $y = C_1y_1 + C_2y_2$, where

$$y_1 = 1 - \frac{n(n + 1)}{2!}x^2 + \frac{n(n - 2)(n + 1)(n + 3)}{4!}x^4 - \frac{n(n - 2)(n - 4)(n + 1)(n + 3)(n + 5)}{6!}x^6 + \cdots$$

$$y_2 = x - \frac{(n - 1)(n + 2)}{3!}x^3 + \frac{(n - 1)(n - 3)(n + 2)(n + 4)}{5!}x^5 - \frac{(n - 1)(n - 3)(n - 5)(n + 2)(n + 4)(n + 6)}{7!}x^7 + \cdots$$

5. (a) Show that if n is an integer, y_1 or y_2 (Prob. 4) becomes a polynomial.

(b) Considering y_1 and y_2 of Prob. 4 as functions of n, show that

$y_1(0) = 1, \quad y_1(2) = 1 - 3x^2, \quad y_1(4) = 1 - 10x^2 + \frac{35}{3}x^4, \quad \cdots,$

$y_2(1) = x, \quad y_2(3) = x - \frac{5}{3}x^3, \quad y_2(5) = x - \frac{14}{3}x^3 + \frac{21}{5}x^5, \quad \cdots.$

(c) If $P_n(x)$ is defined as $y_1(n)$ or $y_2(n)$, according as n is even or odd, multiplied by a proper numerical factor to make $P_n(1) = 1$, for every n, show that

$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}, \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x,$

$P_4(x) = \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}; \quad P_5(x) = \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x, \quad \cdots.$

These are called *Legendre polynomials.*

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